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Interior capacities of condensers with countably many plates in locally compact spaces

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Abstract. The study deals with the theory of interior capacities of condensers in a locally compact space, a condenser being treated here as a countable, locally finite collection of arbitrary sets with the sign $+1$ or -1 prescribed such that the closures of oppositely signed sets are mutually disjoint. We are motivated by the known fact that, in the noncompact case, the main minimum-problem of the theory is in general unsolvable, and this occurs even under very natural assumptions (e.g., for the Newtonian, Green, or Riesz kernels in $\mathbf{R}^n$, $n \geqslant 2$, and closed condensers of finitely many plates). Therefore it was particularly interesting to find statements of variational problems dual to the main minimum-problem (and hence providing new equivalent definitions to the capacity), but now always solvable (e.g., even for nonclosed, unbounded condensers of infinitely many plates). For all positive definite kernels satisfying Fuglede's condition of consistency between the strong and vague ($=$ weak$*$) topologies, problems with the desired properties are posed and solved. Their solutions provide a natural generalization of the well-known notion of interior equilibrium measures associated with a set. We give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties. Such results are new even for classical kernels in $\mathbf{R}^n$, which is important in applications.

1. Introduction

The article is devoted to further development of the theory of interior capacities of condensers in a locally compact space. A condenser will be treated here as a countable, locally finite collection of arbitrary sets with the sign $+1$ or -1 prescribed such that the closures of oppositely signed sets are mutually disjoint. For a background of the theory for condensers of finitely many plates we refer the reader to [21]–[25]; see also M. Ohtsuka's study [19], where the condensers were additionally assumed to be compact. The reader is expected to be familiar with the principal notions and results of the theory

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of measures and integration; its exposition can be found in [2, 3, 8] (see also [10, 22] for a brief survey).

The theory of interior capacities of condensers provides a natural extension of the well-known theory of interior capacities of sets, developed by O. Frostman [9], H. Cartan [4], and Vallée-Poussin [20] for classical kernels in $\mathbf{R}^n$ and later on generalized by B. Fuglede [10] for general kernels in a locally compact space X. However, those two theories—for sets and, on the other hand, condensers—are drastically different. To illustrate this, it is enough to note that, in the noncompact case, the main minimum-problem of the theory of interior capacities of condensers is in general *unsolvable*, and this phenomenon occurs even under very natural assumptions (e.g., for the Newtonian, Green, or Riesz kernels in $\mathbf{R}^n$, $n \geqslant 2$, and closed condensers of finitely many plates); compare with [4, 10]. Necessary and sufficient conditions for the problem to be solvable have been given in [23, 25]; see Sec. 5 below for a brief survey.

Therefore it was particularly interesting to find statements of variational problems *dual* to the main minimum-problem of the theory of interior capacities of condensers, but in contrast to the last one, now *always solvable*—e.g., even for nonclosed, unbounded condensers of infinitely many plates. (When speaking on duality of variational problems, we mean their extremal values to be equal.)

In all that follows, X denotes a locally compact Hausdorff space and $\mathfrak{M} = \mathfrak{M}(\mathrm{X})$ the linear space of all real-valued Radon measures ν on X equipped with the *vague* (= *weak*$*$) topology, i.e., the topology of pointwise convergence on the class $\mathrm{C}_0(\mathrm{X})$ of all real-valued continuous functions on X with compact support.

A *kernel* κ on X is meant to be an element from $\Phi(\mathrm{X} \times \mathrm{X})$, where $\Phi(\mathrm{Y})$ consists of all lower semicontinuous functions $\psi : \mathrm{Y} \to (-\infty, \infty]$ such that $\psi \geqslant 0$ unless Y is compact. The *energy* and the *potential* of a measure $\nu \in \mathfrak{M}$ with respect to the kernel κ are defined by

$$\kappa(\nu, \nu) := \int \kappa(x, y) d(\nu \otimes \nu)(x, y) \qquad \text{and} \qquad \kappa(\cdot, \nu) := \int \kappa(\cdot, y) d\nu(y),$$

respectively, provided the corresponding integral is well-defined (as a finite number or $\pm\infty$). Let $\mathscr{E} = \mathscr{E}_\kappa(\mathrm{X})$ denote the set of all $\nu \in \mathfrak{M}$ with $-\infty < \kappa(\nu, \nu) < \infty$.

In the present study we shall be concerned with minimal energy problems over certain subclasses of $\mathscr{E}$, properly chosen. For all positive definite kernels satisfying Fuglede's condition of consistency between the strong and vague topologies on $\mathscr{E}$ (see Sec. 2), those variational problems are shown to be *dual* to the main minimum-problem of the theory of interior capacities of condensers (and hence providing some new *equivalent* definitions to the capacity), but now *always solvable*. See Theorems 2, 3, 4 and Corollaries 11, 13 below.

Their solutions provide a natural generalization of the well-known notion of interior equilibrium measures associated with a set (cf. [10]). We give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties; see Sec. 8–11. The results obtained hold true, e.g., for the Newtonian, Green or Riesz kernels in $\mathbf{R}^n$, $n \geqslant 2$, as well as for the restriction of the logarithmic kernel in $\mathbf{R}^2$ to the open unit disk.

2. Preliminaries: topologies, consistent and perfect kernels

Recall that a measure $\nu \geqslant 0$ is said to be *concentrated* on a set $E \subset \mathrm{X}$ if the complement $\complement E := \mathrm{X} \setminus E$ is locally ν-negligible; or, equivalently, if E is ν-measurable and $\nu = \nu_E$, where $\nu_E := \nu|_E$ is the trace of ν upon E.

Let $\mathfrak{M}^+(E)$ be the convex cone of all nonnegative measures concentrated on E, and $\mathscr{E}^+(E) := \mathfrak{M}^+(E) \cap \mathscr{E}$. Also write $\mathfrak{M}^+ := \mathfrak{M}^+(\mathrm{X})$ and $\mathscr{E}^+ := \mathscr{E}^+(\mathrm{X})$.

From now on, the kernel under consideration is always assumed to be *positive definite*, which means that it is symmetric (i.e., $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in \mathrm{X}$) and the energy $\kappa(\nu, \nu)$, $\nu \in \mathfrak{M}$, is nonnegative whenever defined. Then $\mathscr{E}$ forms a pre-Hilbert space with the scalar product

$$\kappa(\nu_1, \nu_2) := \int \kappa(x, y) d(\nu_1 \otimes \nu_2)(x, y)$$

and the seminorm $\|\nu\|_{\mathscr{E}} := \sqrt{\kappa(\nu, \nu)}$; see [10]. A (positive definite) kernel is called *strictly positive definite* if the seminorm $\|\nu\| := \|\nu\|_{\mathscr{E}}$ is a norm.

A measure $\nu \in \mathscr{E}$ is said to be *equivalent in* $\mathscr{E}$ to a given $\nu_0 \in \mathscr{E}$ if $\|\nu - \nu_0\| = 0$; the equivalence class, consisting of all those ν, will be denoted by $[\nu_0]_{\mathscr{E}}$.

In addition to the *strong* topology on $\mathscr{E}$, determined by the seminorm $\|\cdot\|$, it is often useful to consider the *weak* topology on $\mathscr{E}$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \mu)|$, $\mu \in \mathscr{E}$ (see [10]). The Cauchy-Schwarz inequality

$$|\kappa(\nu, \mu)| \leqslant \|\nu\| \, \|\mu\|, \qquad \nu, \mu \in \mathscr{E},$$

implies immediately that the strong topology on $\mathscr{E}$ is finer than the weak one.

In [10], Fuglede introduced the following two properties of *consistency* between the induced strong, weak, and vague topologies on $\mathscr{E}^+$:

(C) *Every strong Cauchy net in* $\mathscr{E}^+$ *converges strongly to any of its vague cluster points;*

(CW) *Every strongly bounded and vaguely convergent net in* $\mathscr{E}^+$ *converges weakly to the vague limit;*

in [11], the properties (C) and (CW) were shown to be *equivalent*.

Definition 1. Following Fuglede [10], we call a kernel κ *consistent* if it satisfies either of the properties (C) and (CW), and *perfect* if, in addition, it is strictly positive definite.

Remark 1. *One has to consider nets or filters in* $\mathfrak{M}^+$ *instead of sequences, since the vague topology in general does not satisfy the first axiom of countability. We follow Moore's and Smith's theory of convergence, based on the concept of nets (see* [16]*; cf. also* [8, *Chap.* 0] *and* [14, *Chap.* 2]*). However, if* X *is metrizable and countable at infinity, then* $\mathfrak{M}^+$ *satisfies the first axiom of countability (see* [10, *Lemma* 1.2.1]*) and the use of nets may be avoided.*

Theorem 1 (Fuglede [10]). *A kernel* κ *is perfect if and only if* $\mathscr{E}^+$ *is strongly complete and the strong topology on* $\mathscr{E}^+$ *is finer than the vague one.*

Example. *In* $\mathbf{R}^n$, $n \geqslant 3$, *the Newtonian kernel* $|x-y|^{2-n}$ *is perfect* [4]. *So are the Riesz kernels* $|x-y|^{\alpha-n}$, $0<\alpha<n$, *in* $\mathbf{R}^n$, $n \geqslant 2$ [5, 6], *and the restriction of the logarithmic kernel* $-\log|x-y|$ *in* $\mathbf{R}^2$ *to the open unit disk* [15]. *Furthermore, if* D *is an open set in* $\mathbf{R}^n$, $n \geqslant 2$, *and its generalized Green function* g_D *exists (see, e.g.,* [13, *Th.* 5.24]*), then* g_D *is perfect as well* [7].

Remark 2. *As is seen from Theorem* 1, *the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over classes of nonnegative measures with finite energy. Indeed, the theory of capacities of sets has been developed in* [10] *exactly for those kernels. We shall show below that this concept is efficient, as well, in minimal energy problems over classes of signed measures associated with a condenser. This is guaranteed by a theorem on the strong completeness of proper subspaces of* $\mathscr{E}$, *to be stated in Sec.* 12.

3. Condensers of countably many plates. Measures associated with a condenser; their energies and potentials

3.1. Let I^+ and I^- be countable (finite or infinite) disjoint sets of indices $i \in \mathbf{N}$, the latter being allowed to be empty, and let I denote their union. Assume that to every $i \in I$ there corresponds a nonempty set $A_i \subset \mathrm{X}$.

Definition 2. A collection $\mathbf{A} = (A_i)_{i \in I}$ is called an (I^+, I^-)-*condenser* (or simply a *condenser*) in X if every compact subset of X intersects with at most finitely many A_i and

$$\overline{A_i} \cap \overline{A_j} = \varnothing \qquad \text{for all } i \in I^+,\ j \in I^-. \tag{1}$$

The sets A_i, $i \in I^+$, and A_j, $j \in I^-$, are said to be the *positive* and, respectively, the *negative plates* of the condenser $\mathbf{A}$. Note that any two equally signed plates can intersect each other.

Given I^+ and I^-, let $\mathfrak{C} = \mathfrak{C}(I^+, I^-)$ be the class of all (I^+, I^-)-condensers in X. A condenser $\mathbf{A} \in \mathfrak{C}$ is called *closed* or *compact* if all A_i, $i \in I$, are closed or, respectively, compact. Similarly, we call it *universally measurable* if all the plates are universally measurable—that is, measurable with respect to every $\nu \in \mathfrak{M}^+$. Next, $\mathbf{A} = (A_i)_{i \in I}$ is said to be *finite* if so is I.

Given $\mathbf{A} = (A_i)_{i \in I}$, write $\overline{\mathbf{A}} := (\overline{A_i})_{i \in I}$. Then, due to (1), $\overline{\mathbf{A}}$ is a (closed) (I^+, I^-)-condenser. In the sequel, also the following notation will be used:

$$A := \bigcup_{i \in I} A_i, \qquad A^+ := \bigcup_{i \in I^+} A_i, \qquad A^- := \bigcup_{i \in I^-} A_i.$$

Note that A^+ and A^- might both be noncompact even for a compact $\mathbf{A}$.

3.2. With the preceding notation, write

$$\alpha_i := \begin{cases} +1 & \text{if } i \in I^+, \\ -1 & \text{if } i \in I^-. \end{cases}$$

Given $\mathbf{A} \in \mathfrak{C}$, let $\mathfrak{M}(\mathbf{A})$ consist of all (finite or infinite) *linear combinations*

$$\mu := \sum_{i \in I} \alpha_i \mu^i, \qquad \text{where } \mu^i \in \mathfrak{M}^+(A_i).$$

Any two μ_1 and μ_2 in $\mathfrak{M}(\mathbf{A})$ are regarded to be *identical* ($\mu_1 = \mu_2$) if and only if $\mu_1^i = \mu_2^i$ for all $i \in I$. Observe that, under the relation of identity in $\mathfrak{M}(\mathbf{A})$ thus defined, the following correspondence is one-to-one:

$$\mathfrak{M}(\mathbf{A}) \ni \mu \mapsto (\mu^i)_{i \in I} \in \prod_{i \in I} \mathfrak{M}^+(A_i).$$

We call $\mu \in \mathfrak{M}(\mathbf{A})$ a *measure associated with* $\mathbf{A}$, and μ^i, $i \in I$, its *i-coordinate*.

For measures associated with a condenser, it is therefore natural to introduce the following concept of convergence, actually corresponding to the vague convergence by coordinates. Let S denote a directed set of indices, and let μ_s, $s \in S$, and μ_0 be given elements of the class $\mathfrak{M}(\overline{\mathbf{A}})$.

Definition 3. A net $(\mu_s)_{s \in S}$ is said to converge to μ_0 **A**-*vaguely* if

$$\mu_s^i \to \mu_0^i \qquad \text{vaguely for all } i \in I.$$

Then $\mathfrak{M}(\overline{\mathbf{A}})$, equipped with the topology of **A**-vague convergence, and the product space $\prod_{i \in I} \mathfrak{M}^+(\overline{A_i})$ become homeomorphic. Since $\mathfrak{M}(\mathrm{X})$ is Hausdorff, so are both $\mathfrak{M}(\overline{\mathbf{A}})$ and $\prod_{i \in I} \mathfrak{M}^+(\overline{A_i})$ (see, e.g., [14, Chap. 3, Th. 5]).

Similarly, a set $\mathfrak{F} \subset \mathfrak{M}(\overline{\mathbf{A}})$ is called **A**-*vaguely bounded* if all its i-projections are vaguely bounded—that is, if for every $\varphi \in C_0(X)$ and every $i \in I$,

$$\sup_{\mu \in \mathfrak{F}} |\mu^i(\varphi)| < \infty.$$

Lemma 1. *If $\mathfrak{F} \subset \mathfrak{M}(\overline{\mathbf{A}})$ is **A**-vaguely bounded, then it is **A**-vaguely relatively compact.*

Proof. Since by [2, Chap. III, §2, Prop. 9] any vaguely bounded part of $\mathfrak{M}$ is vaguely relatively compact, the lemma follows from Tychonoff's theorem on the product of compact spaces (see, e.g., [14, Chap. 5, Th. 13]). □

3.3. Since each compact subset of X intersects with at most finitely many A_i, for every $\varphi \in C_0(X)$ only a finite number of $\mu^i(\varphi)$ (where $\mu \in \mathfrak{M}(\mathbf{A})$ is given) are nonzero. This yields that to every $\mu \in \mathfrak{M}(\mathbf{A})$ there corresponds a unique Radon measure $R\mu$ such that

$$R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \qquad \text{for all } \varphi \in C_0(X);$$

its positive and negative parts in Jordan's decomposition can be written in the form

$$R\mu^+ = \sum_{i \in I^+} \mu^i \qquad \text{and} \qquad R\mu^- = \sum_{i \in I^-} \mu^i,$$

respectively. Of course, the mapping $R : \mathfrak{M}(\mathbf{A}) \to \mathfrak{M}$ thus defined is in general non-injective, i.e., one may choose $\mu' \in \mathfrak{M}(\mathbf{A})$ so that $\mu' \neq \mu$, while $R\mu' = R\mu$. (It would be injective if all the plates A_i, $i \in I$, were mutually disjoint.) We shall call $\mu, \mu' \in \mathfrak{M}(\mathbf{A})$ *R-equivalent* whenever their R-images coincide.

Lemma 2. *The **A**-vague convergence of $(\mu_s)_{s \in S}$ to μ_0 implies the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$.*

Proof. This is obvious in view of the fact that the support of any $\varphi \in C_0(X)$ might have points in common with only finitely many $\overline{A_i}$. □

Remark 3. *Lemma* 2 *in general can not be inverted. However, if all the sets $\overline{A_i}$, $i \in I$, are mutually disjoint, then the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$ implies the **A**-vague convergence of $(\mu_s)_{s \in S}$ to μ_0. This can be seen by using the Tietze-Urysohn extension theorem (see, e.g.,* [8, *Th.* 0.2.13]*).*

3.4. To define energies and potentials of linear combinations $\mu \in \mathfrak{M}(\mathbf{A})$, we start with the following two lemmas, the former one being well-known (see [10]).

Lemma 3. *Let* Y *be a locally compact Hausdorff space. If* $\psi \in \Phi(\mathrm{Y})$ *is given, then the map* $\nu \mapsto \int \psi \, d\nu$ *is vaguely lower semicontinuous on* $\mathfrak{M}^+(\mathrm{Y})$.

Lemma 4. *Fix* $\mu \in \mathfrak{M}(\mathbf{A})$ *and* $\psi \in \Phi(\mathrm{X})$. *If* $\int \psi \, dR\mu$ *is well-defined, then*

$$\int \psi \, dR\mu = \sum_{i \in I} \alpha_i \int \psi \, d\mu^i, \tag{2}$$

and $\int \psi \, dR\mu$ *is finite if and only if the series on the right converges absolutely.*

Proof. We can assume ψ to be nonnegative, for if not, then we replace ψ by a function $\psi' \geqslant 0$ obtained by adding to ψ a suitable constant $c > 0$, which is always possible since a lower semicontinuous function is bounded from below on a compact space. Hence,

$$\int \psi \, dR\mu^+ \geqslant \sum_{i \in I^+, i \leqslant N} \int \psi \, d\mu^i \qquad \text{for all } N \in \mathbf{N}.$$

On the other hand, the sum of μ^i over all $i \in I^+$ that do not exceed N approaches $R\mu^+$ vaguely as $N \to \infty$; consequently, by Lemma 3,

$$\int \psi \, dR\mu^+ \leqslant \lim_{N \to \infty} \sum_{i \in I^+, i \leqslant N} \int \psi \, d\mu^i.$$

Combining the last two inequalities and then letting $N \to \infty$, we get

$$\int \psi \, dR\mu^+ = \sum_{i \in I^+} \int \psi \, d\mu^i.$$

Since the same holds true for $R\mu^-$ and I^- instead of $R\mu^+$ and I^+, respectively, the lemma follows. □

Corollary 1. *If* $\mu, \mu_1 \in \mathfrak{M}(\mathbf{A})$ *and* $x \in \mathrm{X}$, *then*

$$\kappa(x, R\mu) = \sum_{i \in I} \alpha_i \kappa(x, \mu^i), \tag{3}$$

$$\kappa(R\mu, R\mu_1) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu_1^j), \tag{4}$$

each of the identities being understood in the sense that its right-hand side is well-defined whenever so is the left-hand one and then they coincide. Furthermore, the left-hand side in (3) *or in* (4) *is finite if and only if the corresponding series on the right converges absolutely.*

Proof. Relation (3) is a direct consequence of (2), while (4) follows from Fubini's theorem (cf. [3, §8, Th. 1]) and Lemma 4 on account of the fact that $\kappa(x, \nu)$, where $\nu \in \mathfrak{M}^+$ is given, is lower semicontinuous on X (see, e.g., [10]). □

Definition 4. Given $\mu, \mu_1 \in \mathfrak{M}(\mathbf{A})$, we shall call $\kappa(\cdot, \mu) := \kappa(\cdot, R\mu)$ the *potential* of μ and $\kappa(\mu, \mu_1) := \kappa(R\mu, R\mu_1)$ the *mutual energy* of μ and μ_1 (of course, provided the right-hand side of the corresponding relation is well-defined). For $\mu = \mu_1$ we get the *energy* $\kappa(\mu, \mu)$ of μ; i.e., if $\kappa(R\mu, R\mu)$ is well-defined, then

$$\kappa(\mu, \mu) := \kappa(R\mu, R\mu) = \sum_{i, j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j). \tag{5}$$

Corollary 2. *For $\mu \in \mathfrak{M}(\mathbf{A})$ to be of finite energy, it is necessary and sufficient that $\mu^i \in \mathscr{E}$ for all $i \in I$ and*

$$\sum_{i \in I} \|\mu^i\|^2 < \infty.$$

Proof. This follows immediately from the definition of $\kappa(\mu, \mu)$ in view of the inequality $2\kappa(\nu_1, \nu_2) \leqslant \|\nu_1\|^2 + \|\nu_2\|^2$ for $\nu_1, \nu_2 \in \mathscr{E}$. □

Remark 4. *Observe that the series in* (5) *actually defines the energy of the vector measure $(\mu^i)_{i \in I}$ relative to the interaction matrix $(\alpha_i \alpha_j)_{i, j \in I}$; compare with* [12] *and* [17, *Chap.* 5, §4]. *However, our approach is essentially based on the fact that, due to the specific interaction matrix, the same value can also be obtained as the energy of the corresponding Radon measure $R\mu$.*

Remark 5. *Since we make no difference between $\mu \in \mathfrak{M}(\mathbf{A})$ and $R\mu$ when dealing with their energies or potentials, we shall sometimes call a measure associated with* **A** *simply a measure—certainly, if this causes no confusion.*

3.5. Let $\mathscr{E}(\mathbf{A})$ consist of all $\mu \in \mathfrak{M}(\mathbf{A})$ of finite energy $\kappa(\mu, \mu)$. Since $\mathfrak{M}(\mathbf{A})$ forms a convex cone, it is seen from Corollary 2 that so does $\mathscr{E}(\mathbf{A})$.

One of the crucial arguments in our approach is that $\mathscr{E}(\mathbf{A})$ can be treated as a *semimetric space* with the semimetric

$$\|\mu_1 - \mu_2\|_{\mathscr{E}(\mathbf{A})} := \|R\mu_1 - R\mu_2\|_{\mathscr{E}}, \qquad \mu_1, \mu_2 \in \mathscr{E}(\mathbf{A}); \tag{6}$$

then $\mathscr{E}(\mathbf{A})$ and its R-image become *isometric*. The topology on $\mathscr{E}(\mathbf{A})$ defined by means of the semimetric $\|\cdot\| := \|\cdot\|_{\mathscr{E}(\mathbf{A})}$ is called *strong*.

Two elements of $\mathscr{E}(\mathbf{A})$, μ_1 and μ_2, are said to be *equivalent in* $\mathscr{E}(\mathbf{A})$ if $\|\mu_1 - \mu_2\| = 0$. Note that the equivalence in $\mathscr{E}(\mathbf{A})$ implies R-equivalence (i.e., then $R\mu_1 = R\mu_2$) provided the kernel κ is strictly positive definite, and it implies the identity (i.e., then $\mu_1 = \mu_2$) if, moreover, all A_i, $i \in I$, are mutually disjoint.

4. Interior capacities of condensers. Elementary properties

4.1. Let $\mathscr{H}$ be a set in the pre-Hilbert space $\mathscr{E}$ or in the semimetric space $\mathscr{E}(\mathbf{A})$, an (I^+, I^-)-condenser $\mathbf{A}$ being given. In either case, let us introduce the quantity

$$\|\mathscr{H}\|^2 := \inf_{\nu \in \mathscr{H}} \|\nu\|^2,$$

interpreted as $+\infty$ if $\mathscr{H}$ is empty. If $\|\mathscr{H}\|^2 < \infty$, then one can consider the variational problem on the existence of $\lambda = \lambda(\mathscr{H}) \in \mathscr{H}$ with minimal energy

$$\|\lambda\|^2 = \|\mathscr{H}\|^2;$$

such a problem will be referred to as the *$\mathscr{H}$-problem*. The $\mathscr{H}$-problem is called *solvable* if a minimizer $\lambda(\mathscr{H})$ exists.

The following lemma is a slight generalization of [10, Lemma 4.1.1].

Lemma 5. *Suppose $\mathscr{H}$ is convex and $\lambda = \lambda(\mathscr{H})$ exists. Then for any $\nu \in \mathscr{H}$,*

$$\|\nu - \lambda\|^2 \leqslant \|\nu\|^2 - \|\lambda\|^2. \tag{7}$$

Proof. Let $\mathscr{H} \subset \mathscr{E}$. For every $h \in (0, 1]$, the measure $\mu := (1-h)\lambda + h\nu$ belongs to $\mathscr{H}$, and therefore $\|\mu\|^2 \geqslant \|\lambda\|^2$. Evaluating $\|\mu\|^2$ and then letting $h \to 0$, we get $\kappa(\nu, \lambda) \geqslant \|\lambda\|^2$, and (7) follows (see [10]).

Suppose now $\mathscr{H} \subset \mathscr{E}(\mathbf{A})$. Then $R\mathscr{H}$ is a convex subset of $\mathscr{E}$, while $R\lambda$ is a minimizer in the $R\mathscr{H}$-problem. What has just been shown therefore yields $\|R\nu - R\lambda\|^2 \leqslant \|R\nu\|^2 - \|R\lambda\|^2$, which gives (7) when combined with (6). $\square$

We shall be concerned with the $\mathscr{H}$-problem for various specific $\mathscr{H}$ related to the notion of *interior capacity* of an (I^+, I^-)-condenser (in particular, of a set); see Sec. 4.2 and Sec. 8 below for the definitions.

4.2. Fix a vector-valued function $\mathbf{g} = (g_i)_{i \in I}$, where all $g_i : \mathrm{X} \to (0, \infty)$ are continuous, and a numerical vector $\mathbf{a} = (a_i)_{i \in I}$ with $a_i > 0$, $i \in I$. Given an (I^+, I^-)-condenser $\mathbf{A}$ in X, write

$$\mathfrak{M}^+(A_i, a_i, g_i) := \left\{ \nu \in \mathfrak{M}^+(A_i) : \int g_i \, d\nu = a_i \right\},$$

and let $\mathfrak{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ consist of all $\mu \in \mathfrak{M}(\mathbf{A})$ with $\mu^i \in \mathfrak{M}^+(A_i, a_i, g_i)$ for all i. Given a kernel κ, also write

$$\mathscr{E}^+(A_i, a_i, g_i) := \mathfrak{M}^+(A_i, a_i, g_i) \cap \mathscr{E}, \qquad \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathscr{E}(\mathbf{A}).$$

Definition 5. We shall call the value

$$\operatorname{cap} \mathbf{A} := \operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \frac{1}{\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2} \tag{8}$$

the (*interior*) *capacity* of an (I^+, I^-)-condenser $\mathbf{A}$ (with respect to κ, $\mathbf{a}$, and $\mathbf{g}$).

Here and in the sequel, we adopt the convention that $1/0 = +\infty$. It follows from the positive definiteness of the kernel that $0 \leqslant \operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leqslant \infty$. (See Sec. 4.4 and Sec. 4.5 below, providing necessary and sufficient conditions for the non-degenerated case $0 < \operatorname{cap} \mathbf{A} < \infty$ to hold.)

Remark 6. *If I is a singleton, then any condenser consists of just one set, say A_1. If, moreover, $g_1 = 1$ and $a_1 = 1$, then the notion of interior capacity of a condenser certainly reduces to the notion of interior capacity of a set (see* [10]*). We denote it by $C(\cdot)$ as well, i.e., $C(A_1) := 1/\|\mathscr{E}^+(A_1, 1, 1)\|^2$.*

Remark 7. *In the case of the Newtonian kernel $|x - y|^{-1}$ in $\mathbf{R}^3$, the notion of capacity of a condenser $\mathbf{A}$ has an evident electrostatic interpretation. In the framework of the corresponding electrostatics problem, the functions g_i, $i \in I$, serve as a characteristic of nonhomogeneity of the conductors A_i.*

4.3. On $\mathfrak{C} = \mathfrak{C}(I^+, I^-)$, it is natural to introduce a partial order relation $\prec$ by declaring $\mathbf{A}' \prec \mathbf{A}$ to mean that $A_i' \subset A_i$ for all $i \in I$. Here, $\mathbf{A}' = (A_i')_{i \in I}$. Then $\operatorname{cap}(\cdot, \mathbf{a}, \mathbf{g})$ is a nondecreasing function of a condenser, namely

$$\operatorname{cap}(\mathbf{A}', \mathbf{a}, \mathbf{g}) \leqslant \operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \qquad \text{whenever } \mathbf{A}' \prec \mathbf{A}. \tag{9}$$

Given $\mathbf{A} \in \mathfrak{C}$, denote by $\{\mathbf{K}\}_{\mathbf{A}}$ the increasing filtering family of all compact condensers $\mathbf{K} = (K_i)_{i \in I} \in \mathfrak{C}$ such that $\mathbf{K} \prec \mathbf{A}$.

Lemma 6. *If $\mathbf{K}$ ranges over $\{\mathbf{K}\}_{\mathbf{A}}$, then*

$$\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{\mathbf{K} \uparrow \mathbf{A}} \operatorname{cap}(\mathbf{K}, \mathbf{a}, \mathbf{g}). \tag{10}$$

Proof. We can assume $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ to be nonzero, since otherwise (10) follows at once from (9). Then the set $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ must be nonempty; fix μ, one of its elements. Given $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$ and $i \in I$, let $\mu_{\mathbf{K}}^i$ denote the trace of μ^i upon K_i, i.e., $\mu_{\mathbf{K}}^i := \mu_{K_i}^i$. Applying Lemma 1.2.2 from [10], we conclude that

$$\int g_i \, d\mu^i = \lim_{\mathbf{K} \uparrow \mathbf{A}} \int g_i \, d\mu_{\mathbf{K}}^i, \qquad i \in I, \tag{11}$$

$$\kappa(\mu^i, \mu^j) = \lim_{\mathbf{K} \uparrow \mathbf{A}} \kappa(\mu_{\mathbf{K}}^i, \mu_{\mathbf{K}}^j), \qquad i, j \in I. \tag{12}$$

Fix $\varepsilon > 0$. It follows from (11) and (12) that for every $i \in I$ one can choose a compact set $K_i^0 \subset A_i$ so that

$$\frac{a_i}{\int g_i \, d\mu^i_{K_i^0}} < 1 + \varepsilon i^{-2}, \tag{13}$$

$$\left| \|\mu^i\|^2 - \|\mu^i_{K_i^0}\|^2 \right| < \varepsilon^2 i^{-4}. \tag{14}$$

Having denoted $\mathbf{K}^0 := (K_i^0)_{i \in I}$, for every $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$ that follows $\mathbf{K}^0$ we therefore have $\int g_i \, d\mu^i_{\mathbf{K}} \neq 0$ and

$$\hat{\mu}_{\mathbf{K}} := \sum_{i \in I} \frac{\alpha_i a_i}{\int g_i \, d\mu^i_{\mathbf{K}}} \mu^i_{\mathbf{K}} \in \mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g}),$$

the finiteness of the energy being obtained from (14) and Corollary 2. Thus,

$$\|\hat{\mu}_{\mathbf{K}}\|^2 \geqslant \|\mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g})\|^2. \tag{15}$$

We next proceed by showing that

$$\|\mu\|^2 = \lim_{\mathbf{K} \uparrow \mathbf{A}} \|\hat{\mu}_{\mathbf{K}}\|^2. \tag{16}$$

To this end, it can be assumed that $\kappa \geqslant 0$; for if not, then A must be finite since $\mathbf{X}$ is compact, and (16) follows from (11) and (12) when substituted into (5). Therefore, for every $\mathbf{K}$ that follows $\mathbf{K}_0$ and every $i \in I$ we obtain

$$\|\mu^i_{\mathbf{K}}\| \leqslant \|\mu^i\| \leqslant \|R\mu^+ + R\mu^-\|, \tag{17}$$

$$\|\mu^i - \mu^i_{\mathbf{K}}\| < \varepsilon i^{-2}, \tag{18}$$

the latter being clear from (14) because of $\kappa(\mu^i_{\mathbf{K}}, \mu^i - \mu^i_{\mathbf{K}}) \geqslant 0$. Also observe that, by (5),

$$\begin{aligned} \left| \|\mu\|^2 - \|\hat{\mu}_{\mathbf{K}}\|^2 \right| &\leqslant \sum_{i,j \in I} \left| \kappa(\mu^i, \mu^j) - \frac{a_i}{\int g_i \, d\mu^i_{\mathbf{K}}} \frac{a_j}{\int g_j \, d\mu^j_{\mathbf{K}}} \kappa(\mu^i_{\mathbf{K}}, \mu^j_{\mathbf{K}}) \right| \\ &\leqslant \sum_{i,j \in I} \Bigg[\kappa(\mu^i - \mu^i_{\mathbf{K}}, \mu^j) + \kappa(\mu^i_{\mathbf{K}}, \mu^j - \mu^j_{\mathbf{K}}) \\ &\qquad + \left(\frac{a_i}{\int g_i \, d\mu^i_{\mathbf{K}}} \frac{a_j}{\int g_j \, d\mu^j_{\mathbf{K}}} - 1 \right) \kappa(\mu^i_{\mathbf{K}}, \mu^j_{\mathbf{K}}) \Bigg]. \end{aligned}$$

When combined with (13), (17), and (18), this yields

$$\left| \|\mu\|^2 - \|\hat{\mu}_{\mathbf{K}}\|^2 \right| \leqslant M\varepsilon \qquad \text{for all } \mathbf{K} \succ \mathbf{K}_0,$$

where M is finite and independent of $\mathbf{K}$, and the required relation (16) follows.

Substituting (15) into (16), in view of the arbitrary choice of $\mu \in \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ we get

$$\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 \geqslant \lim_{\mathbf{K}\uparrow\mathbf{A}} \|\mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g})\|^2.$$

Since the converse inequality is obvious from (9), the proof is complete. □

Let $\mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$ denote the class of all $\mu \in \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ such that, for every $i \in I$, the support $S(\mu^i)$ of μ^i is compact and contained in A_i.

Corollary 3. *The capacity* $\mathrm{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *remains unchanged if the class* $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *in its definition is replaced by* $\mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$. *In other words,*

$$\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 = \|\mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2.$$

Proof. We can assume $\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$ to be finite, for otherwise the corollary follows from $\mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Then, by (9) and (10), for every $\varepsilon > 0$ there is $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$ such that $\|\mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g})\|^2 \leqslant \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 + \varepsilon$. Together with $\|\mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g})\|^2 \geqslant \|\mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 \geqslant \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$, this completes the proof. □

4.4. Unless explicitly stated otherwise, in all that follows it is assumed that

$$\mathrm{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) > 0. \tag{19}$$

Lemma 7. *For* (19) *to hold, it is necessary and sufficient that any of the following three equivalent conditions be satisfied:*

(i) $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is nonempty;*

(ii) $\sum_{i \in I} \|\nu_i\|^2 < \infty$ *for some* $\nu_i \in \mathscr{E}^+(A_i, a_i, g_i)$*;*

(iii) $\sum_{i \in I} \|\mathscr{E}^+(A_i, a_i, g_i)\|^2 < \infty$.

Proof. The equivalence of (19) and (i) is obvious, while that of (i) and (ii) can be obtained directly from Corollary 2. If (iii) holds, then for every $i \in I$ one can choose $\nu_i \in \mathscr{E}^+(A_i, a_i, g_i)$ so that $\|\nu_i\|^2 < \|\mathscr{E}^+(A_i, a_i, g)\|^2 + i^{-2}$, and (ii) follows. Since (iii) is an immediate consequence of (ii), the proof is complete. □

Corollary 4. *For* (19) *to be satisfied, it is necessary that*

$$C(A_i) > 0 \qquad \textit{for all } i \in I. \tag{20}$$

If $\mathbf{A}$ *is finite, then* (19) *and* (20) *are actually equivalent*[1].

[1] However, (19) and (20) are no longer equivalent if $\mathbf{A}$ is infinite—cf. Corollary 5.

PROOF. For Lemma 7, (ii) to hold, it is necessary that, for every $i \in I$, there exist a nonzero measure $\nu \in \mathcal{E}^+(A_i)$, which in turn is equivalent to (20) according to [10, Lemma 2.3.1]. Since the former implication can be inverted whenever $\mathbf{A}$ is finite, the proof is complete. $\square$

Let $g_{i,\inf}$ and $g_{i,\sup}$ be the infimum and the supremum of g_i over A_i, and let

$$\mathbf{g}_{\inf} := \inf_{i \in I} g_{i,\inf}, \qquad \mathbf{g}_{\sup} := \sup_{i \in I} g_{i,\sup}.$$

COROLLARY 5. *If* $0 < \mathbf{g}_{\inf} \leqslant \mathbf{g}_{\sup} < \infty$, *then* (19) *holds if and only if*

$$\sum_{i \in I} \frac{a_i^2}{C(A_i)} < \infty.$$

PROOF. Lemma 7 yields the corollary when combined with the relation

$$\frac{a_i^2}{g_{i,\sup}^2 C(A_i)} \leqslant \|\mathcal{E}^+(A_i, a_i, g_i)\|^2 \leqslant \frac{a_i^2}{g_{i,\inf}^2 C(A_i)}, \tag{21}$$

which can be seen by reasons of homogeneity.

Indeed, to establish (21), we can certainly assume $C(A_i)$ to be nonzero, for otherwise Corollary 4 with $I = \{i\}$ shows that each of the three parts in (21) equals $+\infty$. Therefore, there exists $\theta_i \in \mathcal{E}^+(A_i, 1, 1)$. Since

$$\hat{\theta}_i := \frac{a_i \theta_i}{\int g_i \, d\theta_i} \in \mathcal{E}^+(A_i, a_i, g_i),$$

we get

$$a_i^2 \|\theta_i\|^2 \geqslant g_{i,\inf}^2 \|\hat{\theta}_i\|^2 \geqslant g_{i,\inf}^2 \|\mathcal{E}^+(A_i, a_i, g_i)\|^2,$$

and the right-hand side of (21) is obtained by letting θ_i range over $\mathcal{E}^+(A_i, 1, 1)$.

To verify the left-hand side, fix $\omega_i \in \mathcal{E}^+(A_i, a_i, g_i)$. Then

$$0 < a_i g_{i,\sup}^{-1} \leqslant \omega_i(\mathrm{X}) \leqslant a_i g_{i,\inf}^{-1} < \infty.$$

Hence, $\omega_i / \omega_i(\mathrm{X}) \in \mathcal{E}^+(A_i, 1, 1)$ and

$$\|\omega_i\|^2 \geqslant a_i^2 g_{i,\sup}^{-2} \|\mathcal{E}^+(A_i, 1, 1)\|^2.$$

In view of the arbitrary choice of $\omega_i \in \mathcal{E}^+(A_i, a_i, g_i)$, this completes the proof. $\square$

4.5. In the following assertion, providing necessary conditions for cap $\mathbf{A}$ to be finite, it is assumed that $g_{i,\inf} > 0$ for all $i \in I$.

LEMMA 8. *If* $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty$, *then there exists* $j \in I$ *with* $C(A_j) < \infty$.

Proof. Suppose, on the contrary, $C(A_i) = \infty$ for all $i \in I$. Given $\varepsilon > 0$, for every $i \in I$ one can choose $\nu^i \in \mathscr{E}^+(A_i, 1, 1)$ with compact support so that $\|\nu^i\| \leqslant \varepsilon a_i^{-1} i^{-2} g_{i,\inf}$. Since then

$$\hat{\nu} := \sum_{i \in I} \frac{\alpha_i a_i \nu^i}{\int g_i \, d\nu^i} \in \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$$

and $\|\hat{\nu}\| \leqslant \varepsilon \sum_{i \in I} i^{-2}$, we arrive at a contradiction by letting $\varepsilon \to 0$. □

The following assertion[2], to be proved in Sec. 14, shows that, under proper additional requirements, Lemma 8 can be inverted.

Corollary 6. *Let κ be perfect and either $I^- = \varnothing$ or the following conditions both hold:*

$$\sum_{i \in I} a_i g_{i,\inf}^{-1} < \infty, \tag{22}$$

$$\sup_{x \in A^+, y \in A^-} \kappa(x, y) < \infty. \tag{23}$$

If there exists $j \in I$ such that A_j is closed, $C(A_j) < \infty$, and $g_{j,\sup} < \infty$, then

$$\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty.$$

Remark 8. *Corollary 6 remains valid if, instead of the boundedness of g_j, we require the following restriction on its growth: there exist $r_j \in (1, \infty)$ and $\tau_j \in \mathscr{E}$ such that $g_j^{r_j}(x) \leqslant \kappa(x, \tau_j)$ for all $x \in A_j$.*

5. On the solvability of the main minimum-problem

Because of (19), we are naturally led to the $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$-*problem* (cf. Sec. 4.1), i.e., the problem on the existence of $\lambda \in \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with minimal energy

$$\|\lambda\|^2 = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2;$$

the $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$-problem may certainly be regarded as the main minimum-problem of the theory of interior capacities of condensers. The collection (possibly empty) of all minimizing measures λ in this problem will be denoted by $\mathscr{S}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

If, moreover, $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty$, then let us look at the $\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$-*problem* as well. By reasons of homogeneity, both the $\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$- and

[2] It is in fact a corollary to Lemma 9, to be formulated in Sec. 7 below.

the $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$-problems are simultaneously either solvable or unsolvable, and their extremal values are related to each other by the following law:

$$\frac{1}{\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2} = \|\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})\|^2. \tag{24}$$

Assume for a moment that $\mathbf{A}$ is compact. Since then $\nu \mapsto \int g_i \, d\nu$ is vaguely continuous on $\mathfrak{M}^+(A_i)$, $\mathfrak{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is $\mathbf{A}$-vaguely compact. Therefore, if $\mathbf{A}$ is additionally assumed to be finite, while κ is continuous on $A^+ \times A^-$, then the energy $\|\mu\|^2$ is $\mathbf{A}$-vaguely lower semicontinuous on $\mathscr{E}(\mathbf{A})$ and the solvability of both the problems immediately follows (cf. [19, Th. 2.30]).

But these arguments break down if any of the above-mentioned three assumptions is dropped. In particular, the class $\mathfrak{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is no longer $\mathbf{A}$-vaguely compact whenever $\mathbf{A}$ is noncompact. Moreover, it has been shown by the author that, in the noncompact case, the problems are in general *unsolvable* and this occurs even under very natural assumptions (e.g., for the Newtonian, Green, or Riesz kernels in $\mathbf{R}^n$, $n \geqslant 2$, and finite, closed condensers).

In particular, it was proved in [23] that, if $\mathbf{A}$ is finite and closed, κ is perfect, and bounded and continuous on $A^+ \times A^-$, and satisfies the generalized maximum principle (see [15, Chap. VI]), while $g_i = g_j$ for all i, j and $0 < \mathbf{g}_{\inf} \leqslant \mathbf{g}_{\sup} < \infty$, then either of the $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$- and the $\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$-problems is solvable for any $\mathbf{a}$ if and only if $C(A_i) < \infty$ for all $i \in I$. If, moreover, $C(A_{i_0}) = \infty$ for some $i_0 \in I$, then both the problems are unsolvable for all $\mathbf{a}$ with a_{i_0} sufficiently large.

In [25, Th. 1], the last statement was sharpened. It was shown that if, in addition to all the preceding assumptions, for all $i \neq i_0$,

$$C(A_i) < \infty \qquad \text{and} \qquad A_i \cap A_{i_0} = \varnothing,$$

while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets, then there exists $\Lambda_{i_0} \in [0, \infty)$ such that the problems are unsolvable if and only if $a_{i_0} > \Lambda_{i_0}$. Actually, $\Lambda_{i_0} = \int g_{i_0} \, d\tilde{\lambda}^{i_0}$, where $\tilde{\lambda}$ is a minimizer (it exists) in the auxiliary $\mathscr{H}$-problem with $\mathscr{H} := \{\mu \in \mathscr{E}(\mathbf{A}) : \mu^i \in \mathscr{E}^+(A_i, a_i, g_i) \text{ for all } i \neq i_0\}$.

6. Standing assumptions

In view of the results reviewed in Sec. 5, it was particularly interesting to find statements of variational problems *dual* to the $\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$-problem (and hence providing new *equivalent* definitions to cap $\mathbf{A}$), but now *solvable* for any (I^+, I^-)-condenser $\mathbf{A}$ (e.g., even nonclosed or infinite) and any vector $\mathbf{a}$.

We have succeeded in this under the following conditions, which—together with (19)—will always be tacitly assumed: the kernel κ is assumed to be consistent and either $I^- = \varnothing$, or (22) and (23) both hold.

Remark 9. *These assumptions on a kernel are not too restrictive. In particular, they all are satisfied by the Newtonian, Riesz, or Green kernels in* $\mathbf{R}^n$, $n \geqslant 2$, *provided the Euclidean distance between* A^+ *and* A^- *is nonzero, as well as by the restriction of the logarithmic kernel in* $\mathbf{R}^2$ *to the open unit disk.*

7. A-vague and strong cluster sets of minimizing nets

7.1. To formulate the results obtained, we shall need the following notation. Denote by $\mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ the class of all nets $(\mu_t)_{t \in T} \subset \mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$ such that

$$\lim_{t \in T} \|\mu_t\|^2 = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2. \tag{25}$$

This class is not empty, which is seen from (19) on account of Corollary 3.

Let $\mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ (respectively, $\mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$) consist of all limit points of the nets $(\mu_t)_{t \in T} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ in the $\mathbf{A}$-vague topology of the space $\mathfrak{M}(\overline{\mathbf{A}})$ (respectively, in the strong topology of the semimetric space $\mathscr{E}(\overline{\mathbf{A}})$). Also write

$$\mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) := \left\{ \mu \in \mathfrak{M}(\mathbf{A}) : \int g_i \, d\mu^i \leqslant a_i \text{ for all } i \in I \right\}$$

and $\mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) := \mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) \cap \mathscr{E}(\mathbf{A})$. Then the following lemma, to be proved in Sec. 13 below, holds true.

Lemma 9. *For every* $(\mu_t)_{t \in T} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, *there exist its* $\mathbf{A}$-*vague cluster points; hence,* $\mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is nonempty. Moreover,*

$$\mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a}, \mathbf{g}). \tag{26}$$

Furthermore, for every $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$,

$$\lim_{t \in T} \|\mu_t - \chi\|^2 = 0, \tag{27}$$

and hence $\mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *forms an equivalence class in* $\mathscr{E}(\overline{\mathbf{A}})$.

It follows from (25)–(27) that $\|\zeta\|^2 = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$ for all $\zeta \in \mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Also observe that, if $\mathbf{A} = \mathbf{K}$ is compact, then $\mathscr{M}(\mathbf{K}, \mathbf{a}, \mathbf{g}) \subset \mathfrak{M}(\mathbf{K}, \mathbf{a}, \mathbf{g})$, which together with the preceding relation proves the following assertion.

Corollary 7. *If* $\mathbf{A} = \mathbf{K}$ *is compact, then the* $\mathscr{E}(\mathbf{K}, \mathbf{a}, \mathbf{g})$-*problem is solvable. Actually,*

$$\mathscr{S}(\mathbf{K}, \mathbf{a}, \mathbf{g}) = \mathscr{M}(\mathbf{K}, \mathbf{a}, \mathbf{g}). \tag{28}$$

7.2. When approaching $\mathbf{A}$ by compact condensers $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$, we shall always suppose all those $\mathbf{K}$ to be of capacity nonzero. This involves no loss of generality, which is clear from (19) and Lemma 6. Then Corollary 7 enables us to introduce the (nonempty) class $\mathbf{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})$ of all nets $(\lambda_{\mathbf{K}})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$, where $\lambda_{\mathbf{K}}\in\mathscr{S}(\mathbf{K},\mathbf{a},\mathbf{g})$. Let $\mathscr{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})$ consist of all their $\mathbf{A}$-vague cluster points.

On account of Lemma 6, we have $\mathbf{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})\subset\mathbf{M}(\mathbf{A},\mathbf{a},\mathbf{g})$. Therefore, the following assertion is an immediate consequence of Lemma 9.

Corollary 8. *The class $\mathscr{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})$ is nonempty, and*

$$\mathscr{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})\subset\mathscr{M}(\mathbf{A},\mathbf{a},\mathbf{g})\subset\mathscr{M}'(\mathbf{A},\mathbf{a},\mathbf{g}).$$

Remark 10. *Each of these three cluster sets, $\mathscr{M}_0(\mathbf{A},\mathbf{a},\mathbf{g})$, $\mathscr{M}(\mathbf{A},\mathbf{a},\mathbf{g})$ and $\mathscr{M}'(\mathbf{A},\mathbf{a},\mathbf{g})$, plays an important role in our study. However, if κ is additionally assumed to be strictly positive definite (hence, perfect), while $\overline{A_i}$, $i\in I$, are mutually disjoint, then all these classes coincide and consist of just one element.*

7.3. Also the following notation will be used. Given $\chi\in\mathscr{M}'(\mathbf{A},\mathbf{a},\mathbf{g})$, write

$$\mathscr{M}'_{\mathscr{E}}(\mathbf{A},\mathbf{a},\mathbf{g}):=[R\chi]_{\mathscr{E}}.$$

This equivalence class does not depend on the choice of χ, which is seen from Lemma 9. This lemma also yields that, for any $(\mu_t)_{t\in T}\in\mathbf{M}(\mathbf{A},\mathbf{a},\mathbf{g})$ and any $\nu\in\mathscr{M}'_{\mathscr{E}}(\mathbf{A},\mathbf{a},\mathbf{g})$, $R\mu_t\to\nu$ in the strong topology of the pre-Hilbert space $\mathscr{E}$.

8. Extremal problems dual to the main minimum-problem

Recall that we are keeping all our standing assumptions, stated in Sec. 6.

8.1. A proposition $P(x)$ involving a variable point $x\in\mathrm{X}$ is said to subsist *nearly everywhere* (n.e.) in E, where E is a given subset of X, if the set of all $x\in E$ for which $P(x)$ fails to hold is of interior capacity zero (see, e.g., [10]).

If $C(E)>0$ and f is a universally measurable function bounded from below n.e. in E, then we write

$$\text{“}\inf_{x\in E}\text{”}\, f(x):=\sup\{q: f(x)\geqslant q \text{ n.e. in } E\}.$$

Then

$$f(x)\geqslant\text{“}\inf_{x\in E}\text{”}\, f(x)\qquad\text{n.e. in } E,$$

which can be obtained directly from the following known fact (see the corollary to Lemma 2.3.5 in [10] and the remark attached to it).

Lemma 10 (Fuglede [10]). *A countable union of $U_n\cap E$ with $C(U_n\cap E)=0$ has interior capacity zero as well, provided these U_n are universally measurable*[3].

[3] Whereas E is arbitrary.

8.2. Let $\hat{\Gamma} = \hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ denote the class of all Radon measures $\nu \in \mathscr{E}$ such that there exist real numbers $c_i(\nu)$, $i \in I$, satisfying the relations

$$\alpha_i a_i \kappa(x, \nu) \geqslant c_i(\nu) g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I, \tag{29}$$

$$\sum_{i \in I} c_i(\nu) \geqslant 1. \tag{30}$$

Remark 11. *For any* $\nu \in \hat{\Gamma}$, *the series in* (30) *must converge absolutely. Indeed, due to* (19) *and Corollary* 3, *there exists* $\mu \in \mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$*; then, by* [10, *Lemma* 2.3.1], *the inequality in* (29) *holds* μ^i*-a.e. in* X. *In view of* $\int g_i \, d\mu^i = a_i$, *this gives* $\kappa(\alpha_i \mu^i, \nu) \geqslant c_i(\nu)$ *for all* $i \in I$. *Since, by Fubini's theorem and Lemma* 4, $\sum_{i \in I} \kappa(\alpha_i \mu^i, \nu)$ *converges absolutely, the required conclusion follows.*

We also observe that $\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is *convex*, which can be seen from Lemma 10.

The following assertion, to be proved in Sec. 17 below, holds true.

Theorem 2. *Under the standing assumptions,*

$$\|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 = \operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{31}$$

If $\|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 < \infty$, then we shall be interested in the $\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$*-problem* (cf. Sec. 4.1), i.e., the problem on the existence of $\hat{\omega} \in \hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with

$$\|\hat{\omega}\|^2 = \|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2;$$

the set of all those $\hat{\omega}$ will be denoted by $\hat{\mathscr{G}} = \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

A minimizing measure $\hat{\omega}$ can be shown to be *unique* up to a summand of seminorm zero (and, hence, it is unique whenever the kernel under consideration is strictly positive definite). Actually, the following stronger result holds.

Lemma 11. *If* $\hat{\omega}$ *exists, then* $\hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *forms an equivalence class in* $\mathscr{E}$.

Proof. Since $\hat{\Gamma}$ is convex, Lemma 5 yields that $\hat{\mathscr{G}}$ is contained in an equivalence class in $\mathscr{E}$. To prove that $\hat{\mathscr{G}}$ actually coincides with that equivalence class, it suffices to show that, if ν belongs to $\hat{\Gamma}$, then so do all measures equivalent to ν in $\mathscr{E}$. But this follows at once from Lemma 10 and the fact that the potentials of any two equivalent in $\mathscr{E}$ measures coincide n.e. in X (see [10, Lemma 3.2.1]). □

8.3. Assume for a moment that $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is finite (cf. Sec. 4.5). Then Theorem 2, combined with (8) and (24), shows that the $\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$-problem and, on the other hand, the $\mathscr{E}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$-problem have the same infimum, equal to the capacity $\operatorname{cap} \mathbf{A}$, and so these two variational problems are *dual.*

But what is surprising is that their infimum, $\operatorname{cap}\mathbf{A}$, turns out to be always an actual minimum in the former extremal problem, while this is not the case for the latter one (see Sec. 5). In fact, the following statement on the solvability of the $\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})$-problem, to be proved in Sec. 17 below, holds true.

THEOREM 3. *Under the standing assumptions, if, moreover,* $\operatorname{cap}\mathbf{A} < \infty$, *then the class* $\hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is nonempty and can be given by the formula*

$$\hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g}) = \mathscr{M}'_{\mathscr{E}}(\mathbf{A},\mathbf{a}\operatorname{cap}\mathbf{A},\mathbf{g}). \tag{32}$$

The numbers $c_i(\hat{\omega})$, $i \in I$, *satisfying both* (29) *and* (30) *for* $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$, *are uniquely determined, do not depend on the choice of* $\hat{\omega}$, *and can be written in either of the forms*

$$c_i(\hat{\omega}) = \alpha_i[\operatorname{cap}\mathbf{A}]^{-1}\kappa(\zeta^i,\zeta), \tag{33}$$

$$c_i(\hat{\omega}) = \alpha_i[\operatorname{cap}\mathbf{A}]^{-1}\lim_{s\in S}\kappa(\mu_s^i,\mu_s), \tag{34}$$

$\zeta \in \mathscr{M}(\mathbf{A},\mathbf{a}\operatorname{cap}\mathbf{A},\mathbf{g})$ *and* $(\mu_s)_{s\in S} \in \mathbf{M}(\mathbf{A},\mathbf{a}\operatorname{cap}\mathbf{A},\mathbf{g})$ *being arbitrarily given.*

The following two assertions, providing additional information about $c_i(\hat{\omega})$, can be obtained directly from the preceding theorem.

COROLLARY 9. *For every* $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$, *we have*

$$c_i(\hat{\omega}) = \text{“}\inf_{x\in A_i}\text{”}\,\frac{\alpha_i a_i \kappa(x,\hat{\omega})}{g_i(x)} \qquad \textit{for all } i \in I. \tag{35}$$

COROLLARY 10. *Inequality* (30) *for* $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is actually an equality; i.e.,*

$$\sum_{i\in I} c_i(\hat{\omega}) = 1. \tag{36}$$

REMARK 12. *Assume for a moment that* $C(A_j) = 0$ *for some* $j \in I$. *Then, by Corollary* 4, $\operatorname{cap}\mathbf{A} = 0$. *On the other hand,* $\nu_0 = 0$ *belongs to* $\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})$ *since it satisfies both* (29) *and* (30) *with* $c_i(\nu_0)$, *where* $c_j(\nu_0) \geqslant 1$ *and* $c_i(\nu_0) = 0$ *for all* $i \neq j$. *This means that identity* (31) *holds true in the degenerate case* $C(A_j) = 0$ *as well, and then* $\hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *consists of all* $\nu \in \mathscr{E}$ *of seminorm zero. What then, however, fails to hold is the statement on the uniqueness of* $c_i(\hat{\omega})$.

Let $\hat{\Gamma}_*(\mathbf{A},\mathbf{a},\mathbf{g})$ consist of all $\nu \in \hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})$ for whom inequality (30) is in fact an equality. By arguments similar to those that have been applied above, one can see that $\hat{\Gamma}_*(\mathbf{A},\mathbf{a},\mathbf{g})$ is convex, and hence all the solutions to the minimal energy problem over this class form an equivalence class in $\mathscr{E}$. Combining this with Theorems 2, 3 and Corollary 10 leads to the following assertion.

Corollary 11. *Under the standing assumptions,*

$$\|\hat{\Gamma}_*(\mathbf{A},\mathbf{a},\mathbf{g})\|^2 = \mathrm{cap}(\mathbf{A},\mathbf{a},\mathbf{g}).$$

If, moreover, $\mathrm{cap}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is finite, then the* $\hat{\Gamma}_*(\mathbf{A},\mathbf{a},\mathbf{g})$*-problem is solvable; actually,* $\mathscr{M}'_{\mathscr{E}}(\mathbf{A},\mathbf{a}\,\mathrm{cap}\,\mathbf{A},\mathbf{g})$ *is the class of all its solutions.*

Remark 13. *Theorem* 2 *and Corollary* 11 *(cf. also Theorem* 4 *and Corollary* 13 *below) provide new equivalent definitions to the capacity* $\mathrm{cap}(\mathbf{A},\mathbf{a},\mathbf{g})$. *Note that, in contrast to the initial definition (cf. Sec.* 4.2*), no restrictions on the supports and total masses of measures from the classes* $\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})$ *or* $\hat{\Gamma}_*(\mathbf{A},\mathbf{a},\mathbf{g})$ *have been imposed; the only restriction involves their potentials. These definitions to* $\mathrm{cap}(\mathbf{A},\mathbf{a},\mathbf{g})$ *are new even for a finite, compact condenser; compare with* [19]. *They are not only of obvious academic interest, but turned out also to be important for numerical computations; see* [18].

8.4. Our next purpose is to formulate an $\mathscr{H}$-problem such that it is still dual to the $\mathscr{E}(\mathbf{A},\mathbf{a}\,\mathrm{cap}\,\mathbf{A},\mathbf{g})$-problem and solvable, but now with $\mathscr{H}$ consisting of measures associated with a condenser.

Let $\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})$ consist of all $\mu\in\mathscr{E}(\overline{\mathbf{A}})$ for whom both the relations (29) and (30) hold (with μ in place of ν). In other words, let

$$\Gamma(\mathbf{A},\mathbf{a},\mathbf{g}) := \{\mu\in\mathscr{E}(\overline{\mathbf{A}}) : R\mu\in\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})\}.$$

Observe that the class $\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})$ is *convex* and

$$\|\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})\|^2 \geqslant \|\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})\|^2. \tag{37}$$

We proceed by showing that inequality (37) is in fact an equality and that the minimal energy problem, when considered over $\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})$, is still solvable.

Theorem 4. *Under the standing assumptions,*

$$\|\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})\|^2 = \mathrm{cap}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{38}$$

If, moreover, $\mathrm{cap}\,\mathbf{A}$ *is finite, then the* $\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})$*-problem is solvable and the class* $\mathscr{G}(\mathbf{A},\mathbf{a},\mathbf{g})$ *of all its solutions* ω *is given by the formula*

$$\mathscr{G}(\mathbf{A},\mathbf{a},\mathbf{g}) = \mathscr{M}'(\mathbf{A},\mathbf{a}\,\mathrm{cap}\,\mathbf{A},\mathbf{g}). \tag{39}$$

Proof. We can assume $\mathrm{cap}\,\mathbf{A}$ to be finite, for if not, then (38) is obtained directly from (31) and (37). Then, according to Lemma 9 with $\mathbf{a}\,\mathrm{cap}\,\mathbf{A}$ instead of $\mathbf{a}$, the class $\mathscr{M}'(\mathbf{A},\mathbf{a}\,\mathrm{cap}\,\mathbf{A},\mathbf{g})$ is nonempty; fix χ, one of its elements. It is clear from its definition and identity (32) that $\chi\in\mathscr{E}(\overline{\mathbf{A}})$ and $R\chi\in\hat{\mathscr{G}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Hence, $\chi\in\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})$ and, therefore,

$$\|\hat{\Gamma}(\mathbf{A},\mathbf{a},\mathbf{g})\|^2 = \|\chi\|^2 \geqslant \|\Gamma(\mathbf{A},\mathbf{a},\mathbf{g})\|^2.$$

In view of (31) and (37), this proves (38) and, as well, the inclusion

$$\mathscr{M}'(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}) \subset \mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

But the right-hand side of this inclusion is an equivalence class in $\mathscr{E}(\overline{\mathbf{A}})$, which follows from the convexity of $\Gamma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and Lemma 5 in the same manner as in the proof of Lemma 11. Since, by Lemma 9, also the left-hand side is an equivalence class in $\mathscr{E}(\overline{\mathbf{A}})$, the two sets must actually be equal. □

Corollary 12. *If* $\mathbf{A} = \mathbf{K}$ *is compact and* $\mathrm{cap}\,\mathbf{K} < \infty$, *then any solution to the* $\mathscr{E}(\mathbf{K}, \mathbf{a}\,\mathrm{cap}\,\mathbf{K}, \mathbf{g})$-*problem gives, as well, a solution to the* $\Gamma(\mathbf{K}, \mathbf{a}, \mathbf{g})$-*problem.*

Proof. This is obtained from (39), combined with (26) and (28) for $\mathbf{a}\,\mathrm{cap}\,\mathbf{K}$ in place of $\mathbf{a}$. □

Remark 14. *In case* $\mathrm{cap}\,\mathbf{A} < \infty$, *fix* $\omega \in \mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *and* $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. *Since, by* (32) *and* (39), $\kappa(x, \omega) = \kappa(x, \hat{\omega})$ *n.e. in* X, *the numbers* $c_i(\omega)$, $i \in I$, *satisfying* (29) *and* (30) *for* ω *instead of* ν, *are uniquely determined and equal* $c_i(\hat{\omega})$. *This implies that relations* (33)–(36) *hold, as well, for* ω *in place of* $\hat{\omega}$.

Remark 15. *In Theorems* 3, 4 *and Corollary* 11, *no restrictions on the topology of* A_i, $i \in I$, *have been imposed. So, all the* $\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$-, $\hat{\Gamma}_*(\mathbf{A}, \mathbf{a}, \mathbf{g})$-, *and* $\Gamma(\mathbf{A}, \mathbf{a}, \mathbf{g})$-*problems are solvable even for a nonclosed, infinite condenser* $\mathbf{A}$.

Remark 16. *If* I *is a singleton and* $g_1 = 1$, *then Theorems* 2, 3, 4 *and Corollaries* 11, 12 *can be derived from* [10]. *Moreover, then one can choose* $\omega \in \mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *so that* $\omega(\mathrm{X}) = a_1 C(A_1)$, *and exactly these* ω *are called the interior equilibrium measures associated with the set* A_1 [10]. *However, this fact in general can not be extended to a condenser* $\mathbf{A}$ *consisting of more than one plate; that is, in general,*

$$\mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathscr{E}(\overline{\mathbf{A}}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}) = \varnothing,$$

which is caused by the unsolvability of the $\mathscr{E}(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$-*problem.*

9. Interior equilibrium constants associated with a condenser

9.1. Throughout Sec. 9, it is always required that $\mathrm{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty$. Due to the uniqueness statement in Theorem 3, the following notion naturally arises.

Definition 6. The numbers

$$C_i := C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) := c_i(\hat{\omega}), \qquad i \in I,$$

satisfying both (29) and (30) for $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, are said to be the (*interior*) *equilibrium constants* associated with $\mathbf{A}$.

Corollary 13. *The interior capacity* $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is equal to* $\inf \kappa(\nu, \nu)$, *where* ν *ranges over all* $\nu \in \mathscr{E}$ *(similarly,* $\nu \in \mathscr{E}(\overline{\mathbf{A}})$*) such that, for every* $i \in I$,

$$\alpha_i a_i \kappa(x, \nu) \geqslant C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) g_i(x) \qquad \text{n.e. in } A_i.$$

The infimum is attained at any $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *(respectively,* $\omega \in \mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g})$*), and hence it is an actual minimum.*

Proof. This follows from Theorems 2, 3, 4 and Remark 14. □

9.2. Some properties of the interior equilibrium constants $C_i(\mathbf{A}, \mathbf{a}, \mathbf{g})$, $i \in I$, have already been provided by Theorem 3 and Corollaries 9, 10. Also observe that, if I is a singleton, then certainly $C_1(\mathbf{A}, \mathbf{a}, \mathbf{g}) = 1$ (cf. [10, Th. 4.1]).

Corollary 14. $C_i(\cdot, \mathbf{a}, \mathbf{g})$, $i \in I$, *are continuous under exhaustion of* $\mathbf{A}$ *by the increasing filtering family of all compact condensers* $\mathbf{K} \prec \mathbf{A}$. *Namely,*

$$C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{\mathbf{K} \uparrow \mathbf{A}} C_i(\mathbf{K}, \mathbf{a}, \mathbf{g}).$$

Proof. Under our assumptions, $0 < \operatorname{cap} \mathbf{K} < \infty$ for every $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$, and hence there exists $\lambda_{\mathbf{K}} \in \mathscr{S}(\mathbf{K}, \mathbf{a} \operatorname{cap} \mathbf{K}, \mathbf{g})$. Substituting $\lambda_{\mathbf{K}}$ into (33) yields

$$C_i(\mathbf{K}, \mathbf{a}, \mathbf{g}) = \alpha_i [\operatorname{cap} \mathbf{K}]^{-1} \kappa(\lambda_{\mathbf{K}}^i, \lambda_{\mathbf{K}}). \tag{40}$$

On the other hand, it follows from Lemma 6 that the net $\operatorname{cap} \mathbf{A} [\operatorname{cap} \mathbf{K}]^{-1} \lambda_{\mathbf{K}}$, where $\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}$, belongs to the class $\mathbf{M}(\mathbf{A}, \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$. Substituting it into (34) and then combining the relation obtained with (40), we get the corollary. □

Corollary 15. *Assume that, for some* $j \in I$, $C(A_j) = \infty$ *and* $g_{j,\inf} > 0$. *Then* $C_j(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leqslant 0$.

Proof. Suppose, on the contrary, that $C_j > 0$. Given $\hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, we have $\alpha_j a_j \kappa(x, \hat{\omega}) \geqslant C_j g_{j,\inf} > 0$ n.e. in A_j. Hence, according to Lemma 3.2.2 from [10], $C(A_j) \leqslant a_j^2 \|\hat{\omega}\|^2 C_j^{-2} g_{j,\inf}^{-2} < \infty$, which is a contradiction. □

Remark 17. *Lemma* 8, *which has already been proved by elementary arguments, can also be obtained as a consequence of Corollary* 15. *Indeed, if it were true that* $C(A_i) = \infty$ *for all* $i \in I$, *then, by Corollary* 15, *the sum of* C_i, *where* i *ranges over* I, *would be not greater than* 0, *which is impossible.*

10. Interior equilibrium measures associated with a condenser

As always, we are keeping all our standing assumptions, stated in Sec. 6. Throughout Sec. 10, it is also required that $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty$. Our next purpose is to introduce a notion of interior equilibrium measures $\gamma_{\mathbf{A}}$ associated with the condenser $\mathbf{A}$ such that the characteristics obtained possess properties similar to those of interior equilibrium measures associated with a set. Fuglede's theory of interior capacities of sets [10] serves here as a model case.

10.1. If $\mathbf{A} = \mathbf{K}$ is compact, then, as follows from Theorem 4, Corollary 12 and Remark 14, any minimizer $\lambda_{\mathbf{K}}$ in the $\mathscr{E}(\mathbf{K}, \mathbf{a}\,\mathrm{cap}\,\mathbf{K}, \mathbf{g})$-problem has the desired properties, and so $\gamma_{\mathbf{K}}$ might be defined by the formula

$$\gamma_{\mathbf{K}} := \lambda_{\mathbf{K}}, \qquad \text{where } \lambda_{\mathbf{K}} \in \mathscr{S}(\mathbf{K}, \mathbf{a}\,\mathrm{cap}\,\mathbf{K}, \mathbf{g}).$$

However, as is seen from Remark 16, in the noncompact case the desired notion can not be obtained as just a direct generalization of the corresponding one from the theory of capacities of sets. Having in mind that, similar to our model case, the required distributions should give a solution to the $\Gamma(\mathbf{A}, \mathbf{a}, \mathbf{g})$-problem and be strongly and $\mathbf{A}$-vaguely continuous under exhaustion of $\mathbf{A}$ by compact condensers, we arrive at the following definition.

Definition 7. We shall call $\gamma_{\mathbf{A}} \in \mathscr{E}(\overline{\mathbf{A}})$ an *(interior) equilibrium measure* associated with the condenser $\mathbf{A}$ if there exist a subnet $(\mathbf{K}_s)_{s\in S}$ of $(\mathbf{K})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$ and $\lambda_{\mathbf{K}_s} \in \mathscr{S}(\mathbf{K}_s, \mathbf{a}\,\mathrm{cap}\,\mathbf{K}_s, \mathbf{g})$ such that the net $(\lambda_{\mathbf{K}_s})_{s\in S}$ converges to $\gamma_{\mathbf{A}}$ both $\mathbf{A}$-vaguely and strongly. Let $\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ denote the collection of all those $\gamma_{\mathbf{A}}$.

Lemmas 6 and 9 enable us to rewrite the above definition in the following, apparently weaker, form:

$$\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \mathscr{M}_0(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}). \tag{41}$$

Theorem 5. *$\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty, $\mathbf{A}$-vaguely compact, and it is contained in an equivalence class in $\mathscr{E}(\overline{\mathbf{A}})$. Furthermore,*

$$\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathscr{G}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}). \tag{42}$$

Given $\gamma := \gamma_{\mathbf{A}} \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, we have

$$\|\gamma\|^2 = \mathrm{cap}\,\mathbf{A}, \tag{43}$$

$$\alpha_i a_i \kappa(x, \gamma) \geqslant C_i g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I, \tag{44}$$

where $C_i = C_i(\mathbf{A}, \mathbf{a}, \mathbf{g})$, $i \in I$, are the interior equilibrium constants. Actually,

$$C_i = \frac{\alpha_i \kappa(\gamma^i, \gamma)}{\mathrm{cap}\,\mathbf{A}} = \text{“}\inf_{x\in A_i}\text{”}\, \frac{\alpha_i a_i \kappa(x, \gamma)}{g_i(x)}. \tag{45}$$

In case $I^- \neq \varnothing$, assume moreover that $\kappa(x, y)$ is continuous on $\overline{A^+} \times \overline{A^-}$, while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets. Then, for every $i \in I$,

$$\alpha_i a_i \kappa(x, \gamma) \leqslant C_i g_i(x) \qquad \text{for all } x \in S(\gamma^i), \tag{46}$$

and hence

$$\alpha_i a_i \kappa(x, \gamma) = C_i g_i(x) \qquad \text{n.e. in } A_i \cap S(\gamma^i).$$

Also note that $\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is contained in an R-equivalence class in $\mathfrak{M}(\overline{\mathbf{A}})$ provided the kernel κ is strictly positive definite, and it consists of a *unique* element $\gamma_{\mathbf{A}}$ if, moreover, all $\overline{A_i}$, $i \in I$, are mutually disjoint.

Remark 18. *As is seen from Theorem* 5, *the properties of interior equilibrium measures associated with a condenser are quite similar to those of interior equilibrium measures associated with a set (compare with* [10, *Th.* 4.1]*). The only important difference is that the sign* $\leqslant$ *in* (42) *in general can not be omitted—even for a finite and closed, though noncompact, condenser (cf. Remark* 16*).*

Remark 19. *Like in the theory of interior capacities of sets, in general none of the i-coordinates of* $\gamma_{\mathbf{A}}$ *is concentrated on* A_i *(unless* A_i *is closed). Indeed, consider* $\mathrm{X} = \mathbf{R}^n$, *where* $n \geqslant 3$, $\kappa(x, y) = |x - y|^{2-n}$, $I^+ = \{1\}$, $I^- = \{2\}$, $g_1 = g_2 = 1$, $a_1 = a_2 = 1$, *and let* $A_1 = \{x : |x| < r\}$ *and* $A_2 = \{x : |x| > R\}$, *where* $0 < r < R < \infty$. *Then it can be shown that*

$$\gamma_{\mathbf{A}} = \gamma_{\overline{\mathbf{A}}} = [\theta^+ - \theta^-] \operatorname{cap} \mathbf{A},$$

where θ^+ *and* θ^- *are obtained by the uniform distribution of unit mass over the spheres* $S(0, r)$ *and* $S(0, R)$, *respectively. Hence,* $|\gamma_{\mathbf{A}}|(A) = 0$.

10.2. The purpose of this section is to point out characteristic properties of the interior equilibrium measures and the interior equilibrium constants.

Proposition 1. *Let* $\mu \in \mathscr{E}(\overline{\mathbf{A}})$ *admit the properties*

$$\|\mu\|^2 = \operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}),$$

$$\alpha_i a_i \kappa(x, \mu) \geqslant \frac{\alpha_i \kappa(\mu^i, \mu)}{\operatorname{cap} \mathbf{A}} g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I.$$

Then μ *is equivalent in* $\mathscr{E}(\overline{\mathbf{A}})$ *to every* $\gamma_{\mathbf{A}} \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *and, for all* $i \in I$,

$$C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \frac{\alpha_i \kappa(\mu^i, \mu)}{\operatorname{cap} \mathbf{A}} = \text{“}\inf_{x \in A_i}\text{”} \frac{\alpha_i a_i \kappa(x, \mu)}{g_i(x)}.$$

Actually, there holds the following stronger result, to be proved in Sec. 19.

Proposition 2. *Let* $\nu \in \mathscr{E}(\overline{\mathbf{A}})$ *and* $\beta_i \in \mathbf{R}$, $i \in I$, *satisfy the relations*

$$\alpha_i a_i \kappa(x, \nu) \geqslant \beta_i g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I, \tag{47}$$

$$\sum_{i \in I} \beta_i = \frac{\operatorname{cap} \mathbf{A} + \|\nu\|^2}{2 \operatorname{cap} \mathbf{A}}. \tag{48}$$

Then ν *is equivalent in* $\mathscr{E}(\overline{\mathbf{A}})$ *to every* $\gamma_{\mathbf{A}} \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *and, for all* $i \in I$,

$$\beta_i = C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \text{“}\inf_{x \in A_i}\text{”} \frac{\alpha_i a_i \kappa(x, \nu)}{g_i(x)}. \tag{49}$$

Thus, under the conditions of Proposition 1 or 2, if, moreover, κ is strictly positive definite and all $\overline{A_i}$, $i \in I$, are mutually disjoint, then the measure under consideration is actually the (unique) interior equilibrium measure $\gamma_{\mathbf{A}}$.

11. On continuity of the interior capacities, equilibrium measures, and equilibrium constants

11.1. Given $\mathbf{A}_n = (A_i^n)_{i \in I}$, $n \in \mathbf{N}$, and $\mathbf{A}$ in $\mathfrak{C}$, we shall write $\mathbf{A}_n \uparrow \mathbf{A}$ if $\mathbf{A}_n \prec \mathbf{A}_{n+1}$ for all n and $A_i = \bigcup_{n \in \mathbf{N}} A_i^n$ for all $i \in I$.

Following [1, Chap. 1, § 9], we call a locally compact space *countable at infinity* if it can be written as a countable union of compact sets.

Theorem 6. *Let either* $g_{i,\inf} > 0$ *for all* $i \in I$ *or the space* X *be countable at infinity. If* $\mathbf{A}_n$, $n \in \mathbf{N}$, *are universally measurable and* $\mathbf{A}_n \uparrow \mathbf{A}$, *then*

$$\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{n \in \mathbf{N}} \operatorname{cap}(\mathbf{A}_n, \mathbf{a}, \mathbf{g}). \tag{50}$$

Assume moreover that $\operatorname{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ *is finite, and let* $\gamma_n := \gamma_{\mathbf{A}_n}$, $n \in \mathbf{N}$, *denote an arbitrary interior equilibrium measure associated with* $\mathbf{A}_n$. *If* γ *is any of the* $\mathbf{A}$-*vague limit points of* $(\gamma_n)_{n \in \mathbf{N}}$ *(such a* γ *exists), then* γ *is actually an interior equilibrium measure associated with* $\mathbf{A}$ *and*

$$\lim_{n \in \mathbf{N}} \|\gamma_n - \gamma\|^2 = 0.$$

Furthermore,

$$C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{n \in \mathbf{N}} C_i(\mathbf{A}_n, \mathbf{a}, \mathbf{g}) \qquad \text{for all } i \in I. \tag{51}$$

Thus, if κ is additionally assumed to be strictly positive definite (hence, perfect) and all $\overline{A_i}$, $i \in I$, are mutually disjoint, then the (unique) interior equilibrium measure associated with $\mathbf{A}_n$ approaches the (unique) interior equilibrium measure associated with $\mathbf{A}$ both $\mathbf{A}$-vaguely and strongly.

Remark 20. *Theorem* 6 *remains true if* $(\mathbf{A}_n)_{n \in \mathbf{N}}$ *is replaced by the increasing filtering family of all compact condensers* $\mathbf{K}$ *such that* $\mathbf{K} \prec \mathbf{A}$. *Moreover, then the assumption that either* $g_{i,\inf} > 0$ *for all* $i \in I$ *or* X *is countable at infinity can be omitted. Cf., e.g., Lemma* 6 *and Corollary* 14.

Remark 21. *If I is a singleton and $g_1 = 1$, then Theorem* 6 *has been proved by Fuglede (see* [10, *Th.* 4.2]*).*

11.2. The rest of the article is devoted to proving the results formulated in Sec. 7–11 and is structured as follows. Theorems 2, 3, 5, and 6 will be proved in Sec. 17, 18, and 20. Their proofs utilize a description of the potentials of measures from the classes $\mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\mathscr{M}_0(\mathbf{A}, \mathbf{a}, \mathbf{g})$, to be given in Sec. 15 and 16 by Lemmas 14 and 15, respectively. In turn, Lemmas 14 and 15 use a theorem on the strong completeness of proper subspaces of $\mathscr{E}(\overline{\mathbf{A}})$, which is a subject of the next section.

12. Strong completeness of measures associated with condensers

12.1. Keeping all our standing assumptions on κ, $\mathbf{g}$, $\mathbf{a}$, and $\mathbf{A}$, stated in Sec. 6, we consider $\mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a}, \mathbf{g})$ to be a topological subspace of the semimetric space $\mathscr{E}(\overline{\mathbf{A}})$; the induced topology is likewise called the *strong* topology.

Theorem 7. *If* $\mathbf{A}$ *is closed, then the semimetric space* $\mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *is complete. In more detail, if* $(\mu_s)_{s\in S} \subset \mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *is a strong Cauchy net and* μ *is one of its* $\mathbf{A}$*-vague cluster points (such a* μ *exists), then* $\mu \in \mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *and*

$$\lim_{s\in S} \|\mu_s - \mu\|^2 = 0. \tag{52}$$

Assume, in addition, that the kernel is strictly positive definite (hence, perfect) and all A_i, $i \in I$, are mutually disjoint. If, moreover, $(\mu_s)_{s\in S}$ *converges strongly to* $\mu_0 \in \mathscr{E}(\mathbf{A})$*, then actually* $\mu_0 \in \mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *and* $\mu_s \to \mu_0$ $\mathbf{A}$*-vaguely.*

Remark 22. *In view of the fact that the semimetric space* $\mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *is isometric to its R-image, Theorem* 7 *has thus singled out a strongly complete topological subspace of the pre-Hilbert space* $\mathscr{E}$*, whose elements are signed measures. This is of independent interest since, according to a well-known counterexample by H. Cartan* [4]*, the whole space* $\mathscr{E}$ *is strongly incomplete even for the Newtonian kernel* $|x-y|^{2-n}$ *in* $\mathbf{R}^n$*,* $n \geqslant 3$.

Remark 23. *Let κ be strictly positive definite (hence, perfect). If, moreover,* $I^- = \varnothing$*, then Theorem* 7 *remains true for* $\mathscr{E}(\mathbf{A})$ *in place of* $\mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *(compare with Theorem* 1*). A question still unanswered is whether this is the case if* I^+ *and* I^- *are both nonempty. We can however show that this is really so for the Riesz kernels* $|x-y|^{\alpha-n}$*,* $0 < \alpha < n$*, in* $\mathbf{R}^n$*,* $n \geqslant 2$ *(cf.* [21, *Th.* 1]*). The proof is based on Deny's theorem* [5] *stating that, for the Riesz kernels,* $\mathscr{E}$ *can be completed by making use of distributions of finite energy.*

12.2. We start with the lemmas to be used below in the proof of Theorem 7.

Lemma 12. *$\mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is $\mathbf{A}$-vaguely bounded; hence, it is $\mathbf{A}$-vaguely relatively compact.*

Proof. Fix $i \in I$, and let a compact set $K \subset A_i$ be given. Since g_i is positive and continuous, the relation

$$a_i \geqslant \int g_i \, d\mu^i \geqslant \mu^i(K) \min_{x \in K} g_i(x), \qquad \text{where } \mu \in \mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}),$$

yields

$$\sup_{\mu \in \mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})} \mu^i(K) < \infty.$$

This implies that $\mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is $\mathbf{A}$-vaguely bounded, and hence it is $\mathbf{A}$-vaguely relatively compact by Lemma 1. □

Lemma 13. *Suppose $\mathbf{A}$ is closed. If a net $(\mu_s)_{s \in S} \subset \mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is strongly bounded, then its $\mathbf{A}$-vague cluster set is nonempty and contained in $\mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$.*

Proof. According to Lemma 12, the $\mathbf{A}$-vague cluster set of $(\mu_s)_{s \in S}$ is nonempty, and it is contained in $\mathfrak{M}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ in consequence of Lemma 3. Thus, it is enough to prove that each of its elements has finite energy.

To this end, observe that $(R\mu_s)_{s \in S}$ is strongly bounded by (5). We proceed by showing that so are the nets $(R\mu_s^+)_{s \in S}$ and $(R\mu_s^-)_{s \in S}$, i.e.,

$$\sup_{s \in S} \|R\mu_s^\pm\|^2 < \infty. \tag{53}$$

Of course, this needs to be proved only when $I^- \neq \varnothing$; then, according to the standing assumptions, (22) and (23) both hold. Since $\int g_i \, d\mu_s^i \leqslant a_i$, we get

$$\sup_{s \in S} \mu_s^i(\mathrm{X}) \leqslant a_i g_{i,\inf}^{-1} \qquad \text{for all } i \in I. \tag{54}$$

Consequently, by (22),

$$\sup_{s \in S} R\mu_s^\pm(\mathrm{X}) \leqslant \sum_{i \in I} a_i g_{i,\inf}^{-1} < \infty.$$

Because of (23), this implies that $\kappa(R\mu_s^+, R\mu_s^-)$ remains bounded from above on S; hence, so do $\|R\mu_s^+\|^2$ and $\|R\mu_s^-\|^2$.

Now, if $(\mu_d)_{d \in D}$ is a subnet of $(\mu_s)_{s \in S}$ that converges $\mathbf{A}$-vaguely to some μ, then, by Lemma 2, $(R\mu_d^+)_{d \in D}$ and $(R\mu_d^-)_{d \in D}$ converge vaguely to $R\mu^+$ and $R\mu^-$, respectively. Applying Lemma 3 with $\mathrm{Y} = \mathrm{X} \times \mathrm{X}$ and $\psi = \kappa$, we conclude from (53) that $R\mu^+$ and $R\mu^-$ are of finite energy, which yields $\kappa(\mu, \mu) < \infty$. □

Corollary 16. *If a net* $(\mu_s)_{s\in S}\subset\mathscr{E}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$ *is strongly bounded, then, for every* $i\in I$, $\|\mu_s^i\|^2$ *and* $\kappa(\mu_s^i,\mu_s)$ *are bounded on* S.

Proof. Fix $i\in I$. In view of (53), the required relation

$$\sup_{s\in S}\|\mu_s^i\|^2<\infty \tag{55}$$

will be proved once we establish the inequality

$$\sum_{\ell,j\in I^{\pm}}\kappa(\mu_s^\ell,\mu_s^j)\geqslant C>-\infty \tag{56}$$

with a constant C, independent of s. Since (56) is obvious when $\kappa\geqslant 0$, one can assume X to be compact. Then κ, being lower semicontinuous, is bounded from below on X (say by $-c$, where $c>0$), while $\mathbf{A}$ is finite. Furthermore, then $g_{\ell,\inf}>0$ for every $\ell\in I$ and, therefore, (54) holds. This implies

$$\kappa(\mu_s^\ell,\mu_s^j)\geqslant -a_\ell a_j g_{\ell,\inf}^{-1}g_{j,\inf}^{-1}c\qquad\text{for all }\ell,j\in I,$$

and (56) follows.

These arguments also show that $\kappa(\mu_s^i,R\mu_s^+)$ and $\kappa(\mu_s^i,R\mu_s^-)$ are bounded from below on S. Since these functions of s are bounded from above as well, which is clear from (53) and (55) by the Cauchy-Schwarz inequality, the required boundedness of $\kappa(\mu_s^i,\mu_s)$ follows. □

12.3. Proof of Theorem 7. Suppose $\mathbf{A}$ is closed, and let $(\mu_s)_{s\in S}$ be a strong Cauchy net in $\mathscr{E}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$. Since such a net converges strongly to each of its strong cluster points, $(\mu_s)_{s\in S}$ can certainly be assumed to be strongly bounded. Then, by Lemma 13, there exists an $\mathbf{A}$-vague cluster point μ of $(\mu_s)_{s\in S}$ and

$$\mu\in\mathscr{E}(\mathbf{A},\leqslant\mathbf{a},\mathbf{g}). \tag{57}$$

We next proceed by verifying (52). Of course, there is no loss of generality in assuming $(\mu_s)_{s\in S}$ to converge $\mathbf{A}$-vaguely to μ. Then, by Lemma 2, $(R\mu_s^+)_{s\in S}$ and $(R\mu_s^-)_{s\in S}$ converge vaguely to $R\mu^+$ and $R\mu^-$, respectively. Since, by (53), these nets are strongly bounded in $\mathscr{E}^+$, the property (CW) (see Sec. 2) shows that they approach $R\mu^+$ and $R\mu^-$, respectively, in the weak topology as well, and so $R\mu_s\to R\mu$ weakly. This gives, by (6),

$$\|\mu_s-\mu\|^2=\|R\mu_s-R\mu\|^2=\lim_{\ell\in S}\kappa(R\mu_s-R\mu,R\mu_s-R\mu_\ell)$$

and hence, by the Cauchy-Schwarz inequality,

$$\|\mu_s-\mu\|^2\leqslant\|\mu_s-\mu\|\liminf_{\ell\in S}\|\mu_s-\mu_\ell\|,$$

which proves (52) as required, because $\|\mu_s - \mu_\ell\|$ becomes arbitrarily small when $s, \ell \in S$ are both sufficiently large.

Suppose now that κ is strictly positive definite, while all A_i, $i \in I$, are mutually disjoint, and let the net $(\mu_s)_{s \in S}$ converge strongly to some $\mu_0 \in \mathscr{E}(\mathbf{A})$. Given an $\mathbf{A}$-vague limit point μ of $(\mu_s)_{s \in S}$, we derive from (52) that $\|\mu_0 - \mu\| = 0$, hence $R\mu_0 = R\mu$ since κ is strictly positive definite, and finally $\mu_0 = \mu$ because all the A_i are mutually disjoint. In view of (57), this means that $\mu_0 \in \mathscr{E}(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$, which is a part of the desired conclusion. Moreover, μ_0 has thus been shown to be identical to any $\mathbf{A}$-vague cluster point of $(\mu_s)_{s \in S}$. Since the $\mathbf{A}$-vague topology is Hausdorff, this implies that μ_0 is actually the $\mathbf{A}$-vague limit of $(\mu_s)_{s \in S}$ (cf. [1, Chap. I, §9, n° 1, cor.]), which completes the proof. □

13. Proof of Lemma 9

Fix any $(\mu_s)_{s \in S}$ and $(\nu_t)_{t \in T}$ in $\mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. It follows by standard arguments that

$$\lim_{(s,t) \in S \times T} \|\mu_s - \nu_t\|^2 = 0, \tag{58}$$

where $S \times T$ is the directed product of the directed sets S and T (see, e.g., [14, Chap. 2, §3]). Indeed, by the convexity of $\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$,

$$2\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\| \leqslant \|\mu_s + \nu_t\| \leqslant \|\mu_s\| + \|\nu_t\|.$$

Hence, by (25),

$$\lim_{(s,t) \in S \times T} \|\mu_s + \nu_t\|^2 = 4\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2,$$

and the parallelogram identity, applied to $R\mu_s$ and $R\nu_t$ in $\mathscr{E}$, yields (58).

Relation (58) implies that $(\mu_s)_{s \in S}$ is strongly fundamental. Thus, according to Theorem 7, there exists an $\mathbf{A}$-vague cluster point μ of $(\mu_s)_{s \in S}$ and, moreover, $\mu \in \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a}, \mathbf{g})$ and $\mu_s \to \mu$ strongly. This means that $\mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ are both nonempty and satisfy inclusion (26).

It is left to prove that $\mu_s \to \chi$ strongly, where $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is arbitrarily given. But then one can choose a net in $\mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, say $(\nu_t)_{t \in T}$, that converges to χ strongly, and repeated application of (58) gives immediately the desired conclusion. □

14. Proof of Corollary 6

Note that, under the assumptions made in the corollary, all the requirements from Sec. 6 hold true, and so Lemma 9 is applicable.

Fix $(\mu_s)_{s\in S} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$; then, by Lemma 9, its strong and $\mathbf{A}$-vague cluster sets have some $\mu_0 \in \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a}, \mathbf{g})$ in common. Taking a subnet if necessary, we assume that $\mu_s \to \mu_0$ both strongly and $\mathbf{A}$-vaguely.

Let A_j be closed, $C(A_j) < \infty$, and let g_j either be bounded or satisfy the restriction on the growth, mentioned in Remark 8. Then, applying arguments similar to those from [24] (see the proof of Lemma 13 therein), we get

$$\int g_j \, d\mu_0^j = \lim_{s\in S} \int g_j \, d\mu_s^j,$$

and consequently $\mu_0 \neq 0$. Due to the strict positive definiteness of the kernel, we thus have $\|\mu_0\|^2 \neq 0$. When combined with $\|\mu_0\|^2 = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$, this establishes the required inequality $\operatorname{cap} \mathbf{A} < \infty$. □

15. Potentials of strong cluster points of minimizing nets

15.1. The aim of this section is to provide a description of the potentials of measures from the class $\mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$. As usual, we are keeping all our standing assumptions, stated in Sec. 6.

Lemma 14. *There exist $\eta_i \in \mathbf{R}$, $i \in I$, such that, for every $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$,*

$$\alpha_i a_i \kappa(x, \chi) \geqslant \alpha_i \eta_i g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I, \tag{59}$$

$$\sum_{i\in I} \alpha_i \eta_i = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2. \tag{60}$$

These η_i, $i \in I$, are uniquely determined and can be given by either of the formulas

$$\eta_i = \kappa(\zeta^i, \zeta), \tag{61}$$

$$\eta_i = \lim_{s\in S} \kappa(\mu_s^i, \mu_s), \tag{62}$$

where $\zeta \in \mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $(\mu_s)_{s\in S} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ are arbitrarily chosen.

Proof. Throughout the proof, we shall assume every net of the class $\mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ to be strongly bounded, which certainly involves no loss of generality.

Fix $\zeta \in \mathscr{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and choose $(\mu_t)_{t\in T} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ that converges $\mathbf{A}$-vaguely to ζ. We begin by showing that, for every $i \in I$,

$$\kappa(\zeta^i, \zeta) = \lim_{t\in T} \kappa(\mu_t^i, \mu_t). \tag{63}$$

Since, by Corollary 16, $\|\mu_t^i\|$ is bounded from above on T (say by M_1), while $\mu_t^i \to \zeta^i$ vaguely, the property (CW) yields that μ_t^i approaches ζ^i also weakly. Hence, for every $\varepsilon > 0$,

$$|\kappa(\zeta^i - \mu_t^i, \zeta)| < \varepsilon$$

if $t \in T$ is sufficiently large. Furthermore, by the Cauchy-Schwarz inequality,

$$|\kappa(\mu_t^i, \zeta) - \kappa(\mu_t^i, \mu_t)| = |\kappa(\mu_t^i, R\zeta - R\mu_t)| \leqslant M_1 \|\zeta - \mu_t\|, \qquad t \in T.$$

Since, by Lemma 9, $\mu_t \to \zeta$ strongly, the last two relations combined give (63).

We next proceed by proving that η_i, $i \in I$, defined by means of (61), satisfy both (59) and (60), where $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is arbitrarily given. Indeed, since

$$\sum_{i \in I} \alpha_i \kappa(\zeta^i, \zeta) = \|\zeta\|^2 = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2,$$

identity (60) follows. To establish (59), we assume, on the contrary, that there exist $j \in I$ and a set $E_j \subset A_j$ of interior capacity nonzero such that

$$\alpha_j a_j \kappa(x, \chi) < \alpha_j \eta_j g_j(x) \qquad \text{for all } x \in E_j. \tag{64}$$

Then one can choose $\nu \in \mathscr{E}^+$ with compact support so that $S(\nu) \subset E_j$ and $\int g_j \, d\nu = a_j$. Integrating the inequality in (64) with respect to ν gives

$$\alpha_j [\kappa(\chi, \nu) - \eta_j] < 0. \tag{65}$$

To get a contradiction, for every $h \in (0, 1]$ we write

$$\tilde{\mu}_t^i := \begin{cases} \mu_t^j - h(\mu_t^j - \nu) & \text{if } i = j, \\ \mu_t^i & \text{otherwise.} \end{cases}$$

Clearly,

$$\tilde{\mu}_t := \sum_{i \in I} \alpha_i \tilde{\mu}_t^i \in \mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g}), \qquad t \in T,$$

and consequently

$$\|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 \leqslant \|\tilde{\mu}_t\|^2 = \|\mu_t\|^2 - 2\alpha_j h \kappa(\mu_t, \mu_t^j - \nu) + h^2 \|\mu_t^j - \nu\|^2. \tag{66}$$

The coefficient of h^2 is bounded from above on T (say by M_0), while, according to Lemma 9, $\mu_t \to \chi$ strongly. Combining (61), (63) and then substituting the result obtained into (66) therefore gives

$$0 \leqslant M_0 h^2 + 2\alpha_j h [\kappa(\chi, \nu) - \eta_j].$$

By letting here $h \to 0$, we arrive at a contradiction to (65), which proves (59).

To establish the uniqueness statement, consider some other η_i', $i \in I$, satisfying both (59) and (60). Then they are necessarily finite and, for every i,

$$\alpha_i a_i \kappa(x, \chi) \geqslant \max\{\alpha_i \eta_i, \alpha_i \eta_i'\} g_i(x) \qquad \text{n.e. in } A_i, \tag{67}$$

which is seen from Lemma 10. Since μ_t^i is concentrated on A_i and has finite energy and compact support, [10, Lemma 2.3.1] shows that the inequality in (67) holds μ_t^i-a.e. in X. Integrating it with respect to μ_t^i and then summing up over all $i \in I$, in view of $\int g_i \, d\mu_t^i = a_i$ we have

$$\kappa(\mu_t, \chi) \geqslant \sum_{i \in I} \max\{\alpha_i \eta_i, \alpha_i \eta_i'\}, \qquad t \in T.$$

Passing here to the limit as t ranges over T, we obtain

$$\|\chi\|^2 = \lim_{t \in T} \kappa(\mu_t, \chi) \geqslant \sum_{i \in I} \max\{\alpha_i \eta_i, \alpha_i \eta_i'\} \geqslant \sum_{i \in I} \alpha_i \eta_i = \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$$

and, hence, $\max\{\alpha_i \eta_i, \alpha_i \eta_i'\} = \alpha_i \eta_i$ for all $i \in I$, since the extreme left and right parts of this chain of inequalities are equal. Applying the same arguments again, but with the roles of η_i and η_i' reversed, we get $\eta_i = \eta_i'$, $i \in I$, as claimed.

What is left is to show that η_i can be written in the form (62), where $(\mu_s)_{s \in S} \in \mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is arbitrarily given. By Corollary 16, $\kappa(\mu_s^i, \mu_s)$ is bounded on S. Choose a cluster point η_i^0 of $\{\kappa(\mu_s^i, \mu_s) : s \in S\}$; then, in consequence of Lemma 12, one can select an $\mathbf{A}$-vaguely convergent subnet $(\mu_d)_{d \in D}$ of $(\mu_s)_{s \in S}$ such that $\lim_{d \in D} \kappa(\mu_d^i, \mu_d) = \eta_i^0$. However, what has already been proved yields $\eta_i^0 = \eta_i$. Since this means that any cluster point of the net $\kappa(\mu_s^i, \mu_s)$, $s \in S$, coincides with η_i, the desired relation (62) follows. □

15.2. From now on, $\eta_i =: \eta_i(\mathbf{A}, \mathbf{a}, \mathbf{g})$, $i \in I$, will always denote the numbers appeared in Lemma 14. They are uniquely determined by relation (59), where $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is arbitrarily chosen, taken together with (60).

This statement on uniqueness can actually be strengthened as follows.

Corollary 17. *Given $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}, \mathbf{g})$, choose $\eta_i' \in \mathbf{R}$, $i \in I$, so that*

$$\sum_{i \in I} \alpha_i \eta_i' \geqslant \|\mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2.$$

If, moreover, (59) *hold for η_i' in place of η_i, then $\eta_i' = \eta_i(\mathbf{A}, \mathbf{a}, \mathbf{g})$ for all $i \in I$.*

Proof. This follows in the same manner as the uniqueness statement in Lemma 14. □

The following assertion is specifying Lemma 14 for a compact condenser $\mathbf{K}$.

Corollary 18. *Let* $\mathbf{A} = \mathbf{K}$ *be compact. For every* $\lambda_{\mathbf{K}} \in \mathscr{S}(\mathbf{K}, \mathbf{a}, \mathbf{g})$, *we have*

$$\alpha_i a_i \kappa(x, \lambda_{\mathbf{K}}) \geqslant \alpha_i \kappa(\lambda_{\mathbf{K}}^i, \lambda_{\mathbf{K}}) g_i(x) \qquad \text{n.e. in } K_i, \tag{68}$$

and hence

$$a_i \kappa(x, \lambda_{\mathbf{K}}) = \kappa(\lambda_{\mathbf{K}}^i, \lambda_{\mathbf{K}}) g_i(x) \qquad \lambda_{\mathbf{K}}^i\text{-a.e. in } \mathrm{X}. \tag{69}$$

Proof. It follows from (28) and (61) that $\eta_i(\mathbf{K}, \mathbf{a}, \mathbf{g}) = \kappa(\lambda_{\mathbf{K}}^i, \lambda_{\mathbf{K}})$, which leads to (68) when substituted into (59). Since $\lambda_{\mathbf{K}}^i$ has finite energy and is supported by K_i, the inequality in (68) holds $\lambda_{\mathbf{K}}^i$-a.e. in X. Hence, (69) must be true, for if not, then we would arrive at a contradiction by integrating the inequality in (68) with respect to $\lambda_{\mathbf{K}}^i$. □

16. Potentials of A-vague cluster points of minimizing nets

In this section we restrict ourselves to measures ξ of the class $\mathscr{M}_0(\mathbf{A}, \mathbf{a}, \mathbf{g})$. It is clear from Corollary 8 that their potentials admit all the properties described in Lemma 14 (see also Corollary 17). Our purpose is to show that, under proper additional restrictions on κ, that description can be sharpened as follows.

Lemma 15. *In the case where* $I^- \neq \varnothing$, *assume moreover that* $\kappa(x, y)$ *is continuous on* $\overline{A^+} \times \overline{A^-}$, *while* $\kappa(\cdot, y) \to 0$ *(as* $y \to \infty$*) uniformly on compact sets. Given* $\xi \in \mathscr{M}_0(\mathbf{A}, \mathbf{a}, \mathbf{g})$, *for all* $i \in I$ *we have*

$$\alpha_i a_i \kappa(x, \xi) \geqslant \alpha_i \kappa(\xi^i, \xi) g_i(x) \qquad \text{n.e. in } A_i, \tag{70}$$

$$\alpha_i a_i \kappa(x, \xi) \leqslant \alpha_i \kappa(\xi^i, \xi) g_i(x) \qquad \text{for all } x \in S(\xi^i), \tag{71}$$

and hence

$$a_i \kappa(x, \xi) = \kappa(\xi^i, \xi) g_i(x) \qquad \text{n.e. in } A_i \cap S(\xi^i).$$

Proof. By definition, one can choose $\lambda_{\mathbf{K}} \in \mathscr{S}(\mathbf{K}, \mathbf{a}, \mathbf{g})$ such that ξ is an **A**-vague cluster point of the net $(\lambda_{\mathbf{K}})_{\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}}$. Since, in consequence of Lemma 6, this net belongs to $\mathbf{M}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, from (61) and (62) we get

$$\eta_i = \kappa(\xi^i, \xi) = \lim_{\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}} \kappa(\lambda_{\mathbf{K}}^i, \lambda_{\mathbf{K}}), \qquad i \in I.$$

Substituting this into (59) with ξ in place of χ gives (70) as required.

To establish (71), we fix i (say $i \in I^+$) and $x_0 \in S(\xi^i)$. Without loss of generality it can certainly be assumed that

$$\lambda_{\mathbf{K}} \to \xi \qquad \mathbf{A}\text{-vaguely}. \tag{72}$$

Because of (69) and (72), there is $x_{\mathbf{K}} \in S(\lambda^i_{\mathbf{K}})$ with the properties

$$x_{\mathbf{K}} \to x_0 \qquad \text{as } \mathbf{K} \uparrow \mathbf{A}, \tag{73}$$
$$a_i \kappa(x_{\mathbf{K}}, \lambda_{\mathbf{K}}) = \kappa(\lambda^i_{\mathbf{K}}, \lambda_{\mathbf{K}}) g_i(x_{\mathbf{K}}).$$

Taking into account that, by [10, Lemma 2.2.1], the map $(x, v) \mapsto \kappa(x, v)$ is lower semicontinuous on the product space $\mathrm{X} \times \mathfrak{M}^+$ (where $\mathfrak{M}^+$ is equipped with the vague topology), we conclude from what has just been shown that the desired relation (71) will follow once we prove

$$\kappa(x_0, R\xi^-) = \lim_{\mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}} \kappa(x_{\mathbf{K}}, R\lambda^-_{\mathbf{K}}). \tag{74}$$

The case we are thus left with is $I^- \neq \varnothing$. Then, according to our standing assumptions, (22) holds, and therefore there is $q \in (0, \infty)$ such that

$$R\lambda^-_{\mathbf{K}}(\mathrm{X}) \leqslant q \qquad \text{for all } \mathbf{K} \in \{\mathbf{K}\}_{\mathbf{A}}. \tag{75}$$

Since, by (72) and Lemma 2, $R\lambda^-_{\mathbf{K}} \to R\xi^-$ vaguely, from Lemma 3 we also get

$$R\xi^-(\mathrm{X}) \leqslant q. \tag{76}$$

Fix $\varepsilon > 0$. Under the assumptions of the lemma, one can choose a compact neighborhood W_{x_0} of the point x_0 in $\overline{A^+}$ and a compact neighborhood F of W_{x_0} in X such that $F_* := F \cap \overline{A^-}$ is nonempty and

$$|\kappa(x, y)| < q^{-1}\varepsilon \qquad \text{for all } (x, y) \in W_{x_0} \times \complement F. \tag{77}$$

In the rest of the proof, $\tilde{\complement}$ and $\tilde{\partial}$ denote respectively the complement and the boundary of a set relative to $\overline{A^-}$ (where $\overline{A^-}$ is treated as a topological subspace of X). Having observed that $\kappa|_{W_{x_0} \times \overline{A^-}}$ is continuous, we proceed by constructing a function $\varphi \in \mathrm{C}_0(W_{x_0} \times \overline{A^-})$ that admits the properties

$$\varphi|_{W_{x_0} \times F_*} = \kappa|_{W_{x_0} \times F_*}, \tag{78}$$
$$|\varphi(x, y)| \leqslant q^{-1}\varepsilon \qquad \text{for all } (x, y) \in W_{x_0} \times \tilde{\complement} F_*. \tag{79}$$

To this end, we consider a compact neighborhood V_* of F_* in $\overline{A^-}$ and write

$$f := \begin{cases} \kappa & \text{on } W_{x_0} \times \tilde{\partial} F_*, \\ 0 & \text{on } W_{x_0} \times \tilde{\partial} V_*. \end{cases}$$

Note that $E := (W_{x_0} \times \tilde{\partial} F_*) \cup (W_{x_0} \times \tilde{\partial} V_*)$ is a compact subset of the Hausdorff and compact, hence normal, space $W_{x_0} \times V_*$ and f is continuous on E. By using the Tietze-Urysohn extension theorem, we deduce from (77) that there is a continuous function $\hat{f} : W_{x_0} \times V_* \to [-\varepsilon q^{-1}, \varepsilon q^{-1}]$ such that $\hat{f}|_E = f|_E$.

Thus, the function in question can be defined as follows:

$$\varphi := \begin{cases} \kappa & \text{on } W_{x_0} \times F_*, \\ \hat{f} & \text{on } W_{x_0} \times (V_* \setminus F_*), \\ 0 & \text{on } W_{x_0} \times \complement V_*. \end{cases}$$

Furthermore, since φ is continuous on $W_{x_0} \times \overline{A^-}$ and has compact support, one can choose a compact neighborhood U_{x_0} of x_0 in W_{x_0} so that

$$|\varphi(x, y) - \varphi(x_0, y)| < q^{-1}\varepsilon \qquad \text{for all } (x, y) \in U_{x_0} \times \overline{A^-}. \tag{80}$$

Therefore, if $\nu \in \mathfrak{M}^+(\overline{A^-})$ is an arbitrary measure with $\nu(\mathbf{X}) \leqslant q$, then, in consequence of (77)–(80), for all $x \in U_{x_0}$ we get

$$|\kappa(x, \nu|_{\complement F})| \leqslant \varepsilon, \tag{81}$$

$$\kappa(x, \nu|_F) = \int \varphi(x, y) d(\nu - \nu|_{\complement F})(y), \tag{82}$$

$$\left| \int \varphi(x, y) d\nu|_{\complement F}(y) \right| \leqslant \varepsilon, \tag{83}$$

$$\left| \int [\varphi(x, y) - \varphi(x_0, y)] d\nu(y) \right| \leqslant \varepsilon. \tag{84}$$

Finally, let us choose $\mathbf{K}_0 \in \{\mathbf{K}\}_{\mathbf{A}}$ so that for all $\mathbf{K}$ that follow $\mathbf{K}_0$ we have $x_{\mathbf{K}} \in U_{x_0}$ and

$$\left| \int \varphi(x_0, y) d(R\lambda_{\mathbf{K}}^- - R\xi^-)(y) \right| < \varepsilon;$$

such a $\mathbf{K}_0$ exists by reason of (72) and (73).

Applying now (81)–(84) to each of the measures $R\lambda_{\mathbf{K}}^-$ and $R\xi^-$, which is possible due to (75) and (76), for all $\mathbf{K}$ that follow $\mathbf{K}_0$ we therefore obtain

$$\begin{aligned} |\kappa(x_{\mathbf{K}}, R\lambda_{\mathbf{K}}^-) - \kappa(x_0, R\xi^-)| &\leqslant |\kappa(x_{\mathbf{K}}, R\lambda_{\mathbf{K}}^-|_F) - \kappa(x_0, R\xi^-|_F)| + 2\varepsilon \\ &\leqslant \left| \int \varphi(x_{\mathbf{K}}, y) dR\lambda_{\mathbf{K}}^-(y) - \int \varphi(x_0, y) dR\xi^-(y) \right| + 4\varepsilon \\ &\leqslant \left| \int [\varphi(x_{\mathbf{K}}, y) - \varphi(x_0, y)] dR\lambda_{\mathbf{K}}^-(y) \right| \\ &\quad + \left| \int \varphi(x_0, y) d(R\lambda_{\mathbf{K}}^- - R\xi^-)(y) \right| + 4\varepsilon \\ &\leqslant \varepsilon + \varepsilon + 4\varepsilon = 6\varepsilon, \end{aligned}$$

and (74) follows by letting $\varepsilon \to 0$. The proof is complete. □

17. Proof of Theorems 2 and 3

We begin by showing that

$$\mathrm{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leqslant \|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2. \tag{85}$$

To this end, $\|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2$ can certainly be assumed to be finite. Then there are $\nu \in \hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\mu \in \mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$, the existence of μ being clear from (19) and Corollary 3. By [10, Lemma 2.3.1], the inequality in (29) holds μ^i-a.e. in X. Integrating it with respect to μ^i and then summing up over all $i \in I$, we get

$$\kappa(\nu, \mu) \geqslant \sum_{i \in I} c_i(\nu),$$

hence $\kappa(\nu, \mu) \geqslant 1$ in consequence of (30), and finally

$$\|\nu\|^2 \|\mu\|^2 \geqslant 1$$

by the Cauchy-Schwarz inequality. The last relation, being valid for arbitrary $\nu \in \hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $\mu \in \mathscr{E}^0(\mathbf{A}, \mathbf{a}, \mathbf{g})$, implies (85), which in turn immediately yields Theorem 2 provided $\mathrm{cap}\,\mathbf{A} = \infty$.

We are thus left with establishing both Theorems 2 and 3 in the case where $\mathrm{cap}\,\mathbf{A} < \infty$. Then the $\mathscr{E}(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$-problem can be considered as well. Taking (8) and (24) into account, we deduce from Lemmas 9 and 14 with $\mathbf{a}$ replaced by $\mathbf{a}\,\mathrm{cap}\,\mathbf{A}$ that, for every $\chi \in \mathscr{M}'(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$,

$$\|\chi\|^2 = \mathrm{cap}\,\mathbf{A} \tag{86}$$

and there are unique $\tilde{\eta}_i \in \mathbf{R}$, $i \in I$, such that

$$\alpha_i a_i \kappa(x, \chi) \geqslant \tilde{\eta}_i g_i(x) \qquad \text{n.e. in } A_i, \qquad i \in I, \tag{87}$$

$$\sum_{i \in I} \tilde{\eta}_i = 1. \tag{88}$$

Actually,

$$\tilde{\eta}_i = \alpha_i [\mathrm{cap}\,\mathbf{A}]^{-1} \eta_i(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}), \qquad i \in I, \tag{89}$$

where $\eta_i(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$, $i \in I$, are the numbers uniquely determined in Sec. 15.

Using Lemma 10 and the fact that the potentials of equivalent in $\mathscr{E}$ measures coincide n.e. in X, we conclude from (87) and (88) that

$$\mathscr{M}'_{\mathscr{E}}(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g}) \subset \hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Together with (85) and (86), this implies that, for every $\sigma \in \mathscr{M}'_{\mathscr{E}}(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$,

$$\mathrm{cap}\,\mathbf{A} = \|\sigma\|^2 \geqslant \|\hat{\Gamma}(\mathbf{A}, \mathbf{a}, \mathbf{g})\|^2 \geqslant \mathrm{cap}\,\mathbf{A},$$

which completes the proof of Theorem 2.

The last two relations also yield $\mathscr{M}'_{\mathscr{E}}(\mathbf{A}, \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g}) \subset \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. As both the sides of this inclusion are equivalence classes in $\mathscr{E}$ (see Lemma 11), they must be equal, and (32) follows.

Applying Corollary 17 for $\mathbf{a} \text{ cap } \mathbf{A}$ in place of $\mathbf{a}$, we deduce from (32) that $c_i(\hat{\omega})$, $i \in I$, satisfying (29) and (30) for $\nu = \hat{\omega} \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, are uniquely determined, do not depend on the choice of $\hat{\omega}$, and are actually equal to $\tilde{\eta}_i$. Therefore, substituting (61) and, subsequently, (62) for $\mathbf{a} \text{ cap } \mathbf{A}$ in place of $\mathbf{a}$ into (89), we get (33) and (34). This proves Theorem 3. $\square$

18. Proof of Theorem 5

We start by observing that $\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty, contained in an equivalence class in $\mathscr{E}(\overline{\mathbf{A}})$, and satisfies the inclusions

$$\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathscr{M}(\mathbf{A}, \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g}) \subset \mathscr{M}'(\mathbf{A}, \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g}) \cap \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g}). \quad (90)$$

Indeed, this follows from (41), Corollary 8, and Lemma 9, the last two being taken for $\mathbf{a} \text{ cap } \mathbf{A}$ in place of $\mathbf{a}$.

Substituting (39) into (90) gives (42) as required. Since, by (42), every $\gamma \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is a minimizer in the $\Gamma(\mathbf{A}, \mathbf{a}, \mathbf{g})$-problem, relations (43) and (44) are obtained directly from Theorems 3 and 4 in view of Definition 6. To show that $C_i(\mathbf{A}, \mathbf{a}, \mathbf{g})$ can be given by means of (45), one only needs to substitute γ instead of ζ into (33)—which is possible due to (90)—and to use Corollary 9.

Assume for a moment that, if $I^- \neq \varnothing$, then the kernel $\kappa(x, y)$ is continuous on $\overline{A^+} \times \overline{A^-}$, while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets. To establish (46), it suffices to apply Lemma 15 (with $\mathbf{a} \text{ cap } \mathbf{A}$ in place of $\mathbf{a}$) to γ, which can be done because of (41), and to substitute (45) into the result obtained.

To prove that $\mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is $\mathbf{A}$-vaguely compact, fix $(\gamma_s)_{s \in S} \subset \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. In consequence of (42) and Lemma 12, this net is $\mathbf{A}$-vaguely relatively compact. Let γ_0 denote one of its $\mathbf{A}$-vague cluster points, and let $(\gamma_t)_{t \in T}$ be a subnet of $(\gamma_s)_{s \in S}$ that converges $\mathbf{A}$-vaguely to γ_0. In view of (41), the proof will be completed once we show that

$$\gamma_0 \in \mathscr{M}_0(\mathbf{A}, \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g}). \quad (91)$$

According to (41), for every $t \in T$ one can choose a subnet $(\mathbf{K}_{s_t})_{s_t \in S_t}$ of the net $(\mathbf{K})_{\mathbf{K} \in \{\mathbf{K}\}_\mathbf{A}}$ and $\lambda_{s_t} \in \mathscr{S}(\mathbf{K}_{s_t}, \mathbf{a} \text{ cap } \mathbf{A}, \mathbf{g})$ for all $s_t \in S_t$ such that $(\lambda_{s_t})_{s_t \in S_t}$ converges $\mathbf{A}$-vaguely to γ_t. Consider the Cartesian product $\prod\{S_t : t \in T\}$—that is, the collection of all functions ψ on T with $\psi(t) \in S_t$, and let D denote the directed product $T \times \prod\{S_t : t \in T\}$ (see, e.g., [14, Chap. 2, §3]). For any given $(t, \psi) \in D$, write

$$\mathbf{K}_{(t,\psi)} := \mathbf{K}_{\psi(t)} \qquad \text{and} \qquad \lambda_{(t,\psi)} := \lambda_{\psi(t)}.$$

Then the theorem on iterated limits from [14, Chap. 2, §4] yields that the net $(\lambda_{(t,\psi)})_{(t,\psi)\in D}$ converges $\mathbf{A}$-vaguely to γ_0. Since, as is seen from the above construction, $(\mathbf{K}_{(t,\psi)})_{(t,\psi)\in D}$ forms a subnet of $(\mathbf{K})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$, this proves (91). □

19. Proof of Proposition 2

Consider $\nu \in \mathscr{E}(\overline{\mathbf{A}})$ and $\beta_i \in \mathbf{R}$, $i \in I$, satisfying both (47) and (48), and fix arbitrarily $\gamma_{\mathbf{A}} \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and $(\mu_t)_{t\in T} \in \mathbf{M}(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$. Since μ_t^i is concentrated on A_i and has finite energy and compact support, the inequality in (47) holds μ_t^i-a.e. in X. Integrating it with respect to μ_t^i and then summing up over all $i \in I$, in view of (43) and (48) we obtain

$$2\kappa(\mu_t, \nu) \geqslant \|\gamma_{\mathbf{A}}\|^2 + \|\nu\|^2, \qquad t \in T.$$

But $(\mu_t)_{t\in T}$ converges to $\gamma_{\mathbf{A}}$ in the strong topology of the semimetric space $\mathscr{E}(\overline{\mathbf{A}})$, which is clear from (90) and Lemma 9 with $\mathbf{a}\,\mathrm{cap}\,\mathbf{A}$ instead of $\mathbf{a}$. Therefore, passing in the preceding relation to the limit through T, we get

$$\|\nu - \gamma_{\mathbf{A}}\|^2 = 0,$$

which is a part of the conclusion of the proposition.

In turn, the last relation implies that the right-hand side in (48) is in fact equal to 1 and, as well, that $\nu \in \mathscr{M}'(\mathbf{A}, \mathbf{a}\,\mathrm{cap}\,\mathbf{A}, \mathbf{g})$. Since, by Theorem 3, the latter means that $R\nu \in \hat{\mathscr{G}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, the claimed relation (49) follows. □

20. Proof of Theorem 6

To establish (50), fix $\mu \in \mathscr{E}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Then either $g_{i,\inf} > 0$ for all i, and consequently $\mu^i(\mathrm{X}) < \infty$, or X is countable at infinity; in any case, every A_i is contained in a countable union of μ^i-integrable sets. Therefore, by Propositions 4.14.1 and 4.14.6 from [8] (see also Appendix in [26]),

$$\int g_i \, d\mu^i = \lim_{n\in\mathbf{N}} \int g_i \, d\mu^i_{\mathbf{A}_n}, \qquad i \in I,$$

$$\kappa(\mu^i, \mu^j) = \lim_{n\in\mathbf{N}} \kappa(\mu^i_{\mathbf{A}_n}, \mu^j_{\mathbf{A}_n}), \qquad i, j \in I,$$

where $\mu^i_{\mathbf{A}_n}$ denotes the trace of μ^i upon A_n^i. Applying the same arguments as in the proof of Lemma 6, but now based on the preceding two relations instead of (11) and (12), we arrive at (50) as required.

In view of (19) and (50), $\mathrm{cap}(\mathbf{A}_n, \mathbf{a}, \mathbf{g})$, $n \in \mathbf{N}$, can certainly be assumed to be nonzero. Suppose moreover that $\mathrm{cap}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is finite; then, by (9), so is $\mathrm{cap}(\mathbf{A}_n, \mathbf{a}, \mathbf{g})$. Hence, according to Theorem 5, there exists

$$\gamma_n := \gamma_{\mathbf{A}_n} \in \mathscr{D}(\mathbf{A}_n, \mathbf{a}, \mathbf{g}). \tag{92}$$

Observe that $R\gamma_n$ is a minimizer in the $\hat{\Gamma}(\mathbf{A}_n, \mathbf{a}, \mathbf{g})$-problem, which is clear from (32), (39), and (42). Since, furthemore, $\hat{\Gamma}(\mathbf{A}_{n+1}, \mathbf{a}, \mathbf{g}) \subset \hat{\Gamma}(\mathbf{A}_n, \mathbf{a}, \mathbf{g})$, application of Lemma 5 to $\mathscr{H} = \hat{\Gamma}(\mathbf{A}_n, \mathbf{a}, \mathbf{g})$, $v = R\gamma_{n+1}$, and $\lambda = R\gamma_n$ gives

$$\|\gamma_{n+1} - \gamma_n\|^2 \leqslant \|\gamma_{n+1}\|^2 - \|\gamma_n\|^2.$$

Also note that $\|\gamma_n\|^2$, $n \in \mathbf{N}$, is a Cauchy sequence in $\mathbf{R}$, because, as a result of (50), its limit exists and is finite. Combined with the preceding inequality, this proves that $(\gamma_n)_{n\in\mathbf{N}}$ is a strong Cauchy sequence in $\mathscr{E}(\overline{\mathbf{A}})$.

Besides, since $\operatorname{cap} \mathbf{A}_n \leqslant \operatorname{cap} \mathbf{A}$, (42) yields $(\gamma_n)_{n\in\mathbf{N}} \subset \mathscr{E}(\overline{\mathbf{A}}, \leqslant \mathbf{a} \operatorname{cap} \mathbf{A}, \mathbf{g})$. Hence, by Theorem 7, there exists an $\mathbf{A}$-vague cluster point γ of $(\gamma_n)_{n\in\mathbf{N}}$ and, moreover, $\gamma_n \to \gamma$ strongly. Let $(\gamma_t)_{t\in T}$ denote a subnet of the sequence $(\gamma_n)_{n\in\mathbf{N}}$ that converges $\mathbf{A}$-vaguely and strongly to γ. We next proceed by showing that

$$\gamma \in \mathscr{D}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{93}$$

For every $t \in T$, consider the filtering family $\{\mathbf{K}_t\}_{\mathbf{A}_t}$ of all compact condensers $\mathbf{K}_t \prec \mathbf{A}_t$. Then, by (92), there exist a subnet $(\mathbf{K}_{s_t})_{s_t\in S_t}$ of $(\mathbf{K}_t)_{\mathbf{K}_t \in \{\mathbf{K}_t\}_{\mathbf{A}_t}}$ and $\lambda_{s_t} \in \mathscr{S}(\mathbf{K}_{s_t}, \mathbf{a} \operatorname{cap} \mathbf{K}_{s_t}, \mathbf{g})$ such that $(\lambda_{s_t})_{s_t\in S_t}$ converges both strongly and $\mathbf{A}$-vaguely to γ_t. Consider the Cartesian product $\prod\{S_t : t \in T\}$—that is, the collection of all functions ψ on T with $\psi(t) \in S_t$, and let D denote the directed product $T \times \prod\{S_t : t \in T\}$. Given $(t, \psi) \in D$, write

$$\mathbf{K}_{(t,\psi)} := \mathbf{K}_{\psi(t)} \qquad \text{and} \qquad \lambda_{(t,\psi)} := \lambda_{\psi(t)}.$$

Then the theorem on iterated limits from [14, Chap. 2, §4] yields that the net $(\lambda_{(t,\psi)})_{(t,\psi)\in D}$ converges both strongly and $\mathbf{A}$-vaguely to γ. Since $(\mathbf{K}_{(t,\psi)})_{(t,\psi)\in D}$ forms a subnet of $(\mathbf{K})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$, this proves (93) as required.

What is finally left is to prove (51). By Corollary 14, for every $n \in \mathbf{N}$ one can choose a compact condenser $\mathbf{K}_n^0 \prec \mathbf{A}_n$ so that

$$|C_i(\mathbf{A}_n, \mathbf{a}, \mathbf{g}) - C_i(\mathbf{K}_n^0, \mathbf{a}, \mathbf{g})| < n^{-1}, \qquad i \in I.$$

This $\mathbf{K}_n^0$ can be chosen so large that the sequence obtained, $(\mathbf{K}_n^0)_{n\in\mathbf{N}}$, forms a subnet of $(\mathbf{K})_{\mathbf{K}\in\{\mathbf{K}\}_{\mathbf{A}}}$; therefore, repeated application of Corollary 14 yields

$$\lim_{n\in\mathbf{N}} C_i(\mathbf{K}_n^0, \mathbf{a}, \mathbf{g}) = C_i(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

This leads to (51) when combined with the preceding relation. □

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