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Automorphism groups of vector groups over a field of positive characteristic

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ABSTRACT. We define elementary automorphisms of the n-dimentional vector group over an algebraically closed field of positive characteristic and show that they generate the automorphism group of the vector group. We also give a necessary and sufficient computational condition for a d-tuple of p-polynomials to be a component of an n-tuple of p-polynomials defining an automorphism of the vector group.

1. Introduction

Let k be a field and p its characteristic. Let $\bar{k}$ denote an algebraic closure of k. We consider the direct product $\bar{k}^n$ of the additive group $\bar{k}$ to be an algebraic group, which is denoted by $\mathbf{G}_a^n$. We call $\mathbf{G}_a^n$ an n-dimensional vector group. Let $\mathrm{End}\,\mathbf{G}_a^n$ and $\mathrm{Aut}\,\mathbf{G}_a^n$ denote the k-endomorphism ring and the k-automorphism group of the algebraic group $\mathbf{G}_a^n$ respectively. For every ring R, let $\mathrm{M}_n(R)$ be the ring of $n \times n$ matrices with components in R.

For $1 \le i \le n$, let $\lambda_i = (\lambda_{i1}, \dots, \lambda_{in})$ be a homomorphism from $\mathbf{G}_a$ to $\mathbf{G}_a^n$, where $\lambda_{ii} = 1 \in \mathrm{End}\,\mathbf{G}_a$ and $\lambda_{ij} = 0 \in \mathrm{End}\,\mathbf{G}_a$ for $j \ne i$. Let $\pi_i : \mathbf{G}_a^n \to \mathbf{G}_a$ be the projections. Then $\sum_{i=1}^n \lambda_i \circ \pi_i$ is the identity on $\mathbf{G}_a^n$. Let ϕ be a homomorphism from $\mathbf{G}_a^n$ to $\mathbf{G}_a^d$. Then we may write $\phi = (\pi_1 \circ \phi, \dots, \pi_d \circ \phi)$, and we have

$$\pi_i \circ \phi = \pi_i \circ \phi \circ \left(\sum_{j=1}^n \lambda_j \circ \pi_j \right) = \sum_{j=1}^n (\pi_i \circ \phi \circ \lambda_j) \circ \pi_j \tag{1}$$

for every $1 \le i \le d$. Let $\phi_{ij} = \pi_i \circ \phi \circ \lambda_j$ for every $1 \le i \le d$ and $1 \le j \le n$. If $n = d$, then a mapping $\phi \mapsto (\phi_{ij})$ is a ring isomorphism from $\mathrm{End}\,\mathbf{G}_a^n$ onto $\mathrm{M}_n(\mathrm{End}\,\mathbf{G}_a)$.

Suppose that $p = 0$. Then every k-endomorphism of $\mathbf{G}_a$ is given by a linear polynomial [2, Proposition 12.2], so that $\mathrm{End}\,\mathbf{G}_a$ is isomorphic to k. Denote by $\mathrm{GL}_n(k)$ the unit group $\mathrm{M}_n(k)^*$ of the ring $\mathrm{M}_n(k)$. Then we have

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End $\mathbf{G}_a^n = \mathrm{M}_n(k)$ and Aut $\mathbf{G}_a^n = \mathrm{GL}_n(k)$. Suppose $p > 0$. We call a polynomial of the form

$$\sum_{i=1}^{n} \sum_{r \geq 0} c_{ir} T_i^{p^r} \tag{2}$$

with $c_{ir} \in k$ a *p-polynomial in n variables*. Every k-endomorphism of $\mathbf{G}_a$ is given by a p-polynomial in one variable [2, Proposition 12.2] (in the case where k is algebraically closed, [1, VII, §20.3, Lemma A]). Hence every homomorphism $\psi : \mathbf{G}_a^n \to \mathbf{G}_a^d$ is given by a d-tuple $(f_1, \dots, f_d)$ of p-polynomials in n variables, namely $\psi(x) = (f_1(x), \dots, f_d(x))$ for any $x \in \mathbf{G}_a^n$. In particular, Aut $\mathbf{G}_a$ is isomorphic to $\mathrm{GL}_1(k)$. However Aut $\mathbf{G}_a^n$ with $n \geq 2$ is larger than $\mathrm{GL}_n(k)$.

From now on, we assume that k is an algebraically closed field of characteristic $p > 0$. In this paper, first, we give a subgroup of Aut $\mathbf{G}_a^n$ such that the subgroup and $\mathrm{GL}_n(k)$ generate Aut $\mathbf{G}_a^n$ (Theorem 1). Second, we say that a d-tuple $\psi_1 = (f_1, \dots, f_d)$ of p-polynomials is a component of an automorphism of $\mathbf{G}_a^n$ if there exists an $(n-d)$-tuple $\psi_2 = (f_{d+1}, \dots, f_n)$ of p-polynomials such that $(\psi_1, \psi_2) = (f_1, \dots, f_d, f_{d+1}, \dots, f_n)$ is an automorphism of $\mathbf{G}_a^n$. We give a necessary and sufficient condition for a d-tuple of p-polynomials to be a component of an automorphism of $\mathbf{G}_a^n$ (Theorem 2).

2. Generators of the automorphism group

Let $\sigma : k \to k$ be a ring homomorphism. Let $B^{(i)} = (\sigma^i(b_{st}))$ for $B = (b_{st}) \in \mathrm{M}_n(k)$, where σ^i means the iteration of σ with itself i times and σ^0 is the identity. The set of formal power series $\sum_{i \geq 0} A_i \sigma^i$ is a ring under the additon and multiplication defined as follows:

$$\sum_{i=0}^{\infty} A_i \sigma^i + \sum_{i=0}^{\infty} B_i \sigma^i = \sum_{i=0}^{\infty} (A_i + B_i) \sigma^i, \tag{3}$$

$$\sum_{i=0}^{\infty} A_i \sigma^i \sum_{j=0}^{\infty} B_j \sigma^j = \sum_{m=0}^{\infty} \left(\sum_{i+j=m} A_i B_j^{(i)} \right) \sigma^m. \tag{4}$$

Let $\mathrm{M}_n(k)[[\sigma]]$ denote this ring. It is immediate that $A = \sum_{i=0}^{\infty} A_i \sigma^i$ belongs to the unit group $\mathrm{M}_n(k)[[\sigma]]^*$ if and only if $A_0 \in \mathrm{GL}_n(k)$. Let $\mathrm{M}_n(k)[\sigma]$ be a subset of $\mathrm{M}_n(k)[[\sigma]]$ that consists of formal power series $\sum_{i \geq 0} A_i \sigma^i$ such that $A_i = 0$ for all but finite i. Then $\mathrm{M}_n(k)[\sigma]$ is a subring of $\mathrm{M}_n(k)[[\sigma]]$. We consider the case where σ is the Frobenius map F, namely $F(t) = t^p$ for $t \in k$. There exists a ring isomorphism Φ from End $\mathbf{G}_a^n$ onto $\mathrm{M}_n(k)[F]$ sending an n-tuple $(f_1, \dots, f_n)$ of p-polynomials $f_i(T) = \sum a_{ijr} T_j^{p^r}$ in n variables to $\sum_r A_r F^r$, where A_r is a matrix in $\mathrm{M}_n(k)$ whose ij-component is a_{ijr} for every $r \geq 0$. The inverse Φ^{-1} of Φ is given as follows:

$$\Phi^{-1}(A)(x_1, \dots, x_n) = (x_1, \dots, x_n)\,{}^t A \qquad \text{for } A \in \mathrm{M}_n(k), \tag{5}$$

$$\Phi^{-1}(F)(x_1, \dots, x_n) = (F(x_1), \dots, F(x_n)), \tag{6}$$

where ${}^t A$ is the transpose of A. Hence we can identify the ring End $\mathbf{G}_a^n$ with the subring $\mathrm{M}_n(k)[F]$ of $\mathrm{M}_n(k)[[F]]$. Thus

$$\mathrm{Aut}\,\mathbf{G}_a^n = \{A \in \mathrm{M}_n(k)[[F]]^* \mid A, A^{-1} \in \mathrm{M}_n(k)[F]\}. \tag{7}$$

Let S_n be the symmetric group of degree n. For $\tau \in S_n$, let $\rho(\tau) = (\delta_{i\tau(j)}) \in \mathrm{GL}_n(k)$, where δ is the Kronecker delta. Then $\rho : S_n \to \mathrm{GL}_n(k)$ is an injective homomorphism. Let $\hat{\tau} = \Phi^{-1}(\rho(\tau))$. Then, for $(x_1, \dots, x_n) \in \mathbf{G}_a^n$,

$$\hat{\tau}(x_1, \dots, x_n) = (x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(n)}). \tag{8}$$

Hence $\hat{\tau}$ is regarded as an n-tuple $(T_{\tau^{-1}(1)}, \dots, T_{\tau^{-1}(n)})$ of p-polynomials in n variables.

Lemma 1. *Let*

$$A = A_0 - \sum_{i=1}^{m} A_i F^i \tag{9}$$

be an endomorphism of $\mathbf{G}_a^n$, where $A_0 \in \mathrm{GL}_n(k)$ is a diagonal matrix and $A_i \in \mathrm{M}_n(k)$ for $1 \leq i \leq m$. If A_i is nilpotent and upper (resp. lower) triangular for every $i \geq 1$, then $A \in \mathrm{Aut}\,\mathbf{G}_a^n$ and

$$A^{-1} = A_0^{-1} + \sum_{j=1}^{mn} B_j F^j \tag{10}$$

for some upper (resp. lower) triangular nilpotent matrices B_j.

Proof. Let

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} \in S_n. \tag{11}$$

Suppose that A_i is nilpotent and lower triangular for every $i \geq 1$. Then

$$\hat{\tau}^{-1} A \hat{\tau} = \rho(\tau)^{-1} A_0 \rho(\tau) - \sum_{i=1}^{m} \rho(\tau)^{-1} A_i \rho(\tau) F^i. \tag{12}$$

Besides, $\rho(\tau)^{-1}A_0\rho(\tau)$ is diagonal, and all $\rho(\tau)^{-1}A_i\rho(\tau)$ are upper triangular and nilpotent. Hence we may assume that A_i is upper triangular and nilpotent for every $i \geq 1$. Since $A_0 \in \mathrm{GL}_n(k)$, we have $A \in \mathrm{M}_n(k)[[F]]^*$. Let

$$B = A^{-1} = \sum_{j=0}^{\infty} B_j F^j. \tag{13}$$

It suffices to show that there exists an integer $j_0 > 0$ such that $B_j = O$ for any $j > j_0$. Let $B_j = 0$ for $j < 0$. Then the condition $AB = E_n$ implies

$$B_0 = A_0^{-1}, \tag{14}$$

$$A_0 B_j = A_1 B_{j-1}^{(1)} + \cdots + A_m B_{j-m}^{(m)} \qquad \text{for } j \geq 1. \tag{15}$$

We can show that B_j is nilpotent and upper triangular for every $i \geq 1$ by induction, and that $B_{lm+1}, \ldots, B_{(l+1)m}$ are the sums of at most m products of at least $l+1$ upper triangular nilpotent matrices by induction on l. Thus $B_j = O$ if $j > mn$. □

Let

$$P_u^n = \left\{ \sum_{i=0}^{m} A_i F^i \,\middle|\, \begin{array}{l} 0 \leq m < \infty, A_0 \in \mathrm{GL}_n(k) \text{ is diagonal and} \\ A_1, \ldots, A_m \text{ are nilpotent and upper triangular} \end{array} \right\}, \tag{16}$$

$$P_l^n = \left\{ \sum_{i=0}^{m} A_i F^i \,\middle|\, \begin{array}{l} 0 \leq m < \infty, A_0 \in \mathrm{GL}_n(k) \text{ is diagonal and} \\ A_1, \ldots, A_m \text{ are nilpotent and lower triangular} \end{array} \right\}, \tag{17}$$

where F^0 is the identity. From Lemma 1 and the argument in its proof, we obtain the following result:

COROLLARY 1. *P_u^n and P_l^n are subgroups of* $\mathrm{Aut}\,\mathbf{G}_a^n$. *Furthermore, if $\tau \in S_n$ is given by* (11), *then* $P_u^n = \hat{\tau}^{-1} P_l^n \hat{\tau}$.

DEFINITION 1. When $n \geq 2$, we call $A \in \mathrm{GL}_n(k) \cup P_l^n$ an *elementary automorphism of the vector group* $\mathbf{G}_a^n$.

We recall the discussion in [1, §20.4]. Let f be a non-zero p-polynomial of the form $\sum_{i=1}^n \sum_{r \geq 0} c_{ir} T_i^{p^r}$, which is regarded as a homomorphism from $\mathbf{G}_a^n$ to $\mathbf{G}_a$. We define the pricipal part $\mathscr{P}(f)$, $\mathrm{Nv}\, f$ and $\mathrm{Deg}\, f$ of the polynomial f as follows. Let $f_i(T_i) = \sum_{r \geq 0} c_{ir} T_i^{p^r}$ for $1 \leq i \leq n$. For each $1 \leq i \leq n$ such that $f_i \neq 0$, let $r(i)$ be the integer that satisfies $p^{r(i)} = \deg f_i$ and $c(i)$ the leading coefficient of f_i. For each $1 \leq i \leq n$ such that $f_i = 0$, let $r(i) = 0$ and $c(i) = 0$. Then let

$$\mathscr{P}(f) = \sum_{i=1}^{n} c(i) T_i^{p^{r(i)}}, \tag{18}$$

$$\mathrm{Nv}(f) = \#\{i \mid f_i \neq 0\}, \tag{19}$$

$$\text{and} \qquad \mathrm{Deg}(f) = \sum_{i=1}^{n} r(i). \tag{20}$$

It is clear that f is a linear polynomial if $\mathrm{Deg}(f) = 0$. Assume that $\mathrm{Nv}(f) \geq 2$ and $\mathrm{Deg}(f) > 0$. First, we consider the case where

$$r(1) \geq \cdots \geq r(n). \tag{21}$$

Then, by assumption, there exist $1 \leq m < m' \leq n$ such that $c(m) \neq 0$ and $c(m') \neq 0$. Since k is algebraically closed, we can choose an element $a \in k^n$ such that $\mathscr{P}(f)(a) = 0$ with $a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, and define the p-polynomials g_j as follows:

$$g_i(T) = T_i \qquad \text{for } i < m, \tag{22}$$

$$g_m(T) = a_m T_m, \tag{23}$$

$$\text{and} \qquad g_j(T) = T_j + a_j T_m^{p^{r(m)-r(j)}} \qquad \text{for } j > m. \tag{24}$$

Then $\phi(x) = (g_1(x), \dots, g_n(x))$ is an elementary automorphism of $\mathbf{G}_a^n$. Define $r'(i)$ and $c'(i)$ for $f \circ \phi$ in the same manner as $r(i)$ and $c(i)$ for f, so that

$$\mathscr{P}(f \circ \phi) = \sum_{j=1}^{n} c'(j) T_j^{p^{r'(j)}}. \tag{25}$$

Then the degree of the polynomial $f \circ \phi(T) = \sum_j f_j(g_j(T))$ in T_m is at most $p^{r(m)}$ and the coefficient of $T_m^{p^{r(m)}}$ is $\mathscr{P}(f)(a) = 0$, that is,

$$\mathscr{P}(f \circ \phi) = c'(m) T_m^{p^{r'(m)}} + \sum_{j \neq m} c(j) T_j^{p^{r(j)}} \tag{26}$$

with $r'(m) < r(m)$. Therefore

$$\mathrm{Nv}(f \circ \phi) \leq \mathrm{Nv}(f), \tag{27}$$

$$\text{and} \qquad \mathrm{Deg}(f \circ \phi) < \mathrm{Deg}(f). \tag{28}$$

In the case where the inequality (21) does not hold, let $\tau \in S_n$ be the permutation such that $r(\tau(1)) \geq \cdots \geq r(\tau(n))$. Then, there exists an elementary automorphism ϕ' of $\mathbf{G}_a^n$ such that

$$\mathrm{Nv}(f \circ \hat{\tau} \circ \phi') \le \mathrm{Nv}(f \circ \hat{\tau}) = \mathrm{Nv}\, f, \tag{29}$$

$$\text{and} \qquad \mathrm{Deg}(f \circ \hat{\tau} \circ \phi') < \mathrm{Deg}(f \circ \hat{\tau}) = \mathrm{Deg}\, f. \tag{30}$$

LEMMA 2. *Let $f(T_1, \dots, T_n)$ be a non-zero p-polynomial in n variables with $n \ge 2$. For every $1 \le v \le n$, there exist a finite number of elementary automorphisms $\phi_1, \dots, \phi_l$ such that $f \circ \phi_1 \circ \cdots \circ \phi_l$ is a p-polynomial in T_v. If the polynomial f is irreducible additionally, then there exists an elementary automorphism ϕ_{l+1} such that $f \circ \phi_1 \circ \cdots \circ \phi_l \circ \phi_{l+1}(T) = T_v$.*

PROOF. Unless $\mathrm{Nv}(f) = 1$ or $\mathrm{Deg}(f) = 0$, we can find a permutation $\tau \in S_n$ and an elementary automorphism ϕ' of $\mathbf{G}_a^n$ described above such that $\mathrm{Nv}(f \circ \phi) \le \mathrm{Nv}(f)$ and $\mathrm{Deg}(f \circ \phi) < \mathrm{Deg}(f)$, where $\phi = \hat{\tau} \circ \phi'$ is a composite of two elementary automorphisms. Thus there exist a finite number of automorphisms $\phi_1, \dots, \phi_{l-1}$ of $\mathbf{G}_a^n$ such that $f' = f \circ \phi_1 \circ \cdots \circ \phi_{l-1}$ satisfies either $\mathrm{Nv}(f') = 1$ or $\mathrm{Deg}(f') = 0$. If $\mathrm{Nv}(f') = 1$ and f' is a polynomial in T_j, let ϕ_l be the transposition $(j, v) \in S_n$ or the identity mapping according as $j \ne v$ or $j = v$. If $\mathrm{Deg}(f') = 0$ and hence $f' = \sum_{i=1}^n b_i T_i$, then let $\phi_l(x) = (\sum_j a_{1j} x_j, \dots, \sum_j a_{nj} x_j)$, where $(a_{ij}) \in \mathrm{GL}_n(k)$ is a matrix satisfying $\sum_i b_i a_{ij} = \delta_{jv}$. Then $f \circ \phi_1 \circ \cdots \circ \phi_l$ is a p-polynomial in T_v.

If f is irreducible, then $f \circ \phi_1 \circ \cdots \circ \phi_l$ is an irreducible p-polynomial in T_v. Since an irreducible p-polynomial in one variable is linear, we have

$$f \circ \phi_1 \circ \cdots \circ \phi_l(T) = aT_v \tag{31}$$

for some $a \in k^*$. Hence let

$$\phi_{l+1}(T) = (T_1, \dots, T_{v-1}, a^{-1} T_v, T_{v+1}, \dots, T_n). \tag{32}$$

The following two Lemmas will be used later:

LEMMA 3. *If $\phi(x) = (f_1(x), \dots, f_n(x)) \in \mathrm{Aut}\, \mathbf{G}_a^n$, then f_i is irreducible for every i.*

PROOF. The map sending $f \in k[T_1, \dots, T_n]$ to $f \circ \phi \in k[T_1, \dots, T_n]$ is an isomorphism of k-algebras whose inverse is $f \mapsto f \circ \phi^{-1}$. Now

$$\phi \circ \phi^{-1}(T) = (f_1 \circ \phi^{-1}(T), \dots, f_n \circ \phi^{-1}(T)) = (T_1, \dots, T_n). \tag{33}$$

Hence f_i is irreducible for every i.

LEMMA 4. *Let X_1 and X_2 be objects of a category $\mathscr{C}$ and $f : X_2 \to X_2$ a morphism. Suppose that there exists the product $X_1 \times X_2$ such that the projection $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ is an epimorphism. Then, f is an isomorphism if* *and only if* $(\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f \circ \mathrm{pr}_2)$ *with components* $\mathrm{id}_{X_1} \circ \mathrm{pr}_1$ *and* $f \circ \mathrm{pr}_2$ *is an isomorphism, in which case* $(\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f \circ \mathrm{pr}_2)^{-1} = (\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f^{-1} \circ \mathrm{pr}_2)$.

Proof. Hom-sets $\hom(X_i, X_i)$ and $\hom(X_1 \times X_2, X_1 \times X_2)$ are monoids with respect to their compositions. We will show that $\gamma : \hom(X_2, X_2) \to \hom(X_1 \times X_2, X_1 \times X_2)$ that sends f to $(\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f \circ \mathrm{pr}_2)$ is an injective monoid homomorphism. Since

$$(\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f \circ \mathrm{pr}_2) \circ (\mathrm{id}_{X_1} \circ \mathrm{pr}_1, g \circ \mathrm{pr}_2) = (\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f \circ g \circ \mathrm{pr}_2), \quad (34)$$

γ is a monoid homomorphism. Since pr_2 is an epimorphism, $f \mapsto f \circ \mathrm{pr}_2$ is an injective mapping from $\hom(X_2, X_2)$ to $\hom(X_1 \times X_2, X_2)$. Moreover $f' \mapsto (\mathrm{id}_{X_1} \circ \mathrm{pr}_1, f')$ is an injective mapping from $\hom(X_1 \times X_2, X_2)$ to $\hom(X_1 \times X_2, X_1 \times X_2)$. Hence γ is injective.

For $T = (T_1, \dots, T_n)$, $x = (x_1, \dots, x_n) \in k^n$ and $1 \le i \le n$, we write $T^{(i)}$ and $x^{(i)}$ for $(T_i, \dots, T_n)$ and $(x_i, \dots, x_n)$ respectively.

Definition 2. Let $n \ge 2$ and $1 \le d \le n$. For $1 \le i \le n$, a d-tuple $(f_1, \dots, f_d)$ of p-polynomials in $k[T_1, \dots, T_n]$ is said to be *sweepable* in T_i if $f_j \in k[T^{(i)}]$ for $1 \le j \le d$ and f_1 is irreducible.

In particular, a d-tuple $(f_1, \dots, f_d)$ of p-polynomials is sweepable in T_1 if and only if f_1 is irreducible.

Lemma 5. *Let* $n \ge 2$ *and* $1 \le d \le n$. *Suppose that* $f = (f_1, \dots, f_d)$ *is sweepable in* T_i *for an integer* $1 \le i < n$. *Then there exist a finite number of elementary automorphisms* $\phi_1, \dots, \phi_l$ *of* $\mathbf{G}_a^{n-i+1}$ *and an elementary automorphism* η *of* $\mathbf{G}_a^d$ *such that*

$$\eta \circ f \circ \phi(T^{(i)}) = (T_i, h_2(T^{(i+1)}), \dots, h_d(T^{(i+1)})), \quad (35)$$

where ϕ *is the composite* $\phi_1 \circ \cdots \circ \phi_l$ *and* $h_2, \dots, h_d$ *are* p*-polynomials.*

Proof. An irreducible p-polynomial in one variable is linear. Hence, by Lemmas 2 and 3, there exists $\phi \in \mathrm{Aut}\, \mathbf{G}_a^{n-i+1}$ that is a composite of finite number of elementary automorphisms and that satisfies

$$f_1 \circ \phi(T^{(i)}) = T_i. \quad (36)$$

Then, for $j > 1$, we may write

$$f_j \circ \phi(T^{(i)}) = g_j(T_i) + h_j(T^{(i+1)}), \quad (37)$$

where $g_j \in k[T_i]$ and $h_j \in k[T^{(i+1)}]$ are p-polynomials. Define $\eta \in \mathrm{Aut}\, \mathbf{G}_a^d$ by

$$\eta(x_1, \dots, x_d) = (x_1, x_2 - g_2(x_1), \dots, x_d - g_d(x_1)), \quad (38)$$

which is desired.

THEOREM 1. *Let* $n \geq 2$. *Then* $\mathrm{Aut}\,\mathbf{G}_a^n$ *is generated by* $\mathrm{GL}_n(k) \cup P_l^n$.

PROOF. Let $\phi = (f_1, \dots, f_n) \in \mathrm{Aut}\,\mathbf{G}_a^n$, where f_i are p-polynomials in n variables. By Lemma 3, the n-tuple $(f_1, \dots, f_n)$ is sweepable in T_1. Hence, by Lemma 5, there exist elementary automorphisms $\psi_1, \dots, \psi_l$, η of $\mathbf{G}_a^n$ such that the composite $\xi = \eta \circ \phi \circ \psi$ satisfies

$$\xi(x) = (x_1, f_2'(x'), \dots, f_n'(x')), \tag{39}$$

where $\psi = \psi_1 \circ \cdots \circ \psi_l$ and $x' = (x_2, \dots, x_n)$. Let $\xi' = (f_2', \dots, f_n')$. Then, it follows from Lemma 4 that ξ' belongs to $\mathrm{Aut}\,\mathbf{G}_a^{n-1}$.

If $n = 2$, then $f_2'(T_2) = aT_2$ with $a \in k^*$, since $f_2' \in \mathrm{Aut}\,\mathbf{G}_a$. Hence $\phi = \eta^{-1} \circ A \circ \psi^{-1}$, where $A(x_1, x_2) = (x_1, ax_2)$. Now that Theorem 1 holds when $n = 2$, we can proceed by induction on n. Assume that $\mathrm{Aut}\,\mathbf{G}_a^{n-1}$ is generated by $\mathrm{GL}_{n-1}(k) \cup P_l^{n-1}$. Then $\xi' = \xi_1' \circ \cdots \circ \xi_m'$, where ξ_i' are elementary automorphisms of $\mathbf{G}_a^{n-1}$. Let $\xi_i(x) = (x_1, \xi_i'(x'))$. Then ξ_i are elementary automorphisms of $\mathbf{G}_a^n$ and $\xi = \xi_1 \circ \cdots \circ \xi_m$. Since $\xi = \eta \circ \phi \circ \psi$, ϕ is a composite of elementary automorphisms of $\mathbf{G}_a^n$. □

3. Components of an automorphism

We say that a d-tuple $\psi_1 = (f_1, \dots, f_d)$ of p-polynomials is a *component of an automorphism of* $\mathbf{G}_a^n$ if there exists an $(n-d)$-tuple $\psi_2 = (f_{d+1}, \dots, f_n)$ such that $(\psi_1, \psi_2) = (f_1, \dots, f_d, f_{d+1}, \dots, f_n)$ is an automorphism of $\mathbf{G}_a^n$.

It is false that every homomorphism from $\mathbf{G}_a^n$ to $\mathbf{G}_a^d$ is a component of an automorphism of $\mathbf{G}_a^n$. We give a necessary and sufficient condition for a d-tuple of p-polynomials to be a component of an automorphism of $\mathbf{G}_a^n$. Clearly an n-tuple $f = (f_1, \dots, f_n)$ of p-polynomials in n variables is a component of an automorphism of $\mathbf{G}_a^n$ if and only if $f \in \mathrm{Aut}\,\mathbf{G}_a^n$. The following theorem gives a computational criterion for the n-tuple f to belong to $\mathrm{Aut}\,\mathbf{G}_a^n$ in particular.

THEOREM 2. *Let* $n \geq 2$ *be an integer,* d *an integer with* $1 \leq d \leq n$, *and* $f_1, \dots, f_d \in k[T_1, \dots, T_n]$ *p-polynomials. A* d*-tuple* $f = (f_1, \dots, f_d)$ *is a component of an automorphism of* $\mathbf{G}_a^n$ *if and only if there exists a sequence* $(f^{(1)}, \dots, f^{(d)})$ *of* $(d-i+1)$*-tuples* $f^{(i)}$ *of p-polynomials in* $k[T^{(i)}]$ *that satisfies the following:*

(1) $f^{(1)} = f$

(2) $f^{(i)}$ *are sweepable in* T_i *for* $1 \leq i \leq d$

(3) *There exist* $\phi_i \in \mathrm{Aut}\,\mathbf{G}_a^{n-i+1}$ *and* $\eta_i' \in \mathrm{Aut}\,\mathbf{G}_a^{d-i+1}$ *such that*

$$\eta_i' \circ f^{(i)} \circ \phi_i(T^{(i)}) = (T_i, f^{(i+1)}(T^{(i+1)})) \tag{40}$$

for $1 \leq i \leq d$.

In particular, a single (f_1) *is a component of an automorphism of* $\mathbf{G}_a^n$ *if and only if* f_1 *is irreducible.*

Proof. We already know that if an n-tuple $(f_1, \dots, f_n)$ of p-polynomials defines an automorphism of $\mathbf{G}_a^n$, then each f_i is irreducible by Lemma 3.

First, assume that f is a component of an automorphism of $\mathbf{G}_a^n$. Let $h_i^{(1)} = f_i$ for $1 \le i \le d$. Then there exist p-polynomials $h_{d+1}^{(1)}, \dots, h_n^{(1)}$ such that $h^{(1)} = (h_1^{(1)}, \dots, h_n^{(1)}) \in \operatorname{Aut} \mathbf{G}_a^n$. By Lemma 3, $h^{(1)}$ is sweepable in T_1. Hence, by Lemma 5, there exists an automorphism $\phi_1, \eta_1 \in \operatorname{Aut} \mathbf{G}_a^n$ such that

$$\eta_1 \circ h^{(1)} \circ \phi_1(T) = (T_1, h_2^{(2)}(T^{(2)}), \dots, h_n^{(2)}(T^{(2)})). \tag{41}$$

By Lemma 4, $h^{(2)}(T^{(2)}) = (h_2^{(2)}(T^{(2)}), \dots, h_n^{(2)}(T^{(2)}))$ belongs to $\operatorname{Aut} \mathbf{G}_a^{n-1}$. Repeating the same argument, we see that there exist $\phi_i, \eta_i \in \operatorname{Aut} \mathbf{G}_a^{n-i+1}$ such that

$$\eta_i \circ h^{(i)} \circ \phi_i(T^{(i)}) = (T_i, h_{i+1}^{(i+1)}(T^{(i+1)}), \dots, h_n^{(i+1)}(T^{(i+1)})) \tag{42}$$

for $1 \le i \le d$. Write $\eta_i = (\eta_{ii}, \dots, \eta_{in})$ and $h^{(i)} = (h_i^{(i)}, \dots, h_n^{(i)})$, and let $\eta_i' = (\eta_{ii}, \dots, \eta_{id})$ and $(h')^{(i)} = (h_i^{(i)}, \dots, h_d^{(i)})$. Then $(h')^{(i)}(T^{(i)})$ is sweepable in T_i. Moreover we have $(h')^{(1)} = f^{(1)}$ and

$$\eta_i' \circ (h')^{(i)} \circ \phi_i(T^{(i)}) = (T_i, (h')^{(i+1)}(T^{(i+1)})), \tag{43}$$

since η_{ij} depend only on x_i and x_j. Therefore we may take $(h')^{(i)}$ as $f^{(i)}$.

Conversely, assume there exists $\phi_i \in \operatorname{Aut} \mathbf{G}_a^{n-i+1}$ such that

$$\begin{aligned} &f^{(i)} \circ \phi_i(T^{(i)}) \\ &\quad = (T_i, g_{i+1}^{(i+1)}(T_i) + f_{i+1}^{(i+1)}(T^{(i+1)}), \dots, g_d^{(i+1)}(T_i) + f_d^{(i+1)}(T^{(i+1)})). \end{aligned} \tag{44}$$

Define $\eta_i' \in \operatorname{Aut} \mathbf{G}_a^{d-i+1}$ by

$$\eta_i'(x_i, \dots, x_d) = (x_i, x_{i+1} - g_{i+1}^{(i+1)}(x_i), \dots, x_d - g_d^{(i+1)}(x_i)). \tag{45}$$

Then we have $\eta_i' \circ f^{(i)} \circ \phi_i(T^{(i)}) = (T_i, f^{(i+1)}(T^{(i+1)}))$. Define $\eta_i \in \operatorname{Aut} \mathbf{G}_a^d$ and $\psi_i \in \operatorname{Aut} \mathbf{G}_a^n$ as follows:

$$\eta_i(x_1, \dots, x_d) = (x_1, \dots, x_{i-1}, \eta_i'(x_i, \dots, x_d)), \tag{46}$$

$$\psi_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, \phi_i(x^{(i)})). \tag{47}$$

We can show by induction on i that

$$\eta_i \circ \cdots \circ \eta_1 \circ f^{(1)} \circ \psi_1 \circ \cdots \circ \psi_i(T) = (T_1, \dots, T_i, f^{(i+1)}(T^{(i+1)})). \tag{48}$$

In particular,

$$\eta_d \circ \cdots \circ \eta_1 \circ f^{(1)} \circ \psi_1 \circ \cdots \circ \psi_d(T) = (T_1, \dots, T_d). \tag{49}$$

Let $\eta = \eta_d \circ \cdots \circ \eta_1$, and $\psi = \psi_1 \circ \cdots \circ \psi_d$. Then $(\eta \circ f^{(1)} \circ \psi(T), T_{d+1}, \dots, T_n)$ defines the identity mapping on $\mathbf{G}_a^n$. Let $\xi(x) = (\eta^{-1}(x_1, \dots, x_d), x_{d+1}, \dots, x_n)$. Then $\xi \in \mathrm{Aut}\, \mathbf{G}_a^n$ and $\xi(T) = (f^{(1)} \circ \psi(T), T_{d+1}, \dots, T_n)$. Hence

$$\xi \circ \psi^{-1}(T) = (f^{(1)}(T), h_{d+1}(T), \dots, h_n(T)), \tag{50}$$

where $\psi^{-1}(T) = (h_1(T), \dots, h_n(T))$. Therefore $f^{(1)}$ is a component of an automorphism of $\mathbf{G}_a^n$.

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