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Sharp distortion estimates for p-Bloch functions

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ABSTRACT. Let $p \in (0, \infty)$ and $\mathfrak{B}_1^p$ be the class of analytic functions f in the unit disk $\mathbf{D}$ with $f(0) = 0$ satisfying $|f'(z)| \le 1/(1-|z|^2)^p$. For $z_0, z_1 \in \mathbf{D}$, $w_1 \in \mathbf{C}$ with $z_0 \ne z_1$ and $|w_1| \le 1/(1-|z_1|^2)^p$, put $V^p(z_0; z_1, w_1)$ be the variability region of $f'(z_0)$ when f ranges over the class $\mathfrak{B}_1^p$ with $f'(z_1) = w_1$, i.e., $V^p(z_0; z_1, w_1) = \{f'(z_0) : f \in \mathfrak{B}_1^p$ and $f'(z_1) = w_1\}$. In 1988 M. Bonk showed that $V^1(z_0; z_1, w_1)$ is a convex closed Jordan domain and determined it by giving a parametrization of the simple closed curve $\partial V^1(z_0; z_1, w_1)$. He also derived distortion theorems for $\mathfrak{B}_1^1$ as corollaries. In the present article we shall refine Bonk's method and explicitly determine $V^p(z_0; z_1, w_1)$.

1. Introduction

For $a \in \mathbf{C}$ and $r > 0$, set $\mathbf{D}(a, r) = \{z \in \mathbf{C} : |z - a| < r\}$ and $\mathbf{D} = \mathbf{D}(0, 1)$. For $c \in \mathbf{D}$ and $0 < \rho < 1$ we also set $\Delta(c, \rho) = \{z \in \mathbf{D} : |z - c|/|1 - \bar{c}z| < \rho\}$. Let $\mathscr{H}(\mathbf{D})$ be the class of analytic functions in the unit disk $\mathbf{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbf{D}$. Let $p \in (0, \infty)$. For a function $f \in \mathscr{H}(\mathbf{D})$, we put

$$\mu_p(f, z) = (1 - |z|^2)^p |f'(z)|, \qquad z \in \mathbf{D}.$$

A function f is called a p-Bloch function provided

$$\|f\|_{\mathfrak{B}^p} = \sup_{z \in \mathbf{D}} \mu_p(f, z) = \sup_{z \in \mathbf{D}} (1 - |z|^2)^p |f'(z)|$$

is finite. We denote by $\mathfrak{B}^p$ the complex Banach space consisting of p-Bloch functions f on $\mathbf{D}$ normalized by $f(0) = 0$: $\mathfrak{B}^p = \{f \in \mathscr{H}(\mathbf{D}) : f(0) = 0, \|f\|_{\mathfrak{B}^p} < \infty\}$. We also denote by $\mathfrak{B}_1^p$ the closed unit ball of $\mathfrak{B}^p$, i.e.,

$$\mathfrak{B}_1^p = \{f \in \mathfrak{B}^p : \|f\|_{\mathfrak{B}^p} \le 1\}.$$

In [6] and [2], Avhadiev, Schulte and Wirths studied some extremal problems on $\mathfrak{B}_1^p$ and derived sharp inequalities for the first and the second coefficients of analytic functions $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathfrak{B}_1^p$.

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If $p = 1$, then a function $f \in \mathfrak{B}^1$ is simply called a Bloch function. In [3], Bonk proved that the variability region $V^1(z_0; z_1, w_1) = \{f'(z_0) : f \in \mathfrak{B}^1, f'(z_1) = w_1 \text{ and } \|f\|_{\mathfrak{B}^1} \le 1\}$ for $z_0, z_1 \in \mathbf{D}$, $w_1 \in \mathbf{C}$ with $z_0 \ne z_1$ and $|w_1| \le (1 - |z_1|^2)^{-1}$ is a compact convex subset of $\mathbf{C}$, and that it is a closed Jordan domain. He also determined it by giving a parametrization of the simple closed curve $\partial V^1(z_0; z_1, w_1)$.

In the present article we shall determine the variability region $V^p(z_0; z_1, w_1)$ for $f'(z_0)$ when f ranges over the class $\mathfrak{B}_1^p$ with $f'(z_1) = w_1$, i.e., $V^p(z_0; z_1, w_1) := \{f'(z_0) : f \in \mathfrak{B}_1^p, f'(z_1) = w_1\}$, where $z_0, z_1 \in \mathbf{D}$, $w_1 \in \mathbf{C}$ with $z_0 \ne z_1$ and $|w_1| \le 1/(1 - |z_1|^2)^p$. We shall also give sharp distortion estimates as its corollary. When $p = 1$, for related results see [4], [5], [7] and [8]. For $a \in \mathbf{D}$ set

$$\tau_a(z) = \frac{z - a}{1 - \bar{a}z}, \qquad z \in \mathbf{D}.$$

Then τ_a is a conformal automorphism of $\mathbf{D}$ and $\tau_a^{-1} = \tau_{-a}$. For $a \in \mathbf{D}$ and $f \in \mathscr{H}(\mathbf{D})$ put

$$T_a f(z) = \int_0^z f'(\tau_a(\zeta))\tau_a'(\zeta)^p d\zeta = \int_0^z f'\left(\frac{\zeta - a}{1 - \bar{a}\zeta}\right) \frac{(1 - |a|^2)^p}{(1 - \bar{a}\zeta)^{2p}} d\zeta, \qquad z \in \mathbf{D}.$$

It is easy to see that $T_a^{-1} = T_{-a}$. From the identity $|\tau_a'(z)|/(1 - |\tau_a(z)|^2) = 1/(1 - |z|^2)$ we have

$$\begin{aligned} \mu_p(T_a f, z) &= (1 - |z|^2)^p |f'(\tau_a(z))|\, |\tau_a'(z)|^p \\ &= (1 - |\tau_a(z)|^2)^p |f'(\tau_a(z))| = \mu_p(f, \tau_a(z)). \end{aligned} \tag{1}$$

Therefore T_a acts on $\mathfrak{B}^p$ as an isometry.

It is not difficult to see that for any $\theta, \varphi \in \mathbf{R}$

$$V^p(e^{i\varphi} z_0; e^{i\varphi} z_1, e^{i\theta} w_1) = e^{i\theta} V^p(z_0; z_1, w_1). \tag{2}$$

For any $a \in \mathbf{D}$ we also have

$$V^p(\tau_a(z_0); \tau_a(z_1), w_1/\tau_a'(z_1)^p) = \frac{1}{\tau_a'(z_0)^p} V^p(z_0; z_1, w_1). \tag{3}$$

By virtue of (2) and (3), without loss of generality we may assume $z_1 = 0$, $w_1 = \alpha \in [0, 1]$ and $z_0 = r \in (0, 1)$. We put for $z_0 \in \mathbf{D}$ and $0 \le \alpha \le 1$

$$V_\alpha^p(z_0) = V^p(z_0; 0, \alpha) = \{f'(z_0) : f \in \mathfrak{B}_1^p(\alpha)\}, \tag{4}$$

where $\mathfrak{B}_1^p(\alpha) = \{f \in \mathfrak{B}^p : f'(0) = \alpha, \|f\|_{\mathfrak{B}^p} \le 1\}$. Note that $V_\alpha^p(z_0) = V_\alpha^p(|z_0|)$.

For any fixed $\alpha \in [0,1]$ and $r \in (0,1)$ it is easy to see that the set $V_\alpha^p(r)$ is a compact convex subset of $\mathbf{C}$. This is a consequence of the fact that $\mathfrak{B}_1^p(\alpha)$ is also compact and convex in $\mathscr{H}(\mathbf{D})$. We next see that α is an interior point of $V_\alpha^p(r)$. It is proved by using

$$G(z) = \alpha z + \frac{w-\alpha}{3r^2} z^3$$

for w which belongs to $\mathbf{D}(\alpha, pr^2)$. Then we have $G \in \mathfrak{B}_1^p(\alpha)$, because $G(0) = 0$, $G'(0) = \alpha$ and

$$\begin{aligned} |G'(z)|(1-|z|^2)^p &= \left|\alpha + \frac{w-\alpha}{r^2} z^2\right|(1-|z|^2)^p \\ &\le (\alpha + p|z|^2)(1-|z|^2)^p \le (1+p|z|^2)(1-|z|^2)^p \le 1. \end{aligned}$$

Furthermore it follows from

$$G'(r) = \alpha + \frac{w-\alpha}{r^2} r^2 = w$$

that $w \in V_\alpha^p(r)$. As a result, since $\mathbf{D}(\alpha, pr^2) \subset V_\alpha^p(r)$, α is an interior point of $V_\alpha^p(r)$.

Thus $V_\alpha^p(r)$ is a closed Jordan domain, i.e., $\partial V_\alpha^p(r)$ is a simple closed curve and $V_\alpha^p(r)$ is the union of $\partial V_\alpha^p(r)$ and its inner domain.

We note the following trivial but useful fact:

Lemma 1. *If* $|f'(z_0)| = \frac{1}{(1-|z_0|^2)^p}$ *for some* $f \in \mathfrak{B}_1^p(\alpha)$*, then* $f'(z_0) \in \partial V_\alpha^p(z_0)$.

2. Extremal functions and main results

To state our theorem explicitly we need to introduce some functions which are extremal for the results in this article.

Let $M(t) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p t(1-t^2)^p$, $0 \le t \le 1$. Then $M(t)$ is strictly increasing on $[0, 1/\sqrt{2p+1}]$, strictly decreasing on $[1/\sqrt{2p+1}, 1]$ and $M(1/\sqrt{2p+1}) = 1$. The function

$$B(z) = -\frac{\sqrt{2p+1}}{2}\left(\frac{2p+1}{2p}\right)^p z^2, \qquad z \in \mathbf{D}$$

satisfies

$$\mu_p(B, z) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p |z|(1-|z|^2)^p = M(|z|) \le 1, \qquad z \in \mathbf{D}$$

with equality if and only if $|z| = 1/\sqrt{2p+1}$. Let $m : [0,1] \to [0, 1/\sqrt{2p+1}]$ be the inverse function of the restriction $M|_{[0,1/\sqrt{2p+1}]}$. The function m is strictly increasing with $m(0) = 0$, $m(1) = 1/\sqrt{2p+1}$.

A half class of extremal functions is obtained by putting for $\alpha \in [0,1]$

$$\begin{aligned} B_\alpha(z) &= T_{m(\alpha)}B(z) \\ &= \int_0^z B'(\tau_{m(\alpha)}(\zeta))\tau'_{m(\alpha)}(\zeta)^p d\zeta \\ &= \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p \int_0^z \frac{(1-m(\alpha)^2)^p (m(\alpha)-\zeta)}{(1-m(\alpha)\zeta)^{2p+1}}\, d\zeta. \end{aligned} \tag{5}$$

Precisely by integration and $M(m(\alpha)) = \alpha$ we have

$$\begin{aligned} B_\alpha(z) = {} & \frac{\alpha}{2p(2p-1)m(\alpha)^3} \\ & \times \left\{ \frac{1+(2p-1)m(\alpha)^2 - 2pm(\alpha)z}{(1-m(\alpha)z)^{2p}} - 1 - (2p-1)m(\alpha)^2 \right\}, \end{aligned}$$

when $p \neq \frac{1}{2}$, and

$$B_\alpha(z) = \frac{\alpha}{m(\alpha)^3}\left\{ \log\frac{1}{1-m(\alpha)z} - \frac{m(\alpha)(1-m(\alpha)^2)z}{1-m(\alpha)z} \right\},$$

when $p = \frac{1}{2}$. By (1) we have $\mu_p(B_\alpha, z) \le 1$ with equality if and only if $|\tau_{m(\alpha)}(z)| = 1/\sqrt{2p+1}$. From

$$B'_\alpha(z) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p \frac{(1-m(\alpha)^2)^p(m(\alpha)-z)}{(1-m(\alpha)z)^{2p+1}}$$

we obtain

$$B'_\alpha(0) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p m(\alpha)(1-m(\alpha)^2)^p = M(m(\alpha)) = \alpha. \tag{6}$$

Thus for each $\alpha \in [0,1]$ the function B_α satisfies $B_\alpha(0) = 0$, $B'_\alpha(0) = \alpha$ and $\mu_p(B_\alpha, z) \le 1$ on $\mathbf{D}$ with equality if and only if $|\tau_{m(\alpha)}(z)| = 1/\sqrt{2p+1}$. In particular $B_\alpha \in \mathfrak{B}_1^p(\alpha)$.

Theorem 1. *For $z_0 \in \Delta(m(\alpha), 1/\sqrt{2p+1})$ the relation $B'_\alpha(z_0) \in \partial V_\alpha^p(z_0)$ holds. Furthermore for $f \in \mathfrak{B}_1^p(\alpha)$, $f'(z_0) = B'_\alpha(z_0)$ holds if and only if $f = B_\alpha$.*

We shall prove Theorem 1 in Section 3. By Theorem 1 and $V_\alpha^p(z_0) = V_\alpha^p(|z_0|)$ we have for fixed $r \in (0,1)$ the mapping $\theta \mapsto B'_\alpha(re^{i\theta})$ gives an arc contained in $\partial V_\alpha(r)$, whenever $re^{i\theta} \in \Delta(m(\alpha), 1/\sqrt{2p+1})$. By an elementary calculation we have

$$\Delta(m(\alpha), 1/\sqrt{2p+1}) = \mathbf{D}\left(\frac{2pm(\alpha)}{2p+1-m(\alpha)^2}, \frac{\sqrt{2p+1}(1-m(\alpha)^2)}{2p+1-m(\alpha)^2}\right).$$

Hence

$$\mathbf{D}\left(0, \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}\right)$$

is the largest disk with center 0 which is contained in $\Delta(m(\alpha), 1/\sqrt{2p+1})$ and

$$\mathbf{D}\left(0, \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}\right)$$

is the smallest disk with center 0 which contains $\Delta(m(\alpha), 1/\sqrt{2p+1})$. Furthermore if $z_0 \in \partial\Delta(m(\alpha), 1/\sqrt{2p+1})$, then we have

$$\mu_p(B_\alpha, z_0) = \mu_p(T_{m(\alpha)}B, z_0) = \mu_p(B, \tau_{m(\alpha)}(z_0)) = M(|\tau_{m(\alpha)}(z_0)|) = 1. \tag{7}$$

By Lemma 1, this shows $B'_\alpha(z_0) \in \partial V^p_\alpha(z_0)$, when $z_0 \in \partial\Delta(m(\alpha), 1/\sqrt{2p+1})$. From these considerations it seems natural that the following theorem holds.

Theorem 2. *For $\alpha \in [0,1]$ and $r \in (0,1)$ the variability region $V^p_\alpha(r)$ is a convex closed Jordan domain bounded by $\partial V^p_\alpha(r)$.*

(i) *For $0 < r \le \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}$, the boundary $\partial V^p_\alpha(r)$ is given by the mapping $(-\pi, \pi] \ni \theta \mapsto B'_\alpha(re^{i\theta})$.*

(ii) *For $\frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)} < r < \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}$, the boundary $\partial V^p_\alpha(r)$ is the union of two arcs Γ_1 and Γ_2. Here Γ_1 is given by the mapping $[-\theta_\alpha(r), \theta_\alpha(r)] \ni \theta \mapsto B'_\alpha(re^{i\theta})$, where*

$$\theta_\alpha(r) = \arccos\left(\frac{(2p+1-m(\alpha)^2)r^2 + (2p+1)m(\alpha)^2 - 1}{4prm(\alpha)}\right).$$

The arc Γ_2 is the circular arc contained in $\partial\mathbf{D}(0, (1-r^2)^{-p})$ with endpoints $B'_\alpha(re^{i\theta_\alpha(r)})$ and $B'_\alpha(re^{-i\theta_\alpha(r)})$ that passes through the point $(1-r^2)^{-p}$.

(iii) *For $\frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)} \le r < 1$, the boundary $\partial V^p_\alpha(r)$ coincides with the whole circle $\partial\mathbf{D}(0, (1-r^2)^{-p})$.*

Furthermore $f'(r) = B'_\alpha(re^{i\theta})$ holds for some $f \in \mathfrak{B}^p_1(\alpha)$ and

$$(r, \theta) \in \left(0, \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}\right) \times \mathbf{R}$$

or

$$(r,\theta)\in\left[1-\frac{\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)},\frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}\right)\times(-\theta_\alpha(r),\theta_\alpha(r)),$$

if and only if $f(z)=e^{-i\theta}B_\alpha(e^{i\theta}z)$.

To show $\Gamma_2\subset\partial V_\alpha^p(r)$ in case (ii) and $\partial\mathbf{D}(0,(1-r^2)^{-p})=\partial V_\alpha^p(r)$ in case (iii), we shall construct a function $f\in\mathfrak{B}_1^p(\alpha)$ satisfying $f'(r)=w_0$ for any $w_0\in\Gamma_2$ in case (ii) and for any $w_0\in\partial\mathbf{D}(0,(1-r^2)^{-p})$ in case (iii). For $a\in[0,1)$, $\lambda\in(0,p]$ and $\theta\in\mathbf{R}$ put

$$F_{a,\lambda,\theta}(z)=\int_0^z\frac{(1-a^2)^\lambda(a-\zeta)}{h(\lambda,1/\sqrt{2\lambda+1})(1-a\zeta)^{2\lambda+1}\{1-(e^{-i\theta}\zeta)^2\}^{p-\lambda}}\,d\zeta,\tag{8}$$

where $h(\lambda,x)=x(1-x^2)^\lambda$, $0\le x\le 1$, is strictly increasing on $[0,1/\sqrt{2\lambda+1}]$ and strictly decreasing on $[1/\sqrt{2\lambda+1},1]$. We note that

$$F'_{a,\lambda,\theta}(0)=\frac{h(\lambda,a)}{h(\lambda,1/\sqrt{2\lambda+1})}\tag{9}$$

and

$$\begin{aligned}\mu_p(F_{a,\lambda,\theta},z)&=(1-|z|^2)^p\frac{(1-a^2)^\lambda|a-z|}{h(\lambda,1/\sqrt{2\lambda+1})|1-az|^{2\lambda+1}|1-(e^{-i\theta}z)^2|^{p-\lambda}}\\&=(1-|z|^2)^p\frac{|\tau_a(z)|\,|\tau_a'(z)|^\lambda}{h(\lambda,1/\sqrt{2\lambda+1})|1-(e^{-i\theta}z)^2|^{p-\lambda}}\\&=\frac{1}{h(\lambda,1/\sqrt{2\lambda+1})}\frac{(1-|z|^2)^{p-\lambda}}{|1-(e^{-i\theta}z)^2|^{p-\lambda}}|\tau_a(z)|(1-|\tau_a(z)|^2)^\lambda\\&=\frac{h(\lambda,|\tau_a(z)|)}{h(\lambda,1/\sqrt{2\lambda+1})}\left(\frac{1-|z|^2}{|1-(e^{-i\theta}z)^2|}\right)^{p-\lambda}\\&\le 1\end{aligned}\tag{10}$$

with equality if and only if $|\tau_a(z)|=1/\sqrt{2\lambda+1}$ and $e^{-i\theta}z\in\mathbf{R}$.

Proposition 1. *Let* $\alpha\in[0,1]$.

(i) *If* $\alpha\in[0,1)$, *then for any* $z_0\in\mathbf{D}\backslash\Delta(m(\alpha),1/\sqrt{2p+1})$ *there exists a unique pair* $(a(z_0),\lambda(z_0))$ *with* $0<\lambda(z_0)<p$ *and* $0\le a(z_0)<1/\sqrt{2p+1}$ *such that*

$$h(\lambda(z_0), a(z_0)) = \alpha h(\lambda(z_0), 1/\sqrt{2\lambda(z_0)+1}), \tag{11}$$

$$|\tau_{a(z_0)}(z_0)| = \frac{1}{\sqrt{2\lambda(z_0)+1}}. \tag{12}$$

The functions $a(z_0)$ *and* $\lambda(z_0)$ *are continuous on* $\mathbf{D}\backslash\Delta(m(\alpha), 1/\sqrt{2p+1})$.

(ii) *If* $\alpha = 1$, *then for any* $z_0 \in \overline{\mathbf{D}}(1/2, 1/2)\backslash[\Delta(1/\sqrt{2p+1}, 1/\sqrt{2p+1}) \cup \{0,1\}]$ *there uniquely exists* $\lambda(z_0)$ *such that*

$$|\tau_{1/\sqrt{2\lambda(z_0)+1}}(z_0)| = \frac{1}{\sqrt{2\lambda(z_0)+1}}. \tag{13}$$

The function $\lambda(z_0)$ *is continuous on* $\overline{\mathbf{D}}(1/2, 1/2)\backslash[\Delta(1/\sqrt{2p+1}, 1/\sqrt{2p+1}) \cup \{0,1\}]$.

Combining Proposition 1, (9) and (10) it follows that $F_{a(z_0),\lambda(z_0),\theta_0} \in \mathfrak{B}_1^p(\alpha)$ and $F'_{a(z_0),\lambda(z_0),\theta_0}(z_0) \in \partial\mathbf{D}(0, 1/(1-|z_0|^2)^p)$, where $\theta_0 = \arg z_0$. This and some more analysis on behavior of $F'_{a(z_0),\lambda(z_0),\theta_0}(z_0)$ will complete the proof of Theorem 2. See Section 4 for details. Furthermore as a consequence of Theorem 2, we obtain the following corollary.

Corollary 1. *Suppose* $\alpha \in [0,1]$ *and* $f \in \mathfrak{B}_1^p(\alpha)$.

(i) *For* $|z| < \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}$

$$\operatorname{Re} f'(z) \geq B'_\alpha(|z|) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p \frac{(1-m(\alpha)^2)^p(m(\alpha)-|z|)}{(1-m(\alpha)|z|)^{2p+1}}$$

with equality at $z = re^{i\theta}$, $r \in \left(0, \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}\right)$ *if and only if* $f(z) = e^{i\theta}B_\alpha(e^{-i\theta}z)$. *In particular, we have* $\operatorname{Re} f'(z) > 0$ *for* $|z| < m(\alpha)$.

(ii) *For* $|z| < \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}$

$$|f'(z)| \leq B'_\alpha(-|z|) = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p \frac{(1-m(\alpha)^2)^p(m(\alpha)+|z|)}{(1+m(\alpha)|z|)^{2p+1}}$$

with equality at $z = re^{i\theta}$, $r \in \left(0, \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}\right)$, *if and only if* $f(z) = -e^{i\theta}B_\alpha(-e^{-i\theta}z)$.

The proof of Corollary 1 will be given in Section 3. The following corollary is obtained directly by integrating inequalities in Corollary 1.

COROLLARY 2. *Suppose* $\alpha \in [0,1]$ *and* $f \in \mathfrak{B}_1^p(\alpha)$.

(i) *For* $|z| \le \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}$, *we have* $\operatorname{Re} f(z) \ge B_\alpha(|z|)$ *with equality at* $z = re^{i\theta}$, *if and only if* $f(z) = e^{i\theta}B_\alpha(e^{-i\theta}z)$.

(ii) *For* $|z| \le \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}$, *we have* $|f(z)| \le -B_\alpha(-|z|)$ *with equality at* $z = re^{i\theta}$, *if and only if* $f(z) = -e^{i\theta}B_\alpha(-e^{-i\theta}z)$.

From Corollary 1 and the Wolff-Warschawski-Noshiro Theorem it follows that $f \in \mathfrak{B}_1^p(\alpha)$ is univalent in $\mathbf{D}(0,m(\alpha))$, when $0 < \alpha \le 1$. Since $B'_\alpha(m(\alpha)) = 0$, B_α is not univalent in any larger disk $\mathbf{D}(0,m(\alpha)+\varepsilon)$ for any $\varepsilon > 0$.

COROLLARY 3. *The radius of univalence for* $\mathfrak{B}_1^p(\alpha)$ *is* $m(\alpha)$. *More precisely, if* $\alpha \in (0,1]$ *and* $f \in \mathfrak{B}_1^p(\alpha)$, *then* f *is univalent in* $\mathbf{D}(0,r)$ *for some* $r > m(\alpha)$ *unless* $f(z) = e^{i\theta}B_\alpha(e^{-i\theta}z)$ *for some* $\theta \in \mathbf{R}$.

3. Proof of Theorem 1 and Corollary 1

First, we consider the case that $\alpha \in [0,1)$ in Theorem 1. We need the following lemma.

LEMMA 2. *Let* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *be disks with* $c_1 \in \mathbf{D}_1$ *and* $c_2 \in \mathbf{D}_2$. *Suppose that* $F : \mathbf{D}_1 \to \mathbf{D}_2$ *is a conformal mapping with* $F(c_1) = c_2$. *Let* $\delta_{\mathbf{D}_1}$ *and* $\delta_{\mathbf{D}_2}$ *be the hyperbolic distances on* $\mathbf{D}_1$ *and* $\mathbf{D}_2$, *respectively. If* $f : \mathbf{D}_1 \to \mathbf{D}_2$ *is an analytic function with* $f(c_1) = c_2$, *then*

$$\delta_{\mathbf{D}_2}(f(z), c_2) \le \delta_{\mathbf{D}_1}(z, c_1), \qquad z \in \mathbf{D}_1.$$

Furthermore $f(z_0) = F(z_0)$ *at some* $z_0 \in \mathbf{D}_1 \backslash \{c_1\}$ *holds if and only if* $f = F$.

We can easily verify Lemma 2, and so we omit its proof.

PROOF (Proof of Theorem 1 in the case then $0 \le \alpha < 1$). Let $f \in \mathfrak{B}_1^p(\alpha)$. Then from $T_{-m(\alpha)}f \in \mathfrak{B}_1^p(\alpha)$ we have for $|z| < 1/\sqrt{2p+1}$

$$|(T_{-m(\alpha)}f)'(z)| \le \frac{1}{(1-|z|^2)^p} < \left(\frac{2p+1}{2p}\right)^p$$

and by $M(m(\alpha)) = \alpha$, we have

$$(T_{-m(\alpha)}f)'(-m(\alpha)) = \frac{\alpha}{(1-m(\alpha)^2)^p} = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p m(\alpha) = \gamma_p m(\alpha),$$

where $\gamma_p = \sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p$. Letting $\mathbf{D}_1 = \left\{z \in \mathbf{C} : |z| < \frac{1}{\sqrt{2p+1}}\right\}$ and $\mathbf{D}_2 = \left\{w \in \mathbf{C} : |w| < \left(\frac{2p+1}{2p}\right)^p\right\}$, $(T_{-m(\alpha)}f)'$ is an analytic mapping of $\mathbf{D}_1$ into $\mathbf{D}_2$ with

$(T_{-m(\alpha)}f)'(-m(\alpha)) = \gamma_p m(\alpha)$. Applying Lemma 2 we have for $|z| < 1/\sqrt{2p+1}$

$$\delta_{\mathbf{D}_2}((T_{-m(\alpha)}f)'(z), \gamma_p m(\alpha)) \leq \delta_{\mathbf{D}_1}(z, -m(\alpha)). \tag{14}$$

Take $z_0 \in \Delta(m(\alpha), 1/\sqrt{2p+1}) = \tau_{-m(\alpha)}(\mathbf{D}_1)$ with $z_0 \neq 0$ arbitrarily and put $z_1 = \tau_{m(\alpha)}(z_0)(\in \mathbf{D}_1)$. Let $\overline{\mathbf{D}}_0$ be the closed hyperbolic subdisk of $\mathbf{D}_2$ with center $\gamma_p m(\alpha)$ and radius $\delta_{\mathbf{D}_1}(z_1, -m(\alpha))$. Then by (14) we have $(T_{-m(\alpha)}f)'(z_1) \in \overline{\mathbf{D}}_0$. Thus

$$f'(\tau_{-m(\alpha)}(z_1))\tau'_{-m(\alpha)}(z_1)^p \in \overline{\mathbf{D}}_0.$$

This implies

$$f'(z_0) \in \frac{(1-m(\alpha)^2)^p}{(1-m(\alpha)z_0)^{2p}}\overline{\mathbf{D}}_0.$$

Hence we have

$$V_\alpha^p(z_0) \subset \frac{(1-m(\alpha)^2)^p}{(1-m(\alpha)z_0)^{2p}}\overline{\mathbf{D}}_0. \tag{15}$$

Since $B'_\alpha(z) = (T_{m(\alpha)}B)'(z)$, we have $(T_{-m(\alpha)}B_\alpha)'(z) = B'(z) = -\gamma_p z$. Thus $(T_{-m(\alpha)}B_\alpha)'$ is a conformal mapping of $\mathbf{D}_1$ onto $\mathbf{D}_2$ with $(T_{-m(\alpha)}B_\alpha)'(-m(\alpha)) = \gamma_p m(\alpha)$. In particular we have $(T_{-m(\alpha)}B_\alpha)'(z_1) \in \partial\mathbf{D}_0$ and hence

$$B'_\alpha(z_0) \in \frac{(1-m(\alpha)^2)^p}{(1-m(\alpha)z_0)^{2p}}\partial\mathbf{D}_0. \tag{16}$$

Since $B'_\alpha(z_0) \in V_\alpha^p(z_0)$, it follows from (15) and (16) that $B'_\alpha(z_0) \in \partial V_\alpha^p(z_0)$.

Next, we prove the uniqueness. Assume that $z_0 \in \Delta(m(\alpha), 1/\sqrt{2p+1})$ with $z_0 \neq 0$ and $f'(z_0) = B'_\alpha(z_0)$ for some $f \in \mathfrak{B}_1^p(\alpha)$. Then we have $(T_{-m(\alpha)}f)'(z_1) = (T_{-m(\alpha)}B_\alpha)'(z_1)$, where $z_1 = \tau_{m(\alpha)}(z_0)$. Applying the uniqueness part of Lemma 2 at $z_1(\neq -m(\alpha))$ we obtain $(T_{-m(\alpha)}f)' = (T_{-m(\alpha)}B_\alpha)'$ and hence $f' = B'_\alpha$. Since $f(0) = B_\alpha(0) = 0$ we have $f = B_\alpha$. □

Proof (Proof of Corollary 1 in the case that $0 \leq \alpha < 1$). We use the same notation as in the above. Take $z_0 = r \in \left(0, \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}\right)$ and put $z_1 = \tau_{m(\alpha)}(r)$. Since $(T_{-m(\alpha)}B_\alpha)'(z) = -\gamma_p z$ maps $\mathbf{D}_1$ conformally onto $\mathbf{D}_2$ and $-\gamma_p z_1 < \gamma_p m(\alpha)$, we have

$$\min_{w \in \overline{\mathbf{D}}_0} \operatorname{Re} w = -\gamma_p z_1 = -\gamma_p \tau_{m(\alpha)}(r)$$

and hence for $f \in \mathfrak{B}_1^p(\alpha)$

$$\operatorname{Re} f'(\tau_{-m(\alpha)}(z_1))\tau'_{-m(\alpha)}(z_1)^p = \operatorname{Re}(T_{-m(\alpha)}f)'(z_1) \geq -\gamma_p \tau_{m(\alpha)}(r).$$

This implies

$$\operatorname{Re} f'(r) = \operatorname{Re} f'(\tau_{-m(\alpha)}(z_1)) \geq -\gamma_p \tau_{m(\alpha)}(r)\tau'_{m(\alpha)}(r)^p = B'_\alpha(r). \tag{17}$$

It is not difficult to see that equality holds in (17) if and only if $f = B_\alpha$. Now let $f \in \mathfrak{B}_1^p(\alpha)$ and $\zeta_0 = r_0 e^{i\theta_0}$ with $0 < r_0 < \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}$ and $\theta_0 \in \mathbf{R}$. Applying (17) to $\tilde{f}(z) = e^{-i\theta_0} f(e^{i\theta_0} z) \in \mathfrak{B}_1^p(\alpha)$ at r_0 we have

$$\operatorname{Re} f'(\zeta_0) = \operatorname{Re} \tilde{f}'(r_0) \geq B'_\alpha(r_0) = B'_\alpha(|\zeta_0|)$$

with equality if and only if $\tilde{f} = B_\alpha$ i.e., $f(z) = e^{i\theta_0} B_\alpha(e^{-i\theta_0} z)$.

Take $z_0 = -r$ with $r \in \left(0, \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}\right)$ and $z_1 = \tau_{m(\alpha)}(-r)$. Since $0 < \gamma_p m(\alpha) < -\gamma_p z_1$, we have

$$\max_{w \in \overline{\mathbf{D}_0}} |w| = -\gamma_p z_1 = -\gamma_p \tau_{m(\alpha)}(-r).$$

Hence for $f \in \mathfrak{B}_1^p(\alpha)$ we have

$$|f'(\tau_{-m(\alpha)}(z_1))\tau'_{-m(\alpha)}(z_1)^p| \leq -\gamma_p \tau_{m(\alpha)}(-r)$$

and thus

$$|f'(-r)| \leq -\gamma_p \tau_{m(\alpha)}(-r)\tau'_{m(\alpha)}(-r)^p = B'_\alpha(-r) \tag{18}$$

with equality if and only if $f = B_\alpha$.

Let $f \in \mathfrak{B}_1^p(\alpha)$ and $\zeta_0 = -r_0 e^{i\theta_0}$ with $0 < r_0 < \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}$ and $\theta_0 \in \mathbf{R}$. Applying (18) to $\tilde{f}(z) = -e^{-i\theta_0} f(-e^{i\theta_0} z) \in \mathfrak{B}_1^p(\alpha)$ at $-r_0$ we have

$$|f'(\zeta_0)| = |\tilde{f}'(-r_0)| \leq B'_\alpha(-r_0) = B'_\alpha(-|\zeta_0|)$$

with equality if and only if $\tilde{f} = B_\alpha$ i.e., $f(z) = -e^{i\theta_0} B_\alpha(-e^{-i\theta_0} z)$. □

In order to prove Theorem 1 in the case that $\alpha = 1$, we need the following result known as Julia's lemma.

Lemma 3 (Julia). *Let g be an analytic function on $\mathbf{D} \cup \{1\}$ with $g(1) = 1$ and $|g(z)| < 1$ for $z \in \mathbf{D}$. Then $\beta = g'(1) > 0$ and*

$$\frac{|1 - g(z)|^2}{1 - |g(z)|^2} \leq \beta \frac{|1 - z|^2}{1 - |z|^2}, \qquad z \in \mathbf{D}. \tag{19}$$

Equality occurs in (19) at $z_0 \in \mathbf{D}$ with $w_0 = g(z_0)$ if and only if

$$\tau_{w_0}(g(z)) = \frac{1 - w_0}{1 - \overline{w_0}} \frac{1 - \overline{z_0}}{1 - z_0} \tau_{z_0}(z)$$

For a proof of the inequality (19) see [1, Theorem 1-5]. For a proof of the uniqueness part see [8].

Proof (Proof of Theorem 1 in the case that $\alpha = 1$). First we note that $m(1) = 1/\sqrt{2p+1}$. We consider the following composite functions to apply Julia's lemma in the disk $\Delta(m(1), 1/\sqrt{2p+1})$. For $f \in \mathfrak{B}_1^p(1)$ put

$$\begin{aligned} g_f(z) &= \left(\frac{2p}{2p+1}\right)^p (T_{-m(1)} f)' \left(-\frac{z}{\sqrt{2p+1}}\right) \\ &= \left(\frac{2p}{2p+1}\right)^{2p} f'\left(\frac{\sqrt{2p+1}(1-z)}{(2p+1)-z}\right) \frac{1}{(1-z/(2p+1))^{2p}}. \end{aligned} \tag{20}$$

Then we have for $|z| < 1$

$$\begin{aligned} |g_f(z)| &= \left(\frac{2p}{2p+1}\right)^p \left| f'\left(\tau_{-m(1)}\left(-\frac{z}{\sqrt{2p+1}}\right)\right)\right| \left|\tau'_{-m(1)}\left(-\frac{z}{\sqrt{2p+1}}\right)\right|^p \\ &= \frac{(2p/(2p+1))^p \mu_p(f, \tau_{-m(1)}(-z/\sqrt{2p+1}))}{(1-|z/\sqrt{2p+1}|^2)^p} \\ &\leq \frac{(2p/(2p+1))^p}{(1-|z/\sqrt{2p+1}|^2)^p} < 1 \end{aligned}$$

and $g_f(1) = f'(0) = 1$. In view of the inequality

$$1 + \mathrm{Re}(f''(0)z) + \cdots = |f'(z)| \leq \frac{1}{(1-|z|^2)^p} = 1 + p|z|^2 + \cdots,$$

$f''(0) = 0$ holds for each $f \in \mathfrak{B}_1^p(1)$. From this and (20)

$$g_f'(1) = \frac{g_f'(1)}{g_f(1)} = -\frac{\sqrt{2p+1}}{2p}\frac{f''(0)}{f'(0)} + 1 = 1.$$

Applying Lemma 3, we have

$$\frac{|1-g_f(z)|^2}{1-|g_f(z)|^2} \leq \frac{|1-z|^2}{1-|z|^2} = \delta(z). \tag{21}$$

Since we can rewrite (21) as

$$\left| g_f(z) - \frac{1}{1+\delta(z)} \right| \leq \frac{\delta(z)}{1+\delta(z)},$$

it follows that

$$g_f(z) \in \overline{\mathbf{D}}\left(\frac{1}{1+\delta(z)}, \frac{\delta(z)}{1+\delta(z)}\right) \tag{22}$$

for all $z \in \mathbf{D}$. Take $z_0 \in \Delta(m(1), 1/\sqrt{2p+1})$ arbitrarily. Substituting $z = -\sqrt{2p+1}\tau_{m(1)}(z_0)$ in (22), we obtain

$$f'(z_0) \in \left(\frac{2p+1}{2p}\right)^p \tau'_{m(1)}(z_0)^p \overline{\mathbf{D}}\left(\frac{1}{1+\tilde{\delta}(z_0)}, \frac{\tilde{\delta}(z_0)}{1+\tilde{\delta}(z_0)}\right), \tag{23}$$

where

$$\tilde{\delta}(z_0) = \delta(-\sqrt{2p+1}\tau_{m(1)}(z_0)).$$

Thus we have

$$V_1^p(z_0) \subset \left(\frac{2p+1}{2p}\right)^p \tau'_{m(1)}(z_0)^p \overline{\mathbf{D}}\left(\frac{1}{1+\tilde{\delta}(z_0)}, \frac{\tilde{\delta}(z_0)}{1+\tilde{\delta}(z_0)}\right). \tag{24}$$

Now let us consider the case that $f = B_1$. Since $B_1 = T_{m(1)}B$, we have

$$\begin{aligned} g_{B_1}(z) &= \left(\frac{2p}{2p+1}\right)^p (T_{-m(1)}(T_{m(1)}B))'\left(-\frac{z}{\sqrt{2p+1}}\right) \\ &= \left(\frac{2p}{2p+1}\right)^p B'\left(-\frac{z}{\sqrt{2p+1}}\right) = z. \end{aligned}$$

This implies $g_{B_1}(z) \in \partial\mathbf{D}(1/(1+\delta(z)), \delta(z)/(1+\delta(z)))$ for all $z \in \mathbf{D}$. Hence we have

$$B_1'(z_0) \in \left(\frac{2p+1}{2p}\right)^p \tau'_{m(1)}(z_0)^p \partial\overline{\mathbf{D}}\left(\frac{1}{1+\tilde{\delta}(z_0)}, \frac{\tilde{\delta}(z_0)}{1+\tilde{\delta}(z_0)}\right). \tag{25}$$

Since $B_1'(z_0) \in V_1^p(z_0)$, we infer from (24) and (25) that $B_1'(z_0) \in \partial V_1^p(z_0)$.

Finally, we deal with uniqueness. Suppose $f'(z_0) = B_1'(z_0)$ for some $f \in \mathfrak{B}_1^p(1)$ and $z_0 \in \Delta(m(1), 1/\sqrt{2p+1})$. Then we have $g_f(z_1) = g_{B_1}(z_1) = z_1$, where $z_1 = -\sqrt{2p+1}\tau_{m(1)}(z_0)$. By Lemma 3 we obtain $g_f(z) = z$ in $\mathbf{D}$ and hence $f'(z) = B_1'(z)$ in $\Delta(m(1), 1/\sqrt{2p+1})$. By the identity theorem for analytic functions, the relation $f'(z) = B_1'(z)$ holds on $\mathbf{D}$. From this and $f(0) = B_1(0) = 0$ we have $f = B_1$. Therefore we complete the proof of Theorem 1. □

PROOF (Proof of Corollary 1 in the case that $\alpha = 1$). Since $m(1) = \frac{1}{\sqrt{2p+1}}$, (ii) in Corollary 1 never occurs. We use the same notation as in the proof of Theorem 1 in the case that $\alpha = 1$. Let $z_0 = r \in \left(0, \frac{1+\sqrt{2p+1}m(1)}{\sqrt{2p+1}+m(1)}\right) = \left(0, \frac{\sqrt{2p+1}}{p+1}\right)$. Then by (23) we have

$$\operatorname{Re} f'(z) \geq \left(\frac{2p+1}{2p}\right)^p \tau'_{m(1)}(r)^p \frac{1-\tilde{\delta}(r)}{1+\tilde{\delta}(r)}.$$

Since $\frac{1-\delta}{1+\delta} = \frac{\operatorname{Re} z - |z|^2}{1 - \operatorname{Re} z}$ and $\tilde{\delta}(r) = \delta(-\sqrt{2p+1}\tau_{m(1)}(r))$,

$$\operatorname{Re} f'(r) \geq -\sqrt{2p+1}\left(\frac{2p+1}{2p}\right)^p \tau_{m(1)}(r)\tau'_{m(1)}(r)^p = B_1(r).$$

The rest of the proof is quite similar as in the case that $0 \leq \alpha < 1$ and we omit it. □

4. Proofs of Proposition 1 and Theorem 2

We need a technical lemma characterizing a monotone property of a family of subdisks of $\mathbf{D}$.

Lemma 4. *Let $c(t)$ and $\rho(t)$ be continuously differentiable functions on an interval I satisfying $c(t) \in \mathbf{D}$ and $\rho(t) \in (0,1)$ on I. Then the family of disks $\{\Delta(c(t), \rho(t))\}_{t \in I}$ is nondecreasing if and only if*

$$\frac{|c'(t)|}{1 - |c(t)|^2} \leq \frac{\rho'(t)}{1 - \rho(t)^2}$$

on I. Furthermore if $|c'(t)|/(1-|c(t)|^2) < \rho'(t)/(1-\rho(t)^2)$ holds on I, then $\{\Delta(c(t), \rho(t))\}_{t \in I}$ is strictly increasing in the sense that $\overline{\Delta}(c(t_0), \rho(t_0)) \subset \Delta(c(t_1), \rho(t_1))$ for any $t_0, t_1 \in I$ with $t_0 < t_1$.

Proof. Let $t_0, t_1 \in I$ with $t_0 < t_1$. Put

$$\tilde{c}(t) = \tau_{c(t_0)}(c(t)) = \frac{c(t) - c(t_0)}{1 - \overline{c(t_0)}c(t)}, \qquad t \in I.$$

Then we have $\tau_{c(t_0)}(\Delta(c(t), \rho(t))) = \Delta(\tilde{c}(t), \rho(t))$ and $\tilde{c}(t_0) = 0$.

Assume that $\{\Delta(c(t), \rho(t))\}_{t \in I}$ is nondecreasing. Then $\{\Delta(\tilde{c}(t), \rho(t))\}_{t \in I}$ is also nondecreasing and hence

$$\begin{aligned}\mathbf{D}(0, \rho(t_0)) = \Delta(\tilde{c}(t_0), \rho(t_0)) &\subset \Delta(\tilde{c}(t_1), \rho(t_1)) \\ &= \mathbf{D}\left(\frac{(1-\rho(t_1)^2)\tilde{c}(t_1)}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2}, \frac{(1-|\tilde{c}(t_1)|^2)\rho(t_1)}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2}\right).\end{aligned}$$

This implies

$$\rho(t_0) \leq \frac{(1-|\tilde{c}(t_1)|^2)\rho(t_1)}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2} - \frac{(1-\rho(t_1)^2)|\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2} = \frac{\rho(t_1) - |\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)|\rho(t_1)}.$$

From this it follows that

$$|\tilde{c}(t_1) - \tilde{c}(t_0)| = |\tilde{c}(t_1)| \leq \frac{\rho(t_1) - \rho(t_0)}{1 - \rho(t_0)\rho(t_1)}.$$

Dividing both sides of the above inequality by $t_1 - t_0$ and then letting $t_1 \downarrow t_0$ we obtain

$$\frac{|c'(t_0)|}{1 - |c(t_0)|^2} = |\tilde{c}'(t_0)| \leq \frac{\rho'(t_0)}{1 - \rho(t_0)^2}.$$

Conversely assume that $|c'(t)|/(1 - |c(t)|^2) \leq \rho'(t)/(1 - \rho(t)^2)$ holds on I. We note that

$$\frac{|\tilde{c}'(t)|}{1 - |\tilde{c}(t)|^2} = \frac{|c'(t)|}{1 - |c(t)|^2} \leq \frac{\rho(t)}{1 - \rho(t)^2}$$

and $\tilde{c}(t_0) = 0$. Since $\left|\frac{d}{dt}|\tilde{c}(t)|\right| \leq \left|\frac{d}{dt}\tilde{c}(t)\right|$, we have

$$\begin{aligned}
\frac{1}{2}\log\frac{1 + |\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)|} &= \left|\int_{t_0}^{t_1} \frac{1}{2}\frac{d}{dt}\left\{\log\frac{1 + |\tilde{c}(t)|}{1 - |\tilde{c}(t)|}\right\}dt\right| \\
&= \left|\int_{t_0}^{t_1} \frac{\frac{d}{dt}|\tilde{c}(t)|}{1 - |\tilde{c}(t)|^2}\,dt\right| \\
&\leq \int_{t_0}^{t_1} \frac{\left|\frac{d}{dt}\tilde{c}(t)\right|}{1 - |\tilde{c}(t)|^2}\,dt \\
&\leq \int_{t_0}^{t_1} \frac{\rho'(t)}{1 - \rho(t)^2}\,dt = \frac{1}{2}\log\frac{1 + \rho(t_1)}{1 - \rho(t_1)}\frac{1 - \rho(t_0)}{1 + \rho(t_0)}.
\end{aligned}$$

Thus we have

$$\frac{1 + |\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)|} \leq \frac{1 + \rho(t_1)}{1 - \rho(t_1)}\frac{1 - \rho(t_0)}{1 + \rho(t_0)}$$

and hence from an elementary calculation it follows that

$$|\tilde{c}(t_1)| \leq \frac{\rho(t_1) - \rho(t_0)}{1 - \rho(t_0)\rho(t_1)}. \tag{26}$$

Now we put

$$a = \frac{(1 - \rho(t_1)^2)\tilde{c}(t_1)}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2}, \qquad r = \frac{(1 - |\tilde{c}(t_1)|^2)\rho(t_1)}{1 - |\tilde{c}(t_1)|^2\rho(t_1)^2}.$$

Then we have $\Delta(\tilde{c}(t_1), \rho(t_1)) = \mathbf{D}(a, r)$ and

$$r - |a| = \frac{(1 - |\tilde{c}(t_1)|^2)\rho(t_1) - (1 - \rho(t_1)^2)|\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)|^2 \rho(t_1)^2} = \frac{\rho(t_1) - |\tilde{c}(t_1)|}{1 - |\tilde{c}(t_1)\rho(t_1)|}.$$

We claim that $r - |a| \geq \rho(t_0)$ holds, which is a simple consequence of (26). Thus we have

$$\begin{aligned} \tau_{c(t_0)}(\Delta(c(t_0), \rho(t_0))) = \Delta(0, \rho(t_0)) &\subset \mathbf{D}(0, r - |a|) \\ &\subset \mathbf{D}(a, r) \\ &= \Delta(\tilde{c}(t_1), \rho(t_1)) = \tau_{c(t_0)}(\Delta(c(t_1), \rho(t_1))) \end{aligned}$$

and hence $\Delta(c(t_0), \rho(t_0)) \subset \Delta(c(t_1), \rho(t_1))$.

If $|c'(t)|/(1 - |c(t)|^2) < \rho'(t)/(1 - \rho(t)^2)$ holds on I, then it is easy to see that strict inequality sign holds in (26). This implies $r - |a| > \rho(t_0)$ and thus we conclude $\bar{\Delta}(c(t_0), \rho(t_0)) \subset \Delta(c(t_1), \rho(t_1))$. □

Let $E = \{(\lambda, x) : 0 \leq \lambda \leq p$ and $0 \leq x \leq 1/\sqrt{2\lambda + 1}\}$ and

$$h(\lambda, x) = \begin{cases} x(1 - x^2)^{\lambda}, & (\lambda, x) \in E \backslash \{(0, 1)\} \\ 1, & (\lambda, x) = (0, 1). \end{cases}$$

Lemma 5. *The function $h(\lambda, x)$ is nonnegative and continuous on E, and satisfies $h(\lambda, x) \leq 1$ with equality if and only if $(\lambda, x) = (0, 1)$. Furthermore for fixed $\lambda \in [0, p]$, $h(\lambda, x)$ is a strictly increasing function of $x \in [0, 1/\sqrt{2\lambda + 1}]$ and for fixed $x \in (0, 1)$, $h(\lambda, x)$ is a strictly decreasing function of $\lambda \in [0, \min\{2^{-1}(x^{-2} - 1), p\}]$.*

Proof. The monotonic properties of $h(\lambda, x)$ is clear. And it is also clear that $h(\lambda, x)$ is continuous and satisfies $h(\lambda, x) < 1$ on $E \backslash \{(0, 1)\}$. Thus we only have to show that $h(\lambda, x)$ is continuous at $(0, 1)$. To show this let $0 < \delta < 1$. Then for $(\lambda, x) \in E \backslash \{(0, 1)\}$ with $1 - \delta \leq x < 1$ and $0 < \lambda \leq \delta$ we have

$$1 > h(\lambda, x) \geq h(\lambda, 1 - \delta) \geq h(\delta, 1 - \delta) = (1 - \delta)\delta^{\delta}(2 - \delta)^{\delta} \to 1$$

as $\delta \downarrow 0$. □

Let $\alpha \in [0, 1]$. For each fixed $\lambda \in [0, p]$ let $x = c(\lambda)$ be the unique solution of the equation

$$h(\lambda, x) = \alpha h(\lambda, 1/\sqrt{2\lambda + 1}), \qquad 0 \leq x \leq \frac{1}{\sqrt{2\lambda + 1}}.$$

LEMMA 6. *If* $\alpha = 0$ *or* $\alpha = 1$, *then* $c(\lambda) = 0$ *or* $c(\lambda) = 1/\sqrt{2\lambda+1}$, *respectively. If* $0 < \alpha < 1$, *then the function* $c(\lambda)$ *is a continuously differentiable function of* $\lambda \in (0, p)$ *satisfying* $0 < c(\lambda) < 1/\sqrt{2\lambda+1}$ *and* $c'(\lambda) < 0$ *in* $(0, p)$. *Furthermore we have* $c(0) = \lim_{\lambda\downarrow 0} c(\lambda) = \alpha$ *and* $c(p) = \lim_{\lambda\uparrow p} c(\lambda) = m(\alpha)$.

PROOF. We shall only show the assertions, when $0 < \alpha < 1$. In this case it is easy to see that $0 < c(\lambda) < 1/\sqrt{2\lambda+1}$. Put $H(\lambda, x) = h(\lambda, x)/h(\lambda, 1/\sqrt{2\lambda+1})$. Then by Lemma 5, $H(\lambda, x)$ is continuous on E and continuously differentiable in $\operatorname{Int} E$. Since

$$\frac{\partial}{\partial x} H(\lambda, x) = \frac{\{1-(2\lambda+1)x^2\}(1-x^2)^{\lambda-1}}{h(\lambda, 1/\sqrt{2\lambda+1})} > 0 \tag{27}$$

in $\operatorname{Int} E$, it follows from the implicit function theorem that $c(\lambda)$ is continuously differentiable in $(0, p)$. Since

$$h(\lambda, c(\lambda)) = \alpha h(\lambda, 1/\sqrt{2\lambda+1}), \tag{28}$$

we have

$$\log c(\lambda) + \lambda \log(1 - c(\lambda)^2) = \log \alpha - \frac{1}{2}\log(2\lambda+1) + \lambda \log \frac{2\lambda}{2\lambda+1}.$$

By differentiating both sides of the above formula and $0 < c(\lambda) < 1/\sqrt{2\lambda+1}$ we obtain

$$c'(\lambda) = \frac{c(\lambda)(1-c(\lambda)^2)}{1-(2\lambda+1)c(\lambda)^2} \log \frac{2\lambda}{(2\lambda+1)(1-c(\lambda)^2)} < 0. \tag{29}$$

Thus $c(\lambda)$ is strictly decreasing and hence $c(0+) = \lim_{\lambda\downarrow 0} c(\lambda)$ and $c(p-0) = \lim_{\lambda\uparrow p} c(\lambda)$ exist. By continuity of $h(\lambda, x)$ on E we have $c(0+) = \alpha$ and $c(p-0)(1-c(p-0)^2)^p = \alpha h(p, 1/\sqrt{2p+1})$. These imply $c(0+) = \alpha = c(0)$ and $c(p-0) = m(\alpha) = c(p)$. □

LEMMA 7. *If* $0 \le \alpha < 1$, *then the family of disks* $\{\Delta(c(\lambda), 1/\sqrt{2\lambda+1})\}_{0\le\lambda\le p}$ *is strictly decreasing and*

$$\bigcup_{0<\lambda<p} \Delta(c(\lambda), 1/\sqrt{2\lambda+1}) = \Delta(\alpha, 1) = \mathbf{D},$$

$$\bigcap_{0<\lambda<p} \Delta(c(\lambda), 1/\sqrt{2\lambda+1}) = \bar{\Delta}(m(\alpha), 1/\sqrt{2p+1}).$$

If $\alpha = 1$, *then* $c(\lambda) = 1/\sqrt{2\lambda+1}$ *and* $\{\Delta(1/\sqrt{2\lambda+1}, 1/\sqrt{2\lambda+1})\}_{0<\lambda\le p}$ *is decreasing and satisfies*

$$\bigcup_{0<\lambda<p} \Delta(1/\sqrt{2\lambda+1}, 1/\sqrt{2\lambda+1}) = \mathbf{D}(1/2, 1/2),$$

$$\bigcap_{0<\lambda<p} \Delta(1/\sqrt{2\lambda+1}, 1/\sqrt{2\lambda+1}) = \bar{\Delta}(1/\sqrt{2p+1}, 1/\sqrt{2p+1})\backslash\{0\}.$$

Proof. If $\alpha = 1$, then $c(\lambda) = 1/\sqrt{2\lambda+1}$ and it is not difficult to see that the assertion of the lemma holds in this case.

Suppose that $0 \le \alpha < 1$. Put $\rho(\lambda) = 1/\sqrt{2\lambda+1}$, $0 < \lambda \le p$. Applying Lemma 4 to $\{\Delta(c(-t), \rho(-t))\}_{-p\le t\le 0}$, it suffices to show $|c'(\lambda)|/(1-|c(\lambda)|^2) < -\rho'(\lambda)/(1-\rho(\lambda)^2)$. By (29) and $0 < c(\lambda) < \rho(\lambda) = 1/\sqrt{2\lambda+1}$ we obtain

$$\frac{c'(\lambda)}{1-c(\lambda)^2} = -\frac{c(\lambda)\rho(\lambda)^2}{\rho(\lambda)^2 - c(\lambda)^2} \log \frac{1-c(\lambda)^2}{1-\rho(\lambda)^2} < 0.$$

Thus by making use of the inequality $1 + x \le e^x$ and $\rho'(\lambda) = -\rho(\lambda)^3$ we have

$$\begin{aligned}\frac{|c'(\lambda)|}{1-|c(\lambda)|^2} &= \frac{c(\lambda)\rho(\lambda)^2}{\rho(\lambda)^2 - c(\lambda)^2} \log\left(1 + \frac{\rho(\lambda)^2 - c(\lambda)^2}{1-\rho(\lambda)^2}\right) \\ &\le \frac{c(\lambda)\rho(\lambda)^2}{1-\rho(\lambda)^2} \\ &< \frac{\rho(\lambda)^3}{1-\rho(\lambda)^2} = -\frac{\rho'(\lambda)}{1-\rho(\lambda)^2}. \end{aligned}$$ □

Proof (Proof of Proposition 1). Suppose that $0 \le \alpha < 1$. Then by Lemma 7, for any $z \in \mathbf{D}\backslash\Delta(m(\alpha), 1/\sqrt{2p+1})$ there exists a unique $\lambda = \lambda(z) \in (0, p]$ such that

$$z \in \partial\Delta(c(\lambda), 1/\sqrt{2\lambda+1}). \tag{30}$$

We define $a(z) = c(\lambda(z))$. Then by (28) and (30), $(a(z), \lambda(z))$ satisfies (11) and (12). Uniqueness and continuity of $(a(z), \lambda(z))$ on $\mathbf{D}\backslash\Delta(m(\alpha), 1/\sqrt{2p+1})$ follow from the monotone property of the function $[0, 1/\sqrt{2p+1}] \ni x \mapsto h(\lambda, x)$ for fixed $\lambda \in [0, p]$ and the strictly decreasing property of $\{\Delta(c(\lambda), 1/\sqrt{2\lambda+1})\}_{0\le\lambda\le p}$.

Next suppose that $\alpha = 1$. Note that $c(\lambda) = 1/\sqrt{2\lambda+1}$. Then by Lemma 7, for any $z \in \mathbf{D}(1/2, 1/2)\backslash\Delta(1/\sqrt{2p+1}, 1/\sqrt{2p+1})$ there exists a unique $\lambda = \lambda(z) \in (0, p]$ such that

$$z \in \partial\Delta(1/\sqrt{2\lambda+1}, 1/\sqrt{2\lambda+1}).$$

Then it is easy to see that $a(z) = 1/\sqrt{2\lambda+1}$ and $\lambda(z)$ satisfy (11) and (13). For $z \in \partial\mathbf{D}(1/2, 1/2)\backslash\{0,1\}$ we define $\lambda(z) = 0$ and $a(z) = 1$. Then it is not difficult to see that $\lambda(z)$ is unique, and that it is continuous on $\overline{\mathbf{D}}(1/2, 1/2)\backslash [\varDelta(1/\sqrt{2p+1}, 1/\sqrt{2p+1}) \cup \{0,1\}]$. □

PROOF (Proof of Theorem 2). Let $\alpha \in [0,1)$. Suppose that $0 < r < \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}$. Then, since

$$\mathbf{D}\left(0, \frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)}\right) \subset \varDelta(m(\alpha), 1/\sqrt{2p+1}),$$

we have $re^{i\theta} \in \varDelta(m(\alpha), 1/\sqrt{2p+1})$ for all $\theta \in (-\pi, \pi]$. Thus by Theorem 1 and $V_\alpha^p(re^{i\theta}) = V_\alpha^p(r)$, the mapping $(-\pi, \pi] \ni \theta \mapsto B_\alpha'(re^{i\theta})$ gives a closed curve contained in $\partial V_\alpha^p(r)$. We show that it is a simple curve. Assume that $B_\alpha'(re^{i\theta_1}) = B_\alpha'(re^{i\theta_2})$ for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 < \theta_2$. Put $f(z) = e^{-i(\theta_2-\theta_1)}B_\alpha(e^{i(\theta_2-\theta_1)}z)$. Then $f'(re^{i\theta_1}) = B_\alpha'(re^{i\theta_2}) = B_\alpha'(re^{i\theta_1})$. Applying the uniqueness part of Theorem 1 at $re^{i\theta_1}$, we have $f(z) = B_\alpha(z)$ and hence $e^{-i(\theta_2-\theta_1)}B_\alpha(e^{i(\theta_2-\theta_1)}z) = B_\alpha(z)$ on $\mathbf{D}$, which is a contradiction.

Now we have shown that the simple closed curve given by $(-\pi, \pi] \ni \theta \mapsto B_\alpha'(re^{i\theta})$ is contained in the simple closed curve $\partial V_\alpha^p(r)$. Since a simple closed curve cannot contain any simple closed curve other than itself, $\partial V_\alpha^p(r)$ coincides with the curve given as the mapping $(-\pi, \pi] \ni \theta \mapsto B_\alpha'(re^{i\theta})$.

Suppose that $\frac{1-\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}-m(\alpha)} \le r < \frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)}$. Then $re^{i\theta} \in \varDelta(m(\alpha), 1/\sqrt{2p+1})$ if and only if $|\theta| < \theta_\alpha(r)$. As in the above argument, it can be shown that the arc $\varGamma_1$ given by the mapping $[-\theta_\alpha(r), \theta_\alpha(r)] \ni \theta \mapsto B_\alpha'(re^{i\theta})$ is simple and contained in $\partial V_\alpha^p(r)$. We note that from (7) $B_\alpha'(re^{\pm i\theta_\alpha(r)}) \in \partial\mathbf{D}(0, 1/(1-r^2)^p)$.

Combining Proposition 1, (9) and (10) we have for $-\pi < \theta \le -\theta_\alpha(r)$ or $\theta_\alpha(r) \le \theta \le \pi$ that $F_{a(re^{i\theta}), \lambda(re^{i\theta}), \theta} \in \mathfrak{B}_1^p(\alpha)$ and $|F'_{a(re^{i\theta}), \lambda(re^{i\theta}), \theta}(re^{i\theta})| = 1/(1-r^2)^p$. Since $\lambda(re^{\pm i\theta_\alpha(r)}) = p$ and $a(re^{\pm i\theta_\alpha(r)}) = m(\alpha)$, we have

$$F'_{a(re^{\pm i\theta_\alpha(r)}), \lambda(re^{\pm i\theta_\alpha(r)}), \pm\theta_\alpha(r)}(re^{\pm i\theta_\alpha(r)}) = B_\alpha'(re^{\pm i\theta_\alpha(r)}).$$

Furthermore since by (8) we have

$$F'_{a(-r), \lambda(-r), \pi}(-r)$$
$$= \frac{1}{h(\lambda(-r), \sqrt{2\lambda(-r)+1})} \frac{(1-a(-r)^2)^{\lambda(-r)}(a(-r)+r)}{(1+a(-r)r)^{2\lambda(-r)+1}(1-r^2)^{1-\lambda(-r)}} > 0,$$

it follows that $F'_{a(-r), \lambda(-r), \pi}(-r) = 1/(1-r^2)^p$. Thus the circular arc $\varGamma_2(\subset \partial\mathbf{D}(0, 1/(1-r^2)^p))$ with endpoints $B_\alpha'(re^{\pm i\theta_\alpha(r)})$ that passes through $1/(1-r^2)^p$ is contained in $\partial V_\alpha^p(r)$.

Since the union $\Gamma_1 \cup \Gamma_2$ is a simple closed curve contained in $\partial V_\alpha^p(r)$, it coincides with $\partial V_\alpha^p(r)$.

Suppose that $\frac{1+\sqrt{2p+1}m(\alpha)}{\sqrt{2p+1}+m(\alpha)} \le r < 1$. Then as in the above argument we have $|F'_{a(re^{i\theta}),\lambda(re^{i\theta}),\theta}(re^{i\theta})| = 1/(1-r^2)^p$ for all $\theta \in (-\pi,\pi]$ and that $F'_{a(-r),\lambda(-r),\pi}(-r) = 1/(1-r^2)^p$. Since $r \in \partial\Delta(a(r), 1/\sqrt{2\lambda(r)+1})$, we have $a(r) < r$. This implies

$$F'_{a(r),\lambda(r),0}(r) = \frac{1}{h(\lambda(r),\sqrt{2\lambda(r)+1})} \frac{(1-a(r)^2)^{\lambda(r)}(a(r)-r)}{(1-a(r)r)^{2\lambda(r)+1}(1-r^2)^{1-\lambda(r)}} < 0$$

and $F'_{a(r),\lambda(r),0}(r) = -1/(1-r^2)^p$. It is easy to see that $a(re^{-i\theta}) = a(re^{i\theta})$ and $\lambda(re^{-i\theta}) = \lambda(re^{i\theta})$. Thus we have

$$F'_{a(re^{-i\theta}),\lambda(re^{-i\theta}),-\theta}(re^{-i\theta}) = \overline{F'_{a(re^{i\theta}),\lambda(re^{i\theta}),\theta}(re^{i\theta})}. \tag{31}$$

By a continuity argument we infer from

$$F'_{a(r),\lambda(r),0}(r) = -1/(1-r^2)^p, \qquad F'_{a(-r),\lambda(-r),\pi}(-r) = 1/(1-r^2)^p$$

and the above symmetric property, that the image of the mapping $(-\pi,\pi] \ni \theta \mapsto F'_{a(re^{i\theta}),\lambda(re^{i\theta}),\theta}(re^{i\theta})$ contains the circle $\partial\mathbf{D}(0,1/(1-r^2)^p)$ and hence $\partial V_\alpha^p(r) = \partial\mathbf{D}(0,1/(1-r^2)^p)$.

Finally let $\alpha = 1$. Then since $m(1) = 1/\sqrt{2p+1}$, the case (i) in Theorem 2 does not occur. Suppose that $0 < r < \frac{1+\sqrt{2p+1}m(1)}{\sqrt{2p+1}+m(1)} = \frac{\sqrt{2p+1}}{p+1}$. Then as in the case that $0 \le \alpha < 1$, the arc Γ_1 given by the mapping $[-\theta_1(r),\theta_1(r)] \ni \theta \mapsto B_1'(re^{i\theta})$ is simple and contained in $\partial V_1^p(r)$, and $|B_1'(re^{\pm i\theta_1(r)})| = 1/(1-r^2)^p$. For $\theta \in (-\pi,\pi]$, $re^{i\theta} \in \mathbf{D}(1/2,1/2)\backslash\Delta(1/\sqrt{2p+1}, 1/\sqrt{2p+1})$ holds if and only if $\theta_1(r) = \arccos r/(2\sqrt{2p+1}) \le |\theta| \le \arccos r$. Since $\lambda(\theta) \to 0$ as $|\theta| \uparrow \arccos r$, we have

$$F'_{1/\sqrt{2\lambda(re^{i\theta})+1},\lambda(re^{i\theta}),\theta}(re^{i\theta}) \to \frac{1}{(1-r^2)^p}, \qquad \text{as } |\theta| \uparrow \arccos r. \tag{32}$$

Thus the circular arc Γ_2 which has endpoints at $B_1'(re^{\pm i\theta_1(r)})$ and passes through $1/(1-r^2)^p$ is contained in $\partial V_1^p(r)$. Hence the simple closed curve $\Gamma_1 \cup \Gamma_2$ is contained in the simple closed curve $\partial V_1^p(r)$ and we have $\partial V_\alpha^p(r) = \Gamma_1 \cup \Gamma_2$.

Suppose that $\frac{\sqrt{2p+1}}{p+1} \le r < 1$. Then we claim that the image of the mapping $[-\arccos r, \arccos r] \ni \theta \mapsto F'_{1/\sqrt{2\lambda(re^{i\theta})+1},\lambda(re^{i\theta}),\theta}(re^{i\theta})$ contains $\partial\mathbf{D}(0,1/(1-r^2)^p)$, which implies $\partial V_1^p(r) = \partial\mathbf{D}(0,1/(1-r^2)^p)$. This is a consequence of (32), (31) and $F'_{1/\sqrt{2\lambda(r)+1},\lambda(r),0}(r) < 0$.

We note that the uniqueness part of Theorem 2 directly follows from the uniqueness part of Theorem 1. □

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