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Sharp inequalities for the permanental dominance conjecture

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ABSTRACT. For the normalized generalized matrix function $\bar{d}_\chi^G(A)$ for 3×3 positive semi-definite Hermitian matrices A, the permanental dominance conjecture per $A \geq \bar{d}_\chi^G(A)$ is known to hold. In this paper, we show that this inequality is not sharp, and give a sharper bound.

1. Introduction

The normalized generalized matrix function is a complex valued function on $n \times n$ square matrices $M_n(\mathbf{C})$, defined by

$$\bar{d}_\chi^G(A) := \frac{1}{\chi(\mathrm{id})} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where G is a subgroup of the symmetric group $\mathfrak{S}_n$ and χ a character of G. In particular, it is called *immanant* if $G = \mathfrak{S}_n$ and χ is an irreducible character. For example, when $\chi(\sigma) = \operatorname{sgn} \sigma$, the immanant is the determinant, and when $\chi(\sigma) \equiv 1$, the immanant $\bar{d}_1^{\mathfrak{S}_n}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod a_{i\sigma(i)}$ is called *the permanent* of A, denoted by per A.

Note that if its domain is restricted to the (positive semi-definite) Hermitian matrices, the generalized matrix function takes real values. These values have been studied for a long time.

THEOREM 1 (Hadamard [3] 1893). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\det A \leq a_{11} \dots a_{nn}.$$

THEOREM 2 (Fisher [2] 1907). *If $\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ is an $n \times n$ positive semi-definite Hermitian matrix with A_{11} and A_{22} square matrices, then*

$$\det \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \leq (\det A_{11})(\det A_{22}).$$

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These inequalities were generalized by Schur's theorem for the generalized matrix functions.

Theorem 3 (Schur [11] 1918). *If A is an $n \times n$ positive semi-definite Hermitian matrix, G a subgroup of $\mathfrak{S}_n$ and χ a character of G, then*

$$\det A \leq \bar{d}_\chi^G(A).$$

Indeed, Theorems 1, 2 are the special cases, the right-hand sides of which are $\bar{d}_1^{\{\mathrm{id}\}}$, $\bar{d}_{\mathrm{sgn}}^{\mathfrak{S}_k \times \mathfrak{S}_\ell}$, respectively. Schur's theorem says that the determinant is the smallest normalized generalized matrix function. Analogously, the following theorems for the permanent are known.

Theorem 4 (Marcus [9] 1964). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\operatorname{per} A \geq a_{11} \dots a_{nn}.$$

Theorem 5 (Lieb [8] 1966). *If $\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ is an $n \times n$ positive semi-definite Hermitian matrix with A_{11} and A_{22} square matrices, then*

$$\operatorname{per}\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \geq (\operatorname{per} A_{11})(\operatorname{per} A_{22}).$$

From these results, it is natural to expect

Conjecture (Lieb [8] 1966). *(Permanental Dominance Conjecture) If A is an $n \times n$ positive semi-definite Hermitian matrix, G a subgroup of $\mathfrak{S}_n$ and χ a character of G, then*

$$\operatorname{per} A \geq \bar{d}_\chi^G(A).$$

It is known that the permanental dominance conjecture holds for immanants with $n \leq 13$ (see [10]), and for all subgroups of $\mathfrak{S}_n$ and all characters when $n \leq 3$ ([5], [7]).

A stronger result is known for single hook immanants (see [4]), namely

$$\operatorname{per} A = \bar{d}^{\mathfrak{S}_n}_{(n)}(A) \geq \bar{d}^{\mathfrak{S}_n}_{(n-1,1)}(A) \geq \cdots \geq \bar{d}^{\mathfrak{S}_n}_{(2,1^{n-2})}(A) \geq \bar{d}^{\mathfrak{S}_n}_{(1^n)}(A) = \det A.$$

Hence the permanental dominance conjecture for $n = 3$ is already settled. However, in this paper we will show that the inequality for the conjecture is not sharp, and represent sharper bounds by the internally dividing points between the determinant and the permanent. In particular we prove the following theorem for the alternating group $\mathfrak{A}_3$.

Main Theorem. *If A is a 3×3 positive semi-definite Hermitian matrix and ω is a non-trivial irreducible character of $\mathfrak{A}_3$, then*

$$\bar{d}_{\omega}^{\mathfrak{A}_3}(A) \leq 2^{-1/3} \operatorname{per} A + (1 - 2^{-1/3}) \det A.$$

This is an improvement of the inequality in the conjecture for this special case.

2. Proof of Main Theorem

Let G be a subgroup of the symmetric group $\mathfrak{S}_n$ and χ a character of G. We define *the generalized matrix function* by

$$d_{\chi}^{G}(A) := \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

In particular, when $G = \mathfrak{S}_n$ and χ is an irreducible character, $d_{\chi}^{G}(A)$ is called *immanant*. Moreover, *the normalized generalized matrix function* $\bar{d}_{\chi}^{G}$ is defined by

$$\bar{d}_{\chi}^{G}(A) := \frac{1}{\chi(\mathrm{id})} d_{\chi}^{G}(A).$$

For the rest of the paper, we suppose $n = 3$ unless otherwise stated. It is known that the permanental dominance conjecture is true for $n = 3$ ([5], [7]).

In this section, we always write A as $\begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{pmatrix}$. We will display the values of the three normalized immanants for A.

$$\operatorname{per} A = \bar{d}_{(3)}^{\mathfrak{S}_3}(A) = adf + (b\bar{c}e + \bar{b}c\bar{e}) + (a|e|^2 + d|c|^2 + f|b|^2),$$

$$\det A = \bar{d}_{(1,1,1)}^{\mathfrak{S}_3}(A) = adf + (b\bar{c}e + \bar{b}c\bar{e}) - (a|e|^2 + d|c|^2 + f|b|^2),$$

$$\bar{d}_{\yng(2,1)}^{\mathfrak{S}_3}(A) = \bar{d}_{(2,1)}^{\mathfrak{S}_3}(A) = adf - \frac{1}{2}(b\bar{c}e + \bar{b}c\bar{e}).$$

Remark 1. *If A is a positive semi-definite Hermitian matrix, then $a|e|^2 + d|c|^2 + f|b|^2 \geq 0$. Moreover when the equality holds, the values of all the normalized generalized matrix functions coincide.*

The following function plays the key role in this paper.

Definition 1. Define a complex valued function T for 3×3 semi-definite Hermitian matrices A with $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$ by

$$T(A) := \frac{b\bar{c}e}{a|e|^2 + d|c|^2 + f|b|^2}.$$

PROPOSITION 1. *Let A be a positive semi-difinite Hermitian matrix. If $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$, then*

$$\operatorname{Re} T(A) \leq \frac{1}{3}.$$

PROOF. It is a restatement of the inequality $\bar{d}_{(2,1)}(A) - \det A \geq 0$, which is a special case of Schur's theorem.

$$\begin{aligned} 0 &\leq \bar{d}_{(2,1)}(A) - \det A \\ &= \left(adf - \frac{1}{2}(b\bar{c}e + \bar{b}c\bar{e})\right) - (adf + (b\bar{c}e + \bar{b}c\bar{e}) - (a|e|^2 + d|c|^2 + f|b|^2)) \\ &= -3 \operatorname{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2. \end{aligned}$$

Divide both sides by $a|e|^2 + d|c|^2 + f|b|^2$ to obtain $-3 \operatorname{Re} T(A) + 1 \geq 0$. □

REMARK 2. *Conversely, we obtain $\bar{d}_{(2,1)}(A) \geq \det A$ from $\operatorname{Re} T(A) \leq 1/3$. The equality holds for the positive semi-definite Hermitian matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The eigenvalues of this matrix are $3, 0, 0$. More generally, if A has 0 as the eigenvalue with multiplicity 2, then the equality holds.*

Similarly, the inequality $\operatorname{Re} T(A) \geq -1/3$ is equivalent to $\operatorname{per} A \geq \bar{d}_{(2,1)}(A)$, a special case of the permanental dominance conjecture. Conversely, we obtain $\operatorname{per} A \geq \bar{d}_{(2,1)}(A)$ from $\operatorname{Re} T(A) \geq -1/3$. However, this inequality is not sharp:

LEMMA 1. *Let A be a positive semi-difinite Hermitian matrix. If $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$, then*

$$\operatorname{Re} T(A) \geq -\frac{1}{6}.$$

This result was already shown implicitly in [6] and explicitly in [1, 12]. We include a proof for the reader's convenience. We will prove our main theorem by a similar argument.

PROOF (Proof of Lemma 1). As the arithmetic mean is larger than or equal to the geometric mean, we have

$$a|e|^2 + d|c|^2 + f|b|^2 \geq 3\sqrt[3]{adf|bce|^2}.$$

Also as $\det A \geq 0$, we have

$$adf \geq -2 \operatorname{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2.$$

Combining them, we obtain

$$a|e|^2 + d|c|^2 + f|b|^2 \geq 3\sqrt[3]{adf|bce|^2}$$

$$\geq 3\sqrt[3]{(-2\,\mathrm{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2)(\mathrm{Re}(b\bar{c}e))^2}.$$

For simplicity, we write X for $a|e|^2 + d|c|^2 + f|b|^2$ and Z for $\mathrm{Re}(b\bar{c}e)$ so that $\mathrm{Re}\, T(A) = Z/X$. Then we have

$$X \geq 3\sqrt[3]{(-2Z + X)Z^2},$$

$$X^3 \geq -54Z^3 + 27XZ^2,$$

$$54Z^3 - 27XZ^2 + X^3 \geq 0,$$

$$54\left(\frac{Z}{X}\right)^3 - 27\left(\frac{Z}{X}\right)^2 + 1 \geq 0.$$

The graph of the function $Y = 54(Z/X)^3 - 27(Z/X)^2 + 1$ looks like the following.

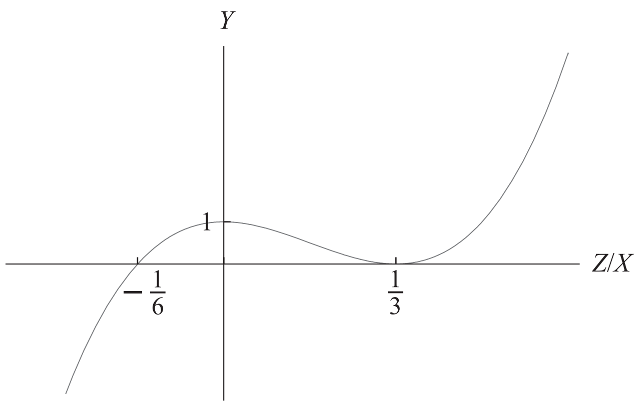

Fig. 1. $Y = 54\left(\frac{Z}{X}\right)^3 - 27\left(\frac{Z}{X}\right)^2 + 1$

From the graph, we can conclude $\mathrm{Re}\, T(A) = Z/X \geq -1/6$. □

The equality holds for the positive semi-definite Hermitian matrices

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & 2 & \sqrt{2} \\ -1 & \sqrt{2} & 4 \end{pmatrix}.$$

The eigenvalues of these matrices are $3, 3, 0$ and $(7 \pm \sqrt{7})/2, 0$, respectively.

The inequality $\operatorname{Re} T(A) \geq -1/6$ immediately implies

$$\bar{d}_{\boxplus}^{\mathfrak{S}_3}(A) \leq \frac{3}{4} \operatorname{per} A + \frac{1}{4} \det A.$$

Corollary 1. *If A is a 3×3 positive semi-definite Hermitian matrix, then*

$$\frac{1}{6} \operatorname{per} A + \frac{5}{6} \det A \leq \bar{d}_1^{\{\mathrm{id}\}}(A) \leq \frac{2}{3} \operatorname{per} A + \frac{1}{3} \det A.$$

These inequalities are sharp, and are improvements of the permanental dominance conjecture and Hadamard's theorem.

From here, we study the values of $T(A)$ in the complex plane. In particular, we obtain the sharper inequalities for $d_{\omega_1}^{\mathfrak{A}_3}$ and $d_{\omega_2}^{\mathfrak{A}_3}$ in terms of the above, where $\mathfrak{A}_3$ is the alternating group and $\omega_i : \mathfrak{A}_3 \to \mathbf{C}$ $(i = 1, 2)$ are the two non-trivial irreducible characters.

Lemma 2. *For the complex number $x + yi$ $(x, y \in \mathbf{R})$, $T(A) = x + yi$ for some positive semi-definite Hermitian matrices if and only if*

$$\begin{cases} 54x(x^2 + y^2) - 27(x^2 + y^2) + 1 \geq 0, \\ -\frac{1}{6} \leq x \leq \frac{1}{3}. \end{cases}$$

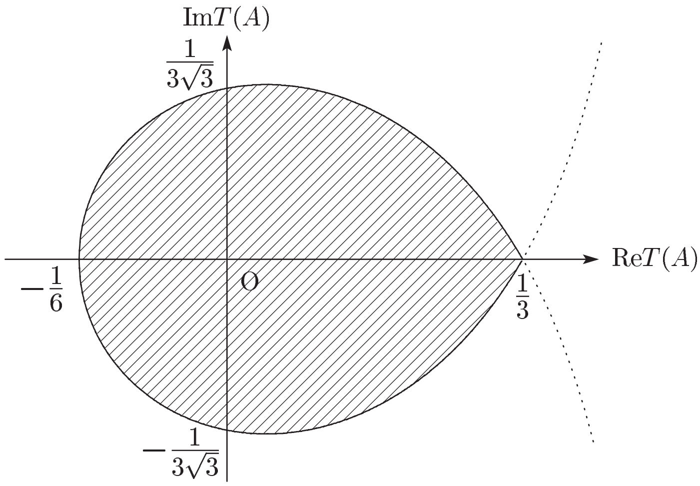

Fig. 2

Proof. We have shown $-1/6 \leq x \leq 1/3$ in Proposition 1 and Lemma 1. Suppose $X = a|e|^2 + d|c|^2 + f|b|^2, R = |bce|, S = \operatorname{Re}(b\bar{c}e)$. Calculating as in the beginning of the proof of Lemma 1, we have

$$X \geq 3(adfR^2)^{1/3} \geq 3((X - 2S)R^2)^{1/3}.$$

Note that $T(A) = x + yi, R/X = (x^2 + y^2)^{1/2}$ and $S/X = x$. An easy calculation shows that

$$54x(x^2 + y^2) - 27(x^2 + y^2) + 1 \geq 0,$$

hence the "only if" part follows.

Conversely for each $x + yi$ satisfying the inequality, the matrix

$$A = \begin{pmatrix} 1 & b & b \\ \bar{b} & 1 & b \\ \bar{b} & \bar{b} & 1 \end{pmatrix}$$

is a positive semi-definite Hermitian matrix with $T(A) = x + yi$, where $b = 3(x + yi)$. □

Proof (Proof of Main Theorem). Let $\omega_0, \omega_1, \omega_2$ denote the characters of the three irreducible representations of $\mathfrak{A}_3$. The following is the character table of $\mathfrak{A}_3$.

$\mathfrak{A}_3$	(1)	(123)	(132)
ω_0	1	1	1
ω_1	1	ω	ω^2
ω_2	1	ω^2	ω

where $\omega = (-1 + \sqrt{3}i)/2$. We prove the assertion only in the case $\omega = \omega_1$. The other case is similar. We want to find μ with $\bar{d}_{\omega_1}^{\mathfrak{A}_3}(A) \leq \mu \operatorname{per} A + (1 - \mu) \det A$. We observe

$$\bar{d}_{\omega_1}^{\mathfrak{A}_3}(A) = adf + 2 \operatorname{Re}(\omega b\bar{c}e),$$

hence $\bar{d}_{\omega_1}^{\mathfrak{A}_3}(A) \leq \mu \operatorname{per} A + (1 - \mu) \det A$ if and only if

$$adf + 2 \operatorname{Re}(\omega b\bar{c}e) \leq adf + 2 \operatorname{Re}(b\bar{c}e) + (2\mu - 1)(a|e|^2 + d|c|^2 + f|b|^2).$$

This inequality can be rewritten as

$$\frac{1}{2} + \operatorname{Re}((\omega - 1)T(A)) \leq \mu.$$

From Lemma 2, finding the boundary point where the slope is $-\sqrt{3}$, calculation shows that

$$T(A) = \frac{1}{6}(2 - \sqrt[3]{4} - \sqrt[3]{2}) - \frac{1}{2\sqrt{3}}(\sqrt[3]{4} - \sqrt[3]{2})i$$

maximizes $\mathrm{Re}((\omega-1)T(A))$ with the value $\mu = 2^{-1/3}$. When $b = 1 - \sqrt[3]{2}\omega^2 - \sqrt[3]{4}\omega$ (hence a root of $b^3 - 3b^2 - 3b - 1 = 0$), this $T(A)$ is realized by $A = \begin{pmatrix} 1 & b & b \\ \bar{b} & 1 & b \\ \bar{b} & \bar{b} & 1 \end{pmatrix}$. □

Remark 3. *For the trivial character ω_0 of $\mathfrak{A}_3$, the following equality always holds:*

$$\bar{d}_{\omega_0}^{\mathfrak{A}_3}(A) = \frac{1}{2}\ \mathrm{per}\ A + \frac{1}{2}\ \det A.$$

Corollary 2. *There are no finitely many test matrices $\{A_1, \dots, A_r\}$ such that for a linear combination*

$$F(A) = \alpha\ \mathrm{Re}(b\bar{c}e) + \beta\ \mathrm{Im}(b\bar{c}e) + \gamma(a|e|^2 + d|c|^2 + f|b|^2),$$

$F(A) \geq 0$ for any positive semi-definite Hermitian matrix A if and only if $F(A_i) \geq 0$ for all $i = 1, 2, \dots, r$.

Proof. Each $F(A_i) \geq 0$ determines a half plane in the complex plane. From Figure 2, the shaded area cannot be written as the intersection of finitely many half planes. □

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