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Isolated singularities, growth of spherical means and Riesz decomposition for superbiharmonic functions

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Abstract. We consider Riesz decomposition theorem for superbiharmonic functions in the punctured ball. In fact, we show that under certain growth condition on surface integrals, superbiharmonic functions are represented as a sum of potentials and biharmonic functions.

1. Introduction

A function u on an open set $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is called biharmonic if $(-\Delta)^2 u = 0$ on Ω. We say that a locally integrable function u on Ω is superbiharmonic in Ω (in the weak sense) if $(-\Delta)^2 u$ is a nonnegative measure on Ω, that is,

$$\int_{\Omega} u(x)(-\Delta)^2 \varphi(x)dx \geq 0 \qquad \text{for all nonnegative } \varphi \in C_0^{\infty}(\Omega).$$

We denote by $\mathscr{H}^2(\Omega)$ and $\mathscr{S}\mathscr{H}^2(\Omega)$ the space of biharmonic functions on Ω and the space of superbiharmonic functions on Ω, respectively. For fundamental properties of biharmonic functions, we refer to [1] and [8].

The open ball and the sphere centered at x with radius r are denoted by $B(x,r)$ and $S(x,r)$. We write $B(r) = B(0,r)$ and $S(r) = S(0,r)$. We also denote by $\mathbf{B}$ and $\mathbf{B}_0$ the unit ball $B(1)$ and the punctured unit ball $\mathbf{B}\backslash\{0\}$, respectively.

For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and a point $x = (x_1, x_2, \dots, x_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$\lambda! = \lambda_1!\lambda_2!\dots\lambda_n!,$$

$$x^{\lambda} = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$$

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and

$$D^{\lambda} = \left(\frac{\partial}{\partial x}\right)^{\lambda} = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

Following the book by Hayman-Kennedy [4], we consider the Riesz kernel of order $2m$ defined by

$$\mathscr{R}_{2m}(x) = \begin{cases} |x|^{2m-n} & \text{if } n \text{ is odd or } n > 2m, \\ |x|^{2m-n} \log(1/|x|) & \text{if } n \text{ is even and } n \le 2m, \end{cases}$$

and the remainder term in the Taylor expansion of $\mathscr{R}_{2m}$, given by

$$\mathscr{R}_{2m,L}(y,x) = \mathscr{R}_{2m}(y-x) - \sum_{|\lambda| \le L} \frac{y^{\lambda}}{\lambda!} (D^{\lambda}\mathscr{R}_{2m})(-x),$$

where L is a real number. Here note that $(-\Delta)^m \mathscr{R}_{2m} = \alpha_m^{-1}\delta_0$ and

$$(-\Delta)^m \mathscr{R}_{2m,L}(\cdot, x) = \alpha_m^{-1}\delta_x$$

with the Dirac measure δ_x at x and a constant $\alpha_m \neq 0$; in fact,

$$\alpha_2^{-1} = \omega_n \begin{cases} -4 & \text{when } n = 2, \\ -2 & \text{when } n = 3, \\ 4 & \text{when } n = 4, \\ 2(4-n)(2-n) & \text{when } n \ge 5, \end{cases}$$

where ω_n denotes the surface area of $S(1)$.

For a Borel measurable function u on $\mathbf{R}^n$, we define the average integral over $S(x,r)$ by

$$M(u,x,r) = \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} u \, dS.$$

If x is the origin, then we write $M(u,r)$ for $M(u,0,r)$.

Let G be a bounded open set in $\mathbf{R}^n$. If u is superbiharmonic in a neighborhood of $\bar{G}$, then Riesz decomposition theorem implies that

$$u(x) = \alpha_2 \int_G \mathscr{R}_4(x-y) d\mu(y) + h_G(x)$$

for almost every $x \in G$, where $\mu = (-\Delta)^2 u$ and h_G is biharmonic in G. Remark that the function u^* defined by the right-hand side is lower semicontinuous and locally integrable on G; further it satisfies

$$u^*(x) = \lim_{r \to 0} M(u^*, x, r) \tag{1}$$

for every $x \in G$. In what follows, superbiharmonic functions are always assumed to be locally integrable, Borel measurable and satisfy the mean value property (1).

Our main result in the present note is the following.

THEOREM 1. *Let* $u \in \mathscr{S}\mathscr{H}^2(2\mathbf{B}_0)$ *and* $\mu = (-\Delta)^2 u$, *where* $2\mathbf{B}_0 = B(0,2)\backslash\{0\}$.

(1) *If* $n = 2$ *and* $M(u, r^2) - 2M(u, r)$ *is bounded above for* $r \in (0,1)$, *then*

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y, x) d\mu(y) + h(x)$$

holds for $x \in \mathbf{B}_0$, *where* $h \in \mathscr{H}^2(\mathbf{B}_0)$.

(2) *If* $n = 3$ *and* $M(u, r/2) - 2M(u, r)$ *is bounded above for* $r \in (0,1)$, *then*

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,0}(y, x) d\mu(y) + h(x)$$

holds for $x \in \mathbf{B}_0$, *where* $h \in \mathscr{H}^2(\mathbf{B}_0)$.

(3) *If* $n = 4$ *and* $M(u, r/2) - 4M(u, r) \leq O(\log(1/r))$ *for* $r \in (0, 1/2)$, *then*

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_4(y - x) d\mu(y) + h(x)$$

holds for $x \in \mathbf{B}_0$, *where* $h \in \mathscr{H}^2(\mathbf{B}_0)$.

(4) *If* $n \geq 5$ *and* $M(u, r/2) - 2^{n-2}M(u, r) \leq O(r^{4-n})$ *for* $r \in (0,1)$, *then*

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_4(y - x) d\mu(y) + h(x)$$

holds for $x \in \mathbf{B}_0$, *where* $h \in \mathscr{H}^2(\mathbf{B}_0)$.

When u is superbiharmonic in $\mathbf{R}^n$, a global representation theorem was given by Kitaura-Mizuta [5], which is an extension of a result by Premalatha [9].

REMARK 2. If $u \in \mathscr{S}\mathscr{H}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2 u$, then, as in the book of Hayman-Kennedy [4], Futamura-Kishi-Mizuta [2] and Futamura-Mizuta [3], u can be represented as

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4, L(|y|)}(y, x) d\mu(y) + h(x),$$

for $x \in \mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B}_0)$ and $L(r)$ is a nonincreasing positive function on $(0,1]$ such that $L(r) \geq 4 - n$.

2. Spherical means for superbiharmonic functions

We write $\Delta^k \mathcal{R}_{2m}(t) = \Delta^k \mathcal{R}_{2m}(x)$ when $t = |x|$. First, in view of Lemma 1 in [2], we note the following result.

LEMMA 3. *If $u \in \mathcal{H}^2(\mathbf{B}_0)$, then*

$$M(u,r) = a + br^2 + c\mathcal{R}_4(r) + d\mathcal{R}_2(r)$$

for $0 < r < 1$, where a, b, c, d are constants independent of r.

COROLLARY 4. *If $u \in \mathcal{H}^2(\mathbf{B}_0)$, then:*
(1) *in case $n = 2$, $M(u,r^2) - 2M(u,r) = O(1)$ as $r \to 0+$;*
(2) *in case $n = 3$, $M(u,r/2) - 2M(u,r) = O(1)$ as $r \to 0+$;*
(3) *in case $n = 4$, $M(u,r/2) - 4M(u,r) = O(\log(1/r))$ as $r \to 0+$;*
(4) *in case $n \geq 5$, $M(u,r/2) - 2^{n-2}M(u,r) = O(r^{4-n})$ as $r \to 0+$.*

For $t > 0$ and $r > 0$, set

$$G(t,r) = \mathcal{R}_4(t) - \mathcal{R}_4(r) + \frac{1}{2n}(r^2\Delta\mathcal{R}_4(t) - t^2\Delta\mathcal{R}_4(r)),$$

that is,

$$G(t,r) = \begin{cases} t^2\log(1/t) - r^2\log(1/r) + r^2(\log(1/t) - 1) - t^2(\log(1/r) - 1) & \text{if } n = 2, \\ \log(1/t) - \log(1/r) - \dfrac{1}{4}(r^2/t^2 - t^2/r^2) & \text{if } n = 4, \\ t^{4-n} - r^{4-n} + \dfrac{4-n}{n}(r^2t^{2-n} - t^2r^{2-n}) & \text{otherwise.} \end{cases}$$

We know that $G(t,r)$ is strictly monotone as a function of t (see [3, Lemma 4.4]).

LEMMA 5. *Let $u \in \mathcal{SH}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2u$. Then for $0 < r < 1$,*

$$M(u,r) = \alpha_2 \int_{\{y: r<|y|<1\}} G(|y|,r)\,d\mu(y) + a + br^2 + c\mathcal{R}_4(r) + d\mathcal{R}_2(r),$$

where a, b, c, d are constants independent of r.

PROOF. For fixed $0 < r_0 < 1$, we write

$$u(x) = \alpha_2 \int_{A(r_0)} \mathcal{R}_{4,2}(y,x)\,d\mu(y) + h_0(x)$$

for $x \in A(r_0) = \{x : r_0 < |x| < 1\}$, where h_0 is biharmonic in $A(r_0)$. Then, as in the proof of Lemma 3 (see [2] and Ligocka [6]), we see that

$$M(h_0, r) = a_0 + b_0 r^2 + c_0 \mathscr{R}_4(r) + d_0 \mathscr{R}_2(r)$$

for $r_0 < r < 1$. Further, using Lemma 4.3 in [3], we find

$$M(u - h_0, r) = \alpha_2 \int_{A(r)} G(|y|, r) d\mu(y),$$

so that

$$M(u, r) = \alpha_2 \int_{A(r)} G(|y|, r) d\mu(y) + a_0 + b_0 r^2 + c_0 \mathscr{R}_4(r) + d_0 \mathscr{R}_2(r)$$

for $r_0 < r < 1$. This implies that the constants a_0, b_0, c_0, d_0 are determined independently of r_0. □

Noting that $|\mathscr{R}_{4,L}(y, x)| \le C|y|^{L+1}$ as $y \to 0$ for fixed $x \in \mathbf{B}_0$, $L \ge -1$ and some constant $C > 0$, we have the following result (cf. [7, Theorem 1]).

Lemma 6. *Let μ be a nonnegative measure on $\mathbf{B}_0$ such that*

$$\int_{\mathbf{B}_0} |y|^{L+1} d\mu(y) < \infty \tag{2}$$

for $L \ge -1$. Then

$$\int_{\mathbf{B}_0} |\mathscr{R}_{4,L}(y, x)| d\mu(y) \not\equiv \infty \qquad \text{on } \mathbf{B}_0,$$

so that $u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,L}(y, x) d\mu(y)$ is superbiharmonic in $\mathbf{B}_0$.

3. Proof of Theorem 1 in case $n = 2$

By Corollary 4 and Lemma 5, we have

$$\begin{aligned} M(u, r^2) - 2M(u, r) = {} & \alpha_2 \int_{\{y : r^2 < |y| < r\}} G(|y|, r^2) d\mu(y) \\ & + \alpha_2 \int_{\{y : r \le |y| < 1\}} \{G(|y|, r^2) - 2G(|y|, r)\} d\mu(y) + O(1). \end{aligned}$$

Here we see that

$$G(t, r) < 0 \qquad \text{for } 0 < r < t < 1$$

and

$$\begin{aligned} G(t,r^2) - 2G(t,r) &= -t^2 \log(1/t) - t^2 + (r^4 - 2r^2)\log(1/t) - 2r^4 \log(1/r) \\ &\quad + 2r^2 \log(1/r) - r^4 + 2r^2 \\ &= r^2\{(s^2 - r^2 + 2)\log s - (s^2 + r^2)\log(1/r) - s^2 - r^2 + 2\} \end{aligned}$$

for $t = rs$. If $r > 0$ is so small that $-(2^{-1} + r^2)\log(1/r) + 1 - r^2 < 0$, then

$$G(t,r^2) - 2G(t,r) < -\frac{1}{2}t^2 \log(1/t)$$

for $r \le t < 1$. (To show the last inequality, by change of variable $s^2 = x$, consider

$$F(x) = (x/2 - r^2 + 2)(\log x)/2 - (x/2 + r^2)\log(1/r) - x - r^2 + 2;$$

then $F(1) = -(2^{-1} + r^2)\log(1/r) + 1 - r^2 < 0$ by our assumption. We see that $F'(1) < 0$, $F'(1/r^2) < 0$ and $F''(x) = \{x - 2(2 - r^2)\}/4x^2$ for $1 < x < 1/r^2$, so that $F'(x) < 0$ and thus $F(x) < 0$ for $1 < x < 1/r^2$.)

Suppose $M(u,r^2) - 2M(u,r)$ is bounded above. Then we see that

$$\int_{\{y: r \le |y| < 1\}} \{G(|y|,r^2) - 2G(|y|,r)\}d\mu(y) \qquad \text{is bounded,}$$

which implies that

$$\int_{\mathbf{B}_0} |y|^2 \log(1/|y|)d\mu(y) < \infty.$$

In view of Lemma 6, $v(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y,x)d\mu(y)$ is superbiharmonic in $\mathbf{B}_0$, so that $h(x) = u(x) - \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y,x)d\mu(y)$ is biharmonic in $\mathbf{B}_0$, as required. □

REMARK 7. Let $u \in \mathscr{S}\mathscr{H}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2 u$, as before. If $n = 2$ and $M(u,r) = O(\log(1/r))$ for $r \in (0,1/2)$, then the above proof shows that

$$\int_{\mathbf{B}_0} |y|^2 d\mu(y) < \infty,$$

so that u is represented as

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y,x)d\mu(y) + h(x)$$

on $\mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B}_0)$ (cf. [3, Theorem 1.3]). Further, if $M(|u|,r) = O(\log(1/r))$ for $r \in (0,1/2)$, then [3, Theorem 1.4] implies that

$$\int_{\mathbf{B}_0} |y|^2 d\mu(y) < \infty$$

and

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y,x)d\mu(y) + h(x) + \sum_{|\lambda| \le 2} C(\lambda) D^{\lambda} \mathscr{R}_4(x)$$

on $\mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B})$ and $C(\lambda)$ are constants.

4. Proof of Theorem 1 in case $n = 3$

By Lemma 5, we have

$$M(u, r/2) - 2M(u, r) = \alpha_2 \int_{\{y : r/2 < |y| < r\}} G(|y|, r/2) d\mu(y)$$
$$+ \alpha_2 \int_{\{y : r \le |y| < 1\}} \{G(|y|, r/2) - 2G(|y|, r)\} d\mu(y) + O(1).$$

We see that

$$G(t, r) < 0 \qquad \text{for } r < t < 1$$

and

$$G(t, r/2) - 2G(t, r) = -t + 3r/2 - 7r^2/(12t) \le -t/28 < 0.$$

Suppose $M(u, r/2) - 2M(u, r)$ is bounded above. Then we see that

$$\int_{\{y : r \le |y| < 1\}} \{G(|y|, r/2) - 2G(|y|, r)\} d\mu(y) \qquad \text{is bounded,}$$

which implies that

$$\int_{\mathbf{B}_0} |y| d\mu(y) < \infty.$$

In view of Lemma 6, $v(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,0}(y,x)d\mu(y)$ is superbiharmonic in $\mathbf{B}_0$, so that $h(x) = u(x) - \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,0}(y,x)d\mu(y)$ is biharmonic in $\mathbf{B}_0$, as required.

Remark 8. Let $u \in \mathscr{SH}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2 u$, as before. If $n = 3$ and $M(u, r) = O(1/r)$ for $r \in (0, 1)$, then the above proof shows that

$$\int_{\mathbf{B}_0} |y|^2 d\mu(y) < \infty,$$

so that u is represented as

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y,x)d\mu(y) + h(x)$$

on $\mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B}_0)$ (cf. [3, Theorem 1.3]).

5. Proof of Theorem 1 in case $n = 4$

By Lemma 5, we have

$$M(u, r/2) - 4M(u, r) = \alpha_2 \int_{\{y: r/2 < |y| < r\}} G(|y|, r/2) d\mu(y)$$
$$+ \alpha_2 \int_{\{y: r \le |y| < 1\}} \{G(|y|, r/2) - 4G(|y|, r)\} d\mu(y)$$
$$+ O(\log(1/r)).$$

We see that

$$G(t, r) > 0 \qquad \text{for } t > r$$

and

$$G(t, r/2) - 4G(t, r) = 3 \log \frac{t}{r} + \frac{15}{16}\left(\frac{r}{t}\right)^2 - \log 2 \ge \log \frac{t}{r} + \frac{15}{16} - \log 2 > 0$$

for $r \le t < 1$. Suppose $\{M(u, r/2) - 4M(u, r)\}/\log(1/r)$ is bounded above. Then

$$\int_{\{y: r \le |y| < 1\}} \{G(|y|, r/2) - 4G(|y|, r)\} d\mu(y) = O(\log(1/r)).$$

Hence it follows that

$$\int_{\{y: r \le |y| < 1\}} \log(|y|/r) d\mu(y) = O(\log(1/r)),$$

which implies that $\mu(\mathbf{B}_0) < \infty$. Consequently, $v(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_4(x - y) d\mu(y)$ is superbiharmonic in $\mathbf{B}_0$, so that we see that $h(x) = u(x) - \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_4(x - y) d\mu(y)$ is biharmonic in $\mathbf{B}_0$, as required.

REMARK 9. Let $u \in \mathscr{SH}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2 u$, as before. If $n = 4$ and $M(u, r) = O(r^{-2})$ for $r \in (0, 1)$, then the above proof shows that

$$\int_{\mathbf{B}_0} |y|^2 d\mu(y) < \infty,$$

so that u is represented as

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y, x) d\mu(y) + h(x)$$

on $\mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B}_0)$ (cf. [3, Theorem 1.3]).

6. Proof of Theorem 1 in case $n \geq 5$

By Lemma 5, we have

$$M(u, r/2) - 2^{n-2} M(u, r) = \alpha_2 \int_{\{y: r/2 < |y| < r\}} G(|y|, r/2) d\mu(y)$$
$$+ \alpha_2 \int_{\{y: r \leq |y| < 1\}} \{G(|y|, r/2) - 2^{n-2} G(|y|, r)\} d\mu(y)$$
$$+ O(r^{4-n}).$$

We see that

$$G(t, r) > 0 \qquad \text{for } t > r$$

and

$$\begin{aligned} &G(t, r/2) - 2^{n-2} G(t, r) \\ &\quad = -(2^{n-2} - 1) t^{4-n} + (2^{n-2} - 2^{n-4}) r^{4-n} + ((n-4)/4n)(2^n - 1) r^2 t^{2-n} \\ &\quad = r^{4-n} \{-(2^{n-2} - 1)(r/t)^{n-4} + 3 \cdot 2^{n-4} + ((n-4)/4n)(2^n - 1)(r/t)^{n-2}\} \\ &\quad \geq r^{4-n} \{2^{n-2}(3n - 16) + 3n + 4\}/(4n) > 0 \end{aligned}$$

for $r \leq t < 1$.

Suppose $\{M(u, r/2) - 2^{n-2} M(u, r)\}/r^{4-n}$ is bounded above. Then

$$\int_{\{y: r \leq |y| < 1\}} \{G(|y|, r/2) - 2^{n-2} G(|y|, r)\} d\mu(y) = O(r^{4-n}).$$

Hence it follows that $\mu(\mathbf{B}_0) < \infty$, so that $h(x) = u(x) - \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_4(x - y) d\mu(y)$ is biharmonic in $\mathbf{B}_0$, as required.

Remark 10. Let $u \in \mathscr{SH}^2(2\mathbf{B}_0)$ and $\mu = (-\Delta)^2 u$, as before. If $n \geq 5$ and $M(u, r) = O(r^{2-n})$ for $r \in (0, 1)$, then the above proof shows that

$$\int_{\mathbf{B}_0} |y|^2 d\mu(y) < \infty,$$

so that u is represented as

$$u(x) = \alpha_2 \int_{\mathbf{B}_0} \mathscr{R}_{4,1}(y, x) d\mu(y) + h(x)$$

on $\mathbf{B}_0$, where $h \in \mathscr{H}^2(\mathbf{B}_0)$ (cf. [3, Theorem 1.3]).

7. The harmonic case

Let $n = 2$ and suppose $u \in \mathscr{SH}(2\mathbf{B}_0)$. If we set $v = (-\Delta)u$, then

$$M(u,r) = \frac{1}{2\pi}\int_{A(r)} \log(r/|y|)\,d\nu(y) + a + b\log(1/r)$$

for $0 < r < 1$, where a and b are constants. Hence we have the following:

(1) If $M(u,r) = O(\log(1/r))$ for $r \in (0,1/2)$, then we can show that $\nu(\mathbf{B}_0) < \infty$ and

$$u(x) = \frac{1}{2\pi}\int_{\mathbf{B}_0} \log(1/|x-y|)\,d\nu(y) + h(x)$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}_0$ (see also [3, Theorem 1.3]). Further, if $M(|u|,r) = O(\log(1/r))$ for $r \in (0,1/2)$, then, in view of [2, Theorem 1] and [3, Theorem 1.4], we can show that $\nu(\mathbf{B}_0) < \infty$ and

$$u(x) = \frac{1}{2\pi}\int_{\mathbf{B}_0} \log(1/|x-y|)\,d\nu(y) + h(x) + a\log(1/|x|)$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}$ and a is a constant.

(2) If $M(u,r^2) - 2M(u,r) \ge O(\log(1/r))$ for $r \in (0,1/2)$, then we can show that

$$u(x) = \frac{1}{2\pi}\int_{\mathbf{B}_0} (\log(1/|x-y|) - \log(1/|x|))\,d\nu(y) + h(x)$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}_0$ (cf. Premalatha [9]).

Let $n \ge 3$ and suppose $u \in \mathscr{SH}(2\mathbf{B}_0)$. If we set $\nu = (-\Delta)u$, then

$$M(u,r) = \alpha_1\int_{A(r)} (r^{2-n} - |y|^{2-n})\,d\nu(y) + a + br^{2-n}$$

for $0 < r < 1$, where $\alpha_1 = -1/((n-2)\omega_n)$, a and b are constants. Hence we have the following:

(1) If $M(u,r) = O(r^{2-n})$ for $r \in (0,1)$, then $\nu(\mathbf{B}_0) < \infty$ and

$$u(x) = -\alpha_1\int_{\mathbf{B}_0} |x-y|^{2-n}\,d\nu(y) + h(x)$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}_0$ (see also [3, Theorem 1.3]). Further, if $M(|u|,r) = O(r^{2-n})$ for $r \in (0,1)$, then [2, Theorem 1] or [3, Theorem 1.4] implies that $\nu(\mathbf{B}_0) < \infty$ and

$$u(x) = -\alpha_1\int_{\mathbf{B}_0} |x-y|^{2-n}\,d\nu(y) + h(x) + a|x|^{2-n}$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}$ and a is a constant.

(2) If $M(u,r/2) - 2^{n-2}M(u,r) \ge O(r^{2-n})$ for $r \in (0,1)$, then we can show that

$$u(x) = -\alpha_1 \int_{\mathbf{B}_0} (|x-y|^{2-n} - |x|^{2-n}) d\nu(y) + h(x)$$

on $\mathbf{B}_0$, where h is harmonic in $\mathbf{B}_0$.

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