HIROSHIMA MATH. J.
38 (2008), 95–124

Large time behavior of solutions to the compressible Navier-Stokes equation in an infinite layer

Yoshiyuki KAGEI

(Received April 9, 2007)
(Revised September 6, 2007)

ABSTRACT. Large time behavior of solutions to the compressible Navier-Stokes equation around a given constant state is considered in an infinite layer $\mathbf{R}^{n-1} \times (0, a)$, $n \geq 2$, under the no slip boundary condition for the velocity. The L^p decay estimates of the solution are established for all $1 \leq p \leq \infty$. It is also shown that the time-asymptotic leading part of the solution is given by a function satisfying the $n-1$ dimensional heat equation. The proof is given by combining a weighted energy method with time-weight functions and the decay estimates for the associated linearized semigroup.

1. Introduction

This paper is concerned with the initial boundary value problem for the compressible Navier-Stokes equation in an infinite layer Ω:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = \mu \Delta v + (\mu + \mu')\nabla \operatorname{div} v, \tag{1.2}$$

$$v|_{x_n=0,a} = 0, \qquad \rho|_{t=0} = \rho_0(x), \qquad v|_{t=0} = v_0(x). \tag{1.3}$$

Here Ω is an n-dimensional infinite layer that is defined by

$$\Omega = \{x = (x', x_n); x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}, 0 < x_n < a\}, \qquad n \geq 2;$$

$\rho = \rho(x,t)$ and $v = (v^1(x,t), \dots, v^n(x,t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; $P = P(\rho)$ is the pressure; μ and μ' are the viscosity coefficients that satisfy $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; and the notation $\operatorname{div}(\rho v \otimes v)$ means that its j-th component is given by $\operatorname{div}(\rho v^j v)$.

We are interested in the large time behavior of solutions to problem (1.1)–(1.3) when the initial value (ρ_0, v_0) is sufficiently close to a given constant state $(\rho_*, 0)$, where ρ_* is a given positive number.

Matsumura and Nishida [22, 23] proved the global in time existence of solutions to the Cauchy problem for (1.1)–(1.2) on the whole space $\mathbf{R}^n$ around

2000 *Mathematics Subject Classification Numbers*. 35Q30, 76N15.
Key words and phrases. Compressible Navier-Stokes equation, asymptotic behavior, infinite layer.

$(\rho_*, 0)$ and obtained the optimal L^2 decay rate of the perturbation $u(t) = (\rho(t) - \rho_*, v(t))$. Kawashima, Matsumura and Nishida [17] then showed that the leading part of $u(t)$ is given by the solution of the linearized problem. (See [16] for the case of a general class of quasilinear hyperbolic-parabolic systems.) The solution of the linearized problem reveals a hyperbolic-parabolic aspect of system (1.1)–(1.2), a typical property of system (1.1)–(1.2). It is written asymptotically in the sum of two terms, one is given by the convolution of the heat kernel and the fundamental solution of the wave equation, which is the so-called *diffusion wave*, and the other is the solution of the heat equation. Hoff and Zumbrun [7, 8] showed that there appears some interesting interaction of hyperbolic and parabolic aspects of the system in the decay properties of L^p norms with $1 \le p \le \infty$. The diffusion wave decays faster than the heat kernel in L^p norm for $p > 2$ while slower for $p < 2$. (See also [20].) This decay property of the diffusion wave also appears in the exterior domain problem [18, 19] and the half space [14, 15].

On the other hand, in contrast to the domains mentioned above, we know that the Poincaré inequality holds for functions on the infinite layer Ω. Therefore, if one considers, for example, the incompressible Navier-Stokes equation on Ω under the no-slip boundary condition for the velocity, it is easily seen that the L^2 norm of the velocity decays exponentially. (See [1, 2, 3] for the L^p decay estimates.) As for problem (1.1)–(1.3), the Poincaré inequality still holds for the velocity $v(t)$ but not for the density part $\phi(t) = \rho(t) - \rho_*$. This leads to that the spectrum of the linearized operator reaches the origin but it is like the one such as the $n-1$ dimensional Laplace operator. As a result, the solution of the linearized problem behaves in large time such as a solution of an $n-1$ dimensional heat equation [11]. In this paper we will prove that the leading part of the solution of the nonlinear problem (1.1)–(1.3) is given by the solution of the linearized problem. More precisely, we will show that under suitable assumptions on the initial value, $u(t)$ satisfies

$$\|u(t) - u^{(0)}(t)\|_{L^p} = O(t^{-((n-1)/2)(1-1/p)-1/2}L(t)) \tag{1.4}$$

for all $1 \le p \le \infty$ as $t \to \infty$. Here $L(t) = 1$ when $n \ge 3$ and $L(t) = \log(1+t)$ when $n = 2$; and $u^{(0)} = (\phi^{(0)}(x', t), 0)$ with $\phi^{(0)}(x', t)$ satisfying

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \qquad \phi^{(0)}|_{t=0} = \frac{1}{a}\int_0^a (\rho_0(x', x_n) - \rho_*)dx_n,$$

where $\kappa = \frac{a^2\gamma^2}{12\nu^2}$, $\nu = \frac{\mu}{\rho_*}$, $\gamma^2 = P'(\rho_*)$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$. We will also establish decay estimates of $\|\partial_x u(t)\|_p$ for all $1 \le p \le \infty$.

The estimate (1.4) means that the leading part of $u(t)$ is given by a solution of the $n-1$ dimensional heat equation and no hyperbolic feature appears in the leading part. We also note that, even in the case of $n = 2$, any effect from the nonlinearity does not appear in the leading part.

As for related works, we mention that the structure of the spectrum of the linearized operator near the origin is quite similar to that of the linearized operator appearing in the free surface problem of viscous incompressible fluid studied in [4]. So, the leading part of $u(t)$ has a similar form to that of the free surface problem. We also mention the work of Benabidallah [5] where the global existence of the solution was proved in the isothermal case under the action of a large potential force such that the density tends to 0 as $|x| \to \infty$.

The proof of (1.4) is similar to that of an analogous result on the half space problem investigated in [15]. It is based on the H^s a priori estimate with time-weight function by the energy method [13, 15, 21, 24] and the decay estimates for the linearized semigroup [10, 11]. There are, however, several aspects different from the half space problem, especially in low-dimensional cases. One thing is that the decay rate of the linearized semigroup is not so fast in the case $n = 2, 3$. Therefore, for these cases, a more detailed treatment of the nonlinearity is needed.

The paper is organized as follows. In Section 2 we state our main results concerning the large time behavior. The proof of the main results is given in Section 3. We first show the asymptotic behavior (1.4) for $p = 2$. We then investigate the asymptotic behavior in L^∞ space by combining the linearized analysis and the decay estimate of the H^s norm. We finally study the asymptotic behavior in L^1 space. In the Appendix we give a proof of the estimates for the solutions of the Stokes problem which are used in the proof of the energy estimates.

2. Main result

We first introduce some notation. For $1 \le p \le \infty$ we denote by L^p the usual Lebesgue space on Ω and its norm is denoted by $\| \cdot \|_p$. The L^2 inner product will be denoted by $(\cdot, \cdot)_2$. Let ℓ be a nonnegative integer. The symbol $W^{\ell,p}$ denotes the ℓ-th order L^p Sobolev space on Ω with norm $\| \cdot \|_{W^{\ell,p}}$. When $p = 2$, the space $W^{\ell,2}$ is denoted by H^ℓ and its norm is denoted by $\| \cdot \|_{H^\ell}$. C_0^ℓ stands for the set of all C^ℓ functions which have compact support in Ω. We denote by H_0^1 the completion of C_0^1 in H^1. The dual space of H_0^1 is denoted by H^{-1}.

We often write $x \in \Omega$ as $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We next rewrite problem (1.1)–(1.3). We set $\phi = \rho - \rho_*$. Then problem (1.1)–(1.3) is reduced to finding $u = (\phi, v)$ that satisfies

$$\partial_t \phi + v \cdot \nabla \phi + \rho \operatorname{div} v = 0, \tag{2.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \operatorname{div} v + P'(\rho)\nabla\phi = 0, \tag{2.2}$$

$$v|_{x_n = 0, a} = 0; \qquad u|_{t=0} = u_0, \tag{2.3}$$

where $\rho = \phi + \rho_*$ and

$$u_0 = (\phi_0, v_0), \qquad \phi_0 = \rho_0 - \rho_*.$$

Here (1.1) is used to obtain (2.2).

In the following we set

$$s_0 \equiv \left[\frac{n}{2}\right] + 1.$$

Here and in what follows $[q]$ denotes the greatest integer less than or equal to q.

For a solution of (2.1)–(2.3) we define some quantities. Let $u = (\phi, v)$ be a solution of (2.1)–(2.3). We define $E_r^\sigma(t)$ and $D_r^\sigma(t)$ by

$$E_r^\sigma(t) = \left(\sup_{0 \le \tau \le t} (1+\tau)^{2r}\{|[\phi(\tau)]|_\sigma^2 + |[v(\tau)]|_\sigma^2\}\right)^{1/2}$$

and

$$D_r^\sigma(t) = \begin{cases} \left(\displaystyle\int_0^t (1+\tau)^{2r} |||Dv|||_0^2 d\tau\right)^{1/2} & \text{for } \sigma = 0, \\ \left(\displaystyle\int_0^t (1+\tau)^{2r}\{|||D\phi|||_{\sigma-1}^2 + |||Dv|||_\sigma^2\} d\tau\right)^{1/2} & \text{for } \sigma \ge 1. \end{cases}$$

Here and in what follows we denote

$$|[\psi(t)]|_\sigma = \left(\sum_{j=0}^{[\sigma/2]} \|\partial_t^j \psi(t)\|_{H^{\sigma-2j}}^2\right)^{1/2},$$

$$|||D\psi(t)|||_\sigma = \begin{cases} \|\partial_x \psi(t)\|_2 & \text{for } \sigma = 0, \\ (|[\partial_x \psi(t)]|_\sigma^2 + |[\partial_t \psi(t)]|_{\sigma-1}^2)^{1/2} & \text{for } \sigma \ge 1. \end{cases}$$

We will look for the solution $u \in \bigcap_{j=0}^{[s/2]} C([0,\infty); H^{s-2j})$ satisfying $E_0^s(t)^2 + D_0^s(t)^2 < \infty$ for all $t \ge 0$ with $s \ge s_0$.

Before stating our main results we mention the compatibility condition. Since we consider strong solutions, we need to require the compatibility condition for the initial value $u_0 = (\phi_0, v_0)$, which is formulated as follows.

Let $u = (\phi, v)$ be a smooth solution of (2.1)–(2.3). Then $\partial_t^j u = (\partial_t^j \phi, \partial_t^j v)$ $(j \ge 1)$ is inductively determined by

$$\partial_t^j \phi = -v \cdot \nabla \partial_t^{j-1} \phi - \rho \operatorname{div} \partial_t^{j-1} v - \{[\partial_t^{j-1}, v \cdot \nabla]\phi + [\partial_t^{j-1}, \rho \operatorname{div}]v\}$$

and

$$\partial_t^j v = -\rho^{-1}\{A\partial_t^{j-1} v + P'(\rho)\nabla \partial_t^{j-1}\phi\} - \rho^{-1}\{[\partial_t^{j-1}, \rho]\partial_t v + [\partial_t^{j-1}, P'(\rho)\nabla]\phi\}$$
$$- \rho^{-1}\partial_t^{j-1}(\rho v \cdot \nabla v).$$

Here $Av = -\mu\Delta v - (\mu + \mu')\nabla \operatorname{div} v$; and $[C, D] = CD - DC$ is the commutator of C and D.

From these relations we see that $(\partial_t^j \phi, \partial_t^j v)|_{t=0}$ is inductively given by (ϕ_0, v_0) in the following way:

$$(\partial_t^j \phi, \partial_t^j v)|_{t=0} = (\phi_j, v_j),$$

where

$$\phi_j = -v_0 \cdot \nabla \phi_{j-1} - \rho_0 \operatorname{div} v_{j-1} - \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \{v_\ell \cdot \nabla \phi_{j-1-\ell} + \phi_\ell \operatorname{div} v_{j-1-\ell}\},$$

$$v_j = -\rho_0^{-1}\{Av_{j-1} + P'(\rho_0)\nabla \phi_{j-1}\}$$
$$- \rho_0^{-1} \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \{\phi_\ell v_{j-\ell} + a_\ell(\phi_0; \phi_1, \dots, \phi_\ell)\phi_{j-1-\ell}\}$$
$$+ \rho_0^{-1} G_{j-1}(\phi_0, v_0, \partial_x v_0; \phi_1, \dots, \phi_{j-1}, v_1, \dots, v_{j-1}, \partial_x v_1, \dots, \partial_x v_{j-1}).$$

Here $\rho_0 = \phi_0 + \rho_*$; $a_\ell(\phi_0; \phi_1, \dots, \phi_\ell)$ is a certain polynomial in $\phi_1, \dots, \phi_\ell$; $\dots\dots$, and so on.

By the boundary condition $v|_{x_n=0,a} = 0$ in (2.3), we necessarily have $\partial_t^j v|_{x_n=0,a} = 0$, and hence,

$$v_j|_{x_n=0,a} = 0.$$

Assume that (ϕ, v) is a solution of (2.1)–(2.3) in $\bigcap_{j=0}^{[s/2]} C([0, T]; H^{s-2j})$ for some $T > 0$. Then, from the above observation, we need the regularity $(\phi_j, v_j) \in H^{s-2j}$ for $j = 0, \dots, [s/2]$, which, indeed, follows from the fact that $(\phi_0, v_0) \in H^s$ with $s \ge s_0$. Furthermore, it is necessary to require that (ϕ_0, v_0) satisfies the *$\hat{s}$-th order compatibility condition*:

$$v_j \in H_0^1 \qquad \text{for } j = 0, 1, \dots, \hat{s} = \left[\frac{s-1}{2}\right].$$

We are ready to state our global existence result.

THEOREM 2.1. *Let s be an integer satisfying $s \geq s_0$ and assume that $P'(\rho_*) > 0$. Then there exists a positive number ε_0 such that if the initial perturbation $u_0 \in H^s$ satisfies $\|u_0\|_{H^s} \leq \varepsilon_0$ and the $\hat{s}$-th order compatibility condition, then there exists a unique global solution $u(t) \in \bigcap_{j=0}^{[s/2]} C([0,\infty); H^{s-2j})$ of problem* (2.1)–(2.3), *which satisfies*

$$E_0^s(t)^2 + D_0^s(t)^2 \leq C\|u_0\|_{H^s}^2$$

for all $t \geq 0$. Furthermore, it holds that $\lim_{t\to\infty}\|u(t)\|_\infty = 0$.

The proof of Theorem 2.1 is similar to that of analogous results in [13, 24]. It is proved by a combination of the local existence and the a priori energy estimate. The local existence can be proved by applying the local solvability result in [12]. The a priori energy estimate can be obtained by the same energy method as in [13, 24]. The decay of the L^∞ norm can also be proved in a similar manner as in [13]. We omit the details. (See Lemma 3.5 below for the energy estimate.)

As for the asymptotic behavior of the solution, we have the following result.

THEOREM 2.2. *Let s be an integer satisfying $s \geq s_0 + 2$ when $n \geq 4$, $s \geq s_0 + 3$ when $n = 3$ and $s \geq s_0 + 4$ when $n = 2$. Assume that $P'(\rho_*) > 0$. In addition to the assumption on u_0 in Theorem* 2.1, *assume also that u_0 belongs to $H^s \cap (W^{2,1} \times W^{1,1})$. Then if u_0 is sufficiently small, the solution $u(t)$ of problem* (2.1)–(2.3) *satisfies*

$$\|u(t)\|_p = O(t^{-((n-1)/2)(1-1/p)}),$$

$$\|\partial_x u(t)\|_p = O(t^{-((n-1)/2)(1-1/p)-1/2} L(t)^{(1/2)(1-2/p)_+})$$

and

$$\|u(t) - u^{(0)}(t)\|_p = O(t^{-((n-1)/2)(1-1/p)-1/2} L(t))$$

for any $1 \leq p \leq \infty$ as $t \to \infty$. Here $\left(1 - \frac{2}{p}\right)_+ = \max\left\{1 - \frac{2}{p}, 0\right\}$; $u^{(0)} = (\phi^{(0)}(x',t), 0)$ and $\phi^{(0)}(x',t)$ is a function satisfying

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \qquad \phi^{(0)}|_{t=0} = \frac{1}{a}\int_0^a (\rho_0(x', x_n) - \rho_*)\,dx_n,$$

where $\kappa = \frac{a^2\gamma^2}{12\nu^2}$, $\nu = \mu/\rho_*$, $\gamma^2 = P'(\rho_*)$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$; and $L(t) = 1$ when $n \geq 3$; and $L(t) = \log(1+t)$ when $n = 2$.

REMARK 2.3. (i) As is well known, $\|u^{(0)}(t)\|_p$ decays exactly in the order $t^{-((n-1)/2)(1-1/p)}$. We thus see that the decay estimate for $u(t)$ in Theorem 2.2 is optimal.

(ii) The regularity assumption on u_0 can be relaxed depending on p. See Theorems 3.4, 3.7, 3.13–3.15 and 3.17 below.

Theorem 2.2 will be proved in the next section.

3. Proof of Theorem 2.2

In this section we prove the asymptotic behavior described in Theorem 2.2. The proof is given by combining the weighted energy estimate (Lemma 3.5) and the estimates for the linearized semigroup (Lemmas 3.1 and 3.2) which were obtained in [10, 11].

We first transform the unknown v into $m = \rho v$. Then (2.1)–(2.3) is written as

$$\partial_t \phi + \operatorname{div} m = 0,$$

$$\partial_t m + \operatorname{div}\left(\frac{m \otimes m}{\rho}\right) + \nabla P(\rho) = \mu \Delta\left(\frac{m}{\rho}\right) + (\mu + \mu')\nabla \operatorname{div}\left(\frac{m}{\rho}\right),$$

$$m|_{x_n = 0, a} = 0; \qquad \phi|_{t=0} = \phi_0(x), \qquad m|_{t=0} = m_0(x),$$

where $m_0 = \rho_0 v_0$ with $\rho_0 = \phi_0 + \rho_*$. We rewrite this problem as

$$\partial_t w + Lw = \operatorname{div} \tilde{\mathcal{N}}, \tag{3.1}$$

$$m|_{\partial\Omega} = 0; \qquad w|_{t=0} = w_0, \tag{3.2}$$

where $w = \begin{pmatrix} \phi \\ m \end{pmatrix}$, $w_0 = \begin{pmatrix} \phi_0 \\ m_0 \end{pmatrix}$ and

$$L = \begin{pmatrix} 0 & \operatorname{div} \\ \gamma^2 \nabla & -\nu\Delta - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix}, \qquad \tilde{\mathcal{N}} = \begin{pmatrix} 0 \\ (\mathcal{N}_{jk})_{1 \le j,k \le n} \end{pmatrix}$$

with $\gamma^2 = P'(\rho_*)$, $\nu = \mu/\rho_*$, $\tilde{\nu} = (\mu + \mu')/\rho_*$ and

$$\begin{aligned} \mathcal{N}_{jk} = {} & -\nu \partial_{x_k}\left(\frac{\phi m_j}{\phi + \rho_*}\right) - \delta_{jk}\tilde{\nu} \operatorname{div}\left(\frac{\phi m}{\phi + \rho_*}\right) \\ & - \delta_{jk}\phi^2 \int_0^1 (1-\theta) P''(\phi\theta + \rho^*) d\theta - \frac{m_j m_k}{\phi + \rho_*}. \end{aligned}$$

Here the j-th component of $\operatorname{div} \mathcal{N}$ is given by $\sum_{k=1}^n \partial_{x_k} \mathcal{N}_{jk}$.

In view of the H^s energy bound in Theorem 2.1, it suffices to prove Theorem 2.2 with $u(t)$ replaced by $w(t)$.

In [10] we showed that the operator $-L$ with domain $D(L) = W^{1,r}(\Omega) \times [W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)]$ generates an analytic semigroup $\mathscr{U}(t)$ on

$W^{1,r}(\Omega) \times L^r(\Omega)$ $(1 < r < \infty)$ and established the estimates of $\mathscr{U}(t)$ for $0 < t \le 2$ stated in Lemma 3.1 below.

In the following we will denote by $\tilde{Q}$ the $(n+1) \times (n+1)$-diagonal matrix $\mathrm{diag}(0, 1, \dots, 1)$. Note that

$$\tilde{Q}w = \begin{pmatrix} 0 \\ m \end{pmatrix} \qquad \text{for } w = \begin{pmatrix} \phi \\ m \end{pmatrix}.$$

Lemma 3.1. *Let $\ell = 0, 1$. Then there hold the estimates*

$$\|\partial_x^\ell \mathscr{U}(t) w_0\|_r \le C t^{-\ell/2} \|w_0\|_{W^{\ell,r} \times L^r}, \qquad 1 < r < \infty,$$

$$\|\partial_x^\ell \mathscr{U}(t) w_0\|_\infty \le C t^{-(1-\varepsilon)} \|w_0\|_{H^{[n/2]+1+\ell} \times H^{[n/2]+\ell}}$$

and

$$\|\partial_x^\ell \mathscr{U}(t) w_0\|_p \le C t^{-\ell/2} \|w_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \qquad p = 1, \infty,$$

for $0 < t \le 2$ with some constant $0 < \varepsilon < 1$, provided that w_0 belongs to the Sobolev spaces indicated on the right-hand side of each inequality above. Furthermore, if $\tilde{Q}w_0|_{x_n = 0, a} = 0$, then

$$\|\partial_x \mathscr{U}(t) w_0\|_1 \le C \|w_0\|_{W^{2,1} \times W^{1,1}}$$

holds for $0 \le t \le 2$.

As for the large time behavior of $\mathscr{U}(t)$, we showed the following result in [11].

Lemma 3.2. *Let $1 < r < \infty$ and let $\mathscr{U}(t)$ be the semigroup generated by $-L$. Suppose that $w_0 = (\phi_0, m_0) \in L^1(\Omega) \cap [W^{1,r}(\Omega) \times L^r(\Omega)]$. Then the solution $w(t) = \mathscr{U}(t) w_0$ of problem (3.1)–(3.2) is decomposed as*

$$\mathscr{U}(t) w_0 = \mathscr{U}^{(0)}(t) w_0 + \mathscr{U}^{(\infty)}(t) w_0,$$

where each term on the right-hand side has the following properties.

(i) *$\mathscr{U}^{(0)}(t) w_0$ is written in the form*

$$\mathscr{U}^{(0)}(t) w_0 = \mathscr{W}^{(0)}(t) w_0 + \mathscr{R}^{(0)}(t) w_0.$$

Here $\mathscr{W}^{(0)}(t) w_0 = \begin{pmatrix} \phi^{(0)}(x', t) \\ 0 \end{pmatrix}$; and $\phi^{(0)}(x', t)$ is a function independent of x_n and satisfies the following heat equation on $\mathbf{R}^{n-1}$:

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \qquad \phi^{(0)}|_{t=0} = \frac{1}{a} \int_0^a \phi_0(x', x_n) dx_n,$$

where $\kappa = \frac{a^2 \gamma^2}{12 \nu}$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$. $\mathscr{W}^{(0)}(t)$ satisfies $\mathscr{W}^{(0)}(t) \tilde{Q} w_0 = 0$, and, furthermore, for any $1 \le p \le \infty$ and $j, \ell = 0, 1$, there exists a positive constant C such that

$$\|\partial_t^j \partial_x^\ell \mathscr{W}^{(0)}(t) w_0\|_p \le C t^{-((n-1)/2)(1-1/p)-j} \|w_0\|_1.$$

The function $\mathscr{R}^{(0)}(t)w_0$ *satisfies the following estimate. For any* $1 \le p \le \infty$ *and* $j, \ell = 0, 1$*, there exists a positive constant* C *such that*

$$\|\partial_t^j \partial_x^\ell \mathscr{R}^{(0)}(t) w_0\|_p \le C t^{-((n-1)/2)(1-1/p)-1/2-j} \|w_0\|_1$$

holds for $t \ge 1$*. Furthermore, it holds that*

$$\|\partial_x \mathscr{R}^{(0)}(t) \tilde{Q} w_0\|_p \le C t^{-((n-1)/2)(1-1/p)-1} \|\tilde{Q} w_0\|_1$$

and

$$\|\mathscr{R}^{(0)}(t)[\partial_x \tilde{Q} w_0]\|_p \le C t^{-((n-1)/2)(1-1/p)-1/2} \|\tilde{Q} w_0\|_1.$$

(ii) *There exists a positive constant* c *such that* $\mathscr{U}^{(\infty)}(t)w_0$ *satisfies*

$$\|\partial_t^j \partial_x^\ell \mathscr{U}^{(\infty)}(t) w_0\|_r \le C e^{-ct} \|w_0\|_{W^{\ell,r} \times L^r}, \qquad j, \ell = 0, 1,$$

for all $t \ge 1$*. Furthermore, the following estimates*

$$\|\partial_x^\ell \mathscr{U}^{(\infty)}(t) w_0\|_\infty \le C e^{-ct} \|w_0\|_{H^{[n/2]+1+\ell} \times H^{[n/2]+\ell}}, \qquad \ell = 0, 1,$$

$$\|\partial_x^\ell \mathscr{U}^{(\infty)}(t) w_0\|_p \le C e^{-ct} \|w_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \qquad p = 1, \infty,\ \ell = 0, 1,$$

hold for all $t \ge 1$*, provided that* w_0 *belongs to the Sobolev spaces on the right of the above inequalities.*

Remark. Although the estimates of the time derivative were not given in [10, 11], it is easy to prove these estimates by tracing the proof in [10, 11].

We first prove the L^2 decay estimates (Theorem 3.4) and then the L^p estimates for $p = \infty$ (Theorems 3.7, 3.13 and 3.14) and $p = 1$ (Theorems 3.15 and 3.17). The L^p estimates for general p can then be obtained by interpolation. To prove the L^p estimates for $p = \infty$ and $p = 1$, we will use the L^2 decay estimates and the H^s energy estimate with a time-weight function.

We define $M_2^{(k)}(t)$ and $M(t)$ by

$$M_2^{(k)}(t) = \sup_{0 \le \tau \le t} (1+\tau)^{(n-1)/4+k/2} \|\partial_x^k w(\tau)\|_2,$$

$$M(t) = M_2^{(0)}(t) + M_2^{(1)}(t).$$

To obtain the decay estimates for L^2 norm we use the following

Lemma 3.3. *Let* $s \ge s_0 + 1$ *and assume that* $\|u_0\|_2 \le \varepsilon_0$*. Then the following inequalities hold.*

(i) $\|\text{div}\, \mathscr{N}\|_1 \le C(1+t)^{-(n-1)/4-1/2} \{E_0^s(t)^{1/2} M(t)^{3/2} + E_0^s(t) M(t)\}$ $(n \ge 3)$.

(ii) $\|\text{div}\, \mathscr{N}\|_1 \le C(1+t)^{-(n-1)/4-1/2} \{\|\partial_x v\|_{H^3}^{1/3} M(t)^{5/3} + E_0^s(t) M(t)\}$ $(n = 2)$.

(iii) $\|\text{div}\, \mathscr{N}\|_2 \le C(1+t)^{-(n-1)/4-1/2} E_0^s(t) M(t)$ $(n \ge 4)$.

(iv) $\|\mathrm{div}\,\mathcal{N}\|_2 \le C(1+t)^{-(n-1)/4-1/2}\{E_0^s(t)^{3/4}M(t)^{5/4}+E_0^s(t)M(t)\}$ $(n=3)$.

(v) $\|\mathrm{div}\,\mathcal{N}\|_2 \le C(1+t)^{-(n-1)/4-1/2}\{E_0^s(t)^{1/2}M(t)^{3/2}+E_0^s(t)M(t)\}$ $(n=2)$.

(vi) *Set* $\mathcal{N}^{(\ell)} = (\mathcal{N}_{jk}^{(\ell)})$, $\ell = 1,2$, *with* $\mathcal{N}_{jk}^{(1)} = -\nu\partial_{x_k}\left(\frac{\phi m_j}{\phi+\rho_*}\right) - \delta_{jk}\tilde{\nu}\,\mathrm{div}\left(\frac{\phi m}{\phi+\rho_*}\right)$ *and* $\mathcal{N}_{jk}^{(2)} = \mathcal{N}_{jk} - \mathcal{N}_{jk}^{(1)}$. *Then*

$$\|\mathcal{N}^{(1)}\|_1 \le C(1+t)^{-(n-1)/2-1/2}M(t)^2,$$

$$\|\mathrm{div}\,\mathcal{N}^{(2)}\|_1 \le C(1+t)^{-(n-1)/2-1/2}M(t)^2.$$

PROOF. The inequalities in Lemma 3.3 follow by a direct application of the Hölder, Poincaré and Gagliardo-Nirenberg-Sobolev inequalities to each term of $\mathrm{div}\,\mathcal{N}$ except $\|J\|_p$ $(p=1,2)$ for $n=2,3$ with $J = -\nu\frac{\phi}{\rho}\Delta m - \tilde{\nu}\frac{\phi}{\rho}\nabla\,\mathrm{div}\,m$. We here estimate it for $n=2$. The case $n=3$ can be treated similarly.

We write $m = \rho v = \phi v + \rho_* v$. Then

$$|\partial_x^2 m| \le C\{|\partial_x^2 v| + |\phi\partial_x^2 v| + |\partial_x\phi\partial_x v| + |\partial_x^2\phi v|\},$$

and whence,

$$\|J\|_1 \le C\{\|\phi\partial_x^2 v\|_1 + \|\phi^2\partial_x^2 v\|_1 + \|\phi\partial_x\phi\partial_x v\|_1 + \|\phi\partial_x^2\phi v\|_1\}.$$

Let us estimate each term on the right-hand side. Since $\partial_x v = \partial_x m/\rho - \partial_x\phi m/\rho^2$, we have

$$|\partial_x v| \le C(1+|m|)|\partial_x w|.$$

Therefore, by the interpolation inequality: $\|\partial_x^2 v\|_2 \le C\|\partial_x v\|_2^{2/3}\|\partial_x v\|_{H^3}^{1/3}$, we have

$$\|\phi\partial_x^2 v\|_1 \le C\|\phi\|_2\|\partial_x v\|_2^{2/3}\|\partial_x v\|_{H^3}^{1/3} \le C\|\phi\|_2\|\partial_x w\|_2^{2/3}\|\partial_x v\|_{H^3}^{1/3}$$
$$\le C(1+t)^{-3/4}\|\partial_x v\|_{H^3}^{1/3}M(t)^{5/3}.$$

Similarly, we have $\|\phi^2\partial_x^2 v\|_1 \le C(1+t)^{-3/4}\|\partial_x v\|_{H^3}^{1/3}M(t)^{5/3}$. The remaining terms can be estimated by using the Hölder and Poincaré inequalities, and, consequently, we obtain

$$\|J\|_1 \le C(1+t)^{-3/4}\{\|\partial_x v\|_{H^3}^{1/3}M(t)^{5/3} + E_0^s(t)M(t)\}.$$

We next consider $\|J\|_2$. We decompose ϕ as

$$\phi = \bar{\phi} + \phi_1, \qquad \bar{\phi} = \frac{1}{a}\int_0^a \phi(x', x_n, t)\,dx_n.$$

Observe that $\bar{\phi}$ does not depend on x_n, namely, $\bar{\phi} = \bar{\phi}(x')$ $(x' \in \mathbf{R})$ (Recall that $n=2$.) Therefore, applying the Gagliardo-Nirenberg inequality for $\bar{\phi} = \bar{\phi}(x')$ $(x' \in \mathbf{R})$, we have

$$\|\bar{\phi}\|_\infty \le C\|\bar{\phi}\|_{L^2_{x'}}^{1/2}\|\partial_{x'}\bar{\phi}\|_{L^2_{x'}}^{1/2} \le C\|\phi\|_2^{1/2}\|\partial_x\phi\|_2^{1/2}.$$

As for ϕ_1, since $\int_0^a \phi_1\, dx_n = 0$ for all (x', t), the Poincaré inequality gives

$$\|\phi_1\|_2 \leq C\|\partial_{x_n}\phi_1\|_2 \leq C\|\partial_x\phi\|_2.$$

It then follows that

$$\begin{aligned}\|\bar{\phi}\partial_x^2 m\|_2 &\leq \|\bar{\phi}\|_\infty \|\partial_x^2 m\|_2 \leq C\|\phi\|_2^{1/2}\|\partial_x\phi\|_2^{1/2}\|\partial_x m\|_2^{1/2}\|\partial_x m\|_{H^2}^{1/2} \\ &\leq C(1+t)^{-1}E_0^s(t)^{1/2}M(t)^{3/2}\end{aligned}$$

and

$$\begin{aligned}\|\phi_1\partial_x^2 m\|_2 \leq \|\phi_1\|_4\|\partial_x^2 m\|_4 &\leq C\|\partial_x\phi_1\|_2\|\partial_x m\|_{H^2} \\ &\leq C(1+t)^{-3/4}E_0^s(t)M(t),\end{aligned}$$

from which the desired inequality for $\|J\|_2$ is obtained. This completes the proof.

In the following we will denote

$$\mathscr{E}_0 = \|w_0\|_{H^s} + \|w_0\|_1.$$

Theorem 3.4. *Let $s \geq s_0 + 1$. Assume that $w_0 \in H^s \cap L^1$. Then there exists a positive number ε_1 such that*

(i) $$\|\partial_x^\ell w(t)\|_2 \leq C(1+t)^{-(n-1)/4-\ell/2}\mathscr{E}_0, \qquad \ell = 0, 1,$$

for all $t \geq 0$, provided that $\mathscr{E}_0 < \varepsilon_1$. Furthermore, it holds

(ii) $$\|w(t) - u^{(0)}(t)\|_2 \leq C(1+t)^{-(n-1)/4-1/2}\mathscr{E}_0.$$

Here $u^{(0)}(t)$ is the function defined in Theorem 2.2.

Proof. To prove (i) we derive a uniform estimate for $M(t)$. By Theorem 2.1, we know that $\|w(t)\|_{H^1} \leq C\mathscr{E}_0$ for all $t \geq 0$. So, it suffices to estimate $M(t)$ for $t \geq 2$.

We write $w(t)$ as

$$\begin{aligned}w(t) &= \mathscr{U}(t)w_0 + \int_0^{t-1}\mathscr{U}(t-\tau)\operatorname{div}\tilde{\mathscr{N}}(\tau)d\tau + \int_{t-1}^t \mathscr{U}(t-\tau)\operatorname{div}\tilde{\mathscr{N}}(\tau)d\tau \\ &\equiv I_0(t) + I_1(t) + I_2(t).\end{aligned}$$

We note that

$$\operatorname{div}\tilde{\mathscr{N}}(\tau) = \begin{pmatrix} 0 \\ \operatorname{div}\mathscr{N}(\tau)\end{pmatrix} = \tilde{Q}\operatorname{div}\tilde{\mathscr{N}}(\tau),$$

and, therefore, $\mathscr{U}^{(0)}\operatorname{div}\tilde{\mathscr{N}}(\tau) = \mathscr{R}^{(0)}\operatorname{div}\tilde{\mathscr{N}}(\tau)$. By Lemma 3.2, we see that

$$\|\partial_x^\ell I_0(t)\|_2 \le C(1+t)^{-(n-1)/4-\ell/2}\mathscr{E}_0.$$

We next apply Lemmas 3.2 and 3.3 to estimate $I_1(t)$. When $n \ge 3$, we have

$$\begin{aligned}\|\partial_x^\ell I_1(t)\|_2 &\le C\int_0^{t-1}(1+t-\tau)^{-(n-1)/4-(\ell+1)/2}\|\mathrm{div}\,\mathscr{N}(\tau)\|_1 d\tau \\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\,\mathscr{N}(\tau)\|_2 d\tau \\ &\le C(1+t)^{-(n-1)/4-\ell/2}\{E_0^s(t)M(t)+E_0^s(t)^{3/4}M(t)^{5/4}\}\end{aligned}$$

for $\ell = 0, 1$. We here used the fact that $\mathrm{div}\,\tilde{\mathscr{N}} = \tilde{Q}\,\mathrm{div}\,\tilde{\mathscr{N}}$.

In case $n = 2$, since $s \ge s_0 + 1 = 3$, we similarly have

$$\begin{aligned}&\|\partial_x^\ell I_1(t)\|_2 \\ &\quad \le C\int_0^{t-1}(1+t-\tau)^{-1/4-(\ell+1)/2}(1+\tau)^{-3/4}(\|\partial_x v\|_{H^3}^{1/3}M(\tau)^{5/3}+E_0^s(\tau)M(\tau))d\tau \\ &\qquad + C\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-3/4}d\tau(E_0^s(t)^{1/2}M(t)^{3/2}+E_0^s(t)M(t)).\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}&\int_0^{t-1}(1+t-\tau)^{-1/4-(\ell+1)/2}(1+\tau)^{-3/4}\|\partial_x v\|_{H^3}^{1/3}d\tau \\ &\quad \le \left(\int_0^{t-1}(1+t-\tau)^{-(6/5)(3/4+\ell/2)}(1+\tau)^{-9/10}d\tau\right)^{5/6}\left(\int_0^{t-1}\|\partial_x v\|_{H^3}^2 d\tau\right)^{1/6} \\ &\quad \le C(1+t)^{-1/4-\ell/2}D_0^s(t)^{1/3}.\end{aligned}$$

It then follows that

$$\|\partial_x^\ell I_1(t)\|_2 \le C(1+t)^{-1/4-\ell/2}\{D_0^s(t)^{1/3}M(t)^{5/3}+E_0^s(t)M(t)+E_0^s(t)^{1/2}M(t)^{3/2}\}$$

for $\ell = 0, 1$.

As for $I_2(t)$, we apply Lemmas 3.1 and 3.3 to obtain

$$\begin{aligned}&\|\partial_x^\ell I_2(t)\|_2 \\ &\quad \le C\int_{t-1}^{t}(t-\tau)^{-\ell/2}\|\mathrm{div}\,\mathscr{N}(\tau)\|_2 d\tau \\ &\quad \le C(1+t)^{-(n-1)/4-1/2}\{E_0^s(t)M(t)+E_0^s(t)^{3/4}M(t)^{5/4}+E_0^s(t)^{1/2}M(t)^{3/2}\}\end{aligned}$$

for $\ell = 0, 1$.

Since $E_0^s(t) + D_0^s(t) \le C\mathscr{E}_0$ for all $t \ge 0$ by Theorem 2.1, it follows from the above estimates that if $\mathscr{E}_0$ is sufficiently small, then

$$M(t) \le C\{\mathscr{E}_0 + \mathscr{E}_0^{1/3} M(t)^{5/3} + \mathscr{E}_0^{3/4} M(t)^{5/4} + \mathscr{E}_0^{1/2} M(t)^{3/2}\}.$$

We thus conclude $M(t) \le C\mathscr{E}_0$, provided that $\mathscr{E}_0$ is sufficiently small. This completes the proof of (i).

We next prove the estimate (ii). By Lemma 3.2, we have

$$\|I_0(t) - u^{(0)}(t)\|_2 \le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}.$$

We already showed that $\|I_2(t)\|_2$ has the desired decay property. As for $I_1(t)$, we write $\mathscr{N} = \mathscr{N}^{(1)} + \mathscr{N}^{(2)}$ as in Lemma 3.3 (vi). It follows from Lemmas 3.2 and 3.3 (vi) that

$$\begin{aligned} \|I_1(t)\|_2 &\le \int_0^{t-1} (1+t-\tau)^{-(n-1)/4-1/2} (\|\mathscr{N}^{(1)}(\tau)\|_1 + \|\mathrm{div}\, \mathscr{N}^{(2)}(\tau)\|_1) d\tau \\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)} \|\mathrm{div}\, \mathscr{N}(\tau)\|_2 d\tau \\ &\le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2} L(t). \end{aligned}$$

We thus obtain the estimate in (ii). This completes the proof.

We next establish L^∞ decay estimates. We first derive a decay estimate of the H^s norm, which will be also used to obtain the L^1 estimate for $\partial_x w(t)$.

Lemma 3.5. *Under the assumption of Theorem* 3.4, *it holds*

$$\|w(t)\|_{H^s} \le C\mathscr{E}_0(1+t)^{-(n-1)/4}(\log(1+t))^{1/2}.$$

Proof. The proof is based on Theorem 3.4 and a weighted energy estimate with a time-weight function. Let $u = (\phi, v)$ be a solution of (2.1)–(2.3). Assume for simplicity that $E_0^s(t) < 1$ for all t. One can then prove that there exists a positive constant C independent of t such that

$$\begin{aligned} E_r^s(t)^2 + D_r^s(t)^2 &\le C\Big\{ \|u_0\|_{H^s}^2 + E_0^s(t) D_r^s(t)^2 \\ &\qquad + rD_0^s(t)^2 + r\int_0^t (1+\tau)^{2r-1} \|u\|_2^2 d\tau \Big\}. \end{aligned} \tag{3.3}$$

The inequality (3.3) is proved in the same way as in the proof of [13, Proposition 3.2] and [15, Propositions 11.2, 11.3], where the half space problem was investigated. In fact, there are only two points to be remarked as compared with the argument in [13, 15]. One is in the estimate of

$E_r^0(t)^2 + D_r^0(t)^2$. Although we can estimate it as in [13], here we can also use the Poincaré inequality. Let $a(\rho) = \sqrt{P'(\rho)/\rho}$. Then we see from (2.1)–(2.2) that

$$\partial_t(a(\rho)\phi) + v \cdot \nabla(a(\rho)\phi) + \rho a(\rho) \operatorname{div} v = -\rho a'(\rho)(\operatorname{div} v)\phi, \tag{3.4}$$

$$\rho(\partial_t v + v \cdot \nabla v) + Av + \nabla(P'(\rho)\phi) = P''(\rho)(\nabla\phi)\phi. \tag{3.5}$$

Taking the L^2 inner product of (3.4) and (3.5) with $(1+t)^{2r}a(\rho)\phi$ and $(1+t)^{2r}v$, respectively, and noting that

$$(\rho a(\rho) \operatorname{div} v, a(\rho)\phi)_2 = -(\nabla(P'(\rho)\phi), v)_2,$$

we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[(1+t)^{2r}(\|a(\rho)\phi(t)\|_2^2 + \|\sqrt{\rho}\, v(t)\|_2^2)] + (1+t)^{2r}\|A^{1/2}v(t)\|_2^2 \\ = r(1+t)^{2r-1}(\|a(\rho)\phi(t)\|_2^2 + \|\sqrt{\rho}\, v(t)\|_2^2) + R(t), \end{aligned} \tag{3.6}$$

where $\|A^{1/2}v\|_2^2 = \mu\|\nabla v\|_2^2 + (\mu+\mu')\|\operatorname{div} v\|_2^2$ and

$$R(t) = -(v \cdot \nabla(a(\rho)\phi), a(\rho)\phi)_2 + (v, \nabla(\rho a'(\rho)a(\rho)\phi^2))_2 + (P''(\rho)(\nabla\rho)\phi, v)_2.$$

For the velocity v, we have the Poincaré inequality: $\|v\|_2 \le C\|\partial_x v\|_2^2$. Therefore, $R(t)$ is estimated as

$$|R(t)| \le C(1 + \|\phi\|_\infty)\|\phi\|_\infty \|v\|_2 \|\partial_x \phi\|_2 \le CE_0^s(t)\|\partial_x u\|_2^2.$$

This, together with (3.6), implies that $E_r^0(t)^2 + D_r^0(t)^2$ is bounded by the right-hand side of (3.3).

The second point is as follows. In deriving (3.3) we use regularity estimates for solutions to the Stokes system. In the case of Ω it is formulated in the following way. Let $(p, v) \in H^{k+1} \times H^{k+2}$ be the solution of the Stokes system

$$\begin{aligned} \operatorname{div} v = f \qquad &\text{in } \Omega \\ -\mu\Delta v + P'(\rho_*)\nabla p = g \qquad &\text{in } \Omega \\ v|_{x_n=0,a} = 0. \end{aligned}$$

Then for any $k \in \mathbf{Z}$, $k \ge 0$, there exists a constant $C > 0$ such that

$$\|\partial_x^{k+2} v\|_2 + \|\partial_x^{k+1} p\|_2 \le C\{\|f\|_{H^{k+1}} + \|g\|_{H^k} + \|\partial_x v\|_2^2\}. \tag{3.7}$$

Here the right-hand side of (3.7) is slightly different from the one for the half space problem, but it does not affect the argument to obtain (3.3). For completeness we will give a proof of (3.7) in the Appendix. The other part of the proof is quite similar to the argument in [13, 15]. We omit the details.

We continue the proof of Lemma 3.5. We see from (3.3) with $r = 0$ that

$$E_0^s(t)^2 + D_0^s(t)^2 \le C\|u_0\|_{H^s}^2, \tag{3.8}$$

provided that $\|u_0\|_{H^s} < \varepsilon_0$ for some small $\varepsilon_0 > 0$. Note that this is just the energy estimate in Theorem 2.1. Since $\|u\|_2 \le C\|w\|_2$, we see from (3.3) and (3.8) that

$$E_r^s(t) \le C\|u_0\|_{H^s} + C\left(\int_0^t (1+\tau)^{2r-1}\|w\|_2^2 d\tau\right)^{1/2},$$

provided that $\|u_0\|_{H^s}$ is sufficiently small. We now take $r = \frac{n-1}{4}$ and apply Theorem 3.4 to obtain

$$E_r^s(t) \le C\mathscr{E}_0\left(\int_0^t (1+\tau)^{-1} d\tau\right)^{1/2} \le C\mathscr{E}_0(\log(1+t))^{1/2}$$

with $r = \frac{n-1}{4}$. The desired estimate now follows since $\|w(t)\|_{H^s} \le C\|u(t)\|_{H^s}$. This completes the proof.

Before proceeding further, we prepare a lemma to estimate the nonlinearity, which follows from [9, Lemma 3.3.1].

LEMMA 3.6. *Let F be a smooth function on* $\mathbf{R}$. *Then*

$$\|\partial_x^k F(f)g\|_2 \le C(1+\|f\|_\infty)^{k-1}(\|f\|_\infty\|g\|_{H^k} + \|g\|_\infty\|f\|_{H^k}).$$

PROOF. The inequality follows by a direct application of [9, Lemma 3.3.1], when Ω is the whole space. The desired inequality can then be obtained by using the extension argument. This completes the proof.

We set

$$M_\infty^{(0)}(t) = \sup_{0\le\tau\le t}(1+\tau)^{(n-1)/2}\|w(\tau)\|_\infty.$$

THEOREM 3.7. *Let $s \ge s_0 + 1$. Then there exists a positive number ε_2 such that*

$$\|w(t)\|_\infty \le C\mathscr{E}_0(1+t)^{-(n-1)/2},$$

provided that $\mathscr{E}_0 < \varepsilon_2$.

PROOF. Since $\|w(t)\|_\infty \le CE_0^s(t) \le C\mathscr{E}_0$ by the Sobolev inequality, it suffices to show $M_\infty^{(0)}(t) \le C\mathscr{E}_0$ for $t \ge 2$.

As in the proof of Theorem 3.4, we write $w(t) = I_0(t) + I_1(t) + I_2(t)$. By Lemma 3.2 we have

$$\|I_0(t)\|_\infty \le C\mathscr{E}_0(1+t)^{-(n-1)/2}.$$

Applying Lemmas 3.5 and 3.6, we see that

$$\|\mathrm{div}\,\mathcal{N}\|_{H^{s_0-1}} \le C\|w\|_\infty\|w\|_{H^s} \le C\mathscr{E}_0(1+t)^{-3(n-1)/4}(\log(1+t))^{1/2}M_\infty^{(0)}(t). \tag{3.9}$$

This, together with Lemmas 3.2 and 3.3, implies that

$$\begin{aligned}
\|I_1(t)\|_\infty &\le C\int_0^{t-1}(1+t-\tau)^{-(n-1)/2}(\|\mathcal{N}^{(1)}(\tau)\|_1 + \|\mathrm{div}\,\mathcal{N}^{(2)}(\tau)\|_1)d\tau \\
&\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\,\mathcal{N}(\tau)\|_{H^{s_0-1}}d\tau \\
&\le C\mathscr{E}_0\int_0^{t-1}(1+t-\tau)^{-(n-1)/2}(1+\tau)^{-(n-1)/2-1/2}d\tau \\
&\quad + C\mathscr{E}_0 M_\infty^{(0)}(t)\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-3(n-1)/4}(\log(1+\tau))^{1/2}d\tau \\
&\le C\mathscr{E}_0\{(1+t)^{-(n-1)/2-1/2}L(t) \\
&\quad + (1+t)^{-3(n-1)/4}(\log(1+t))^{1/2}M_\infty^{(0)}(t)\}.
\end{aligned}$$

As for $I_2(t)$, we see from Lemma 3.1 and (3.9) that

$$\begin{aligned}
\|I_2(t)\|_\infty &\le C\int_{t-1}^{t}(t-\tau)^{-(1-\varepsilon)}\|\mathrm{div}\,\mathcal{N}(\tau)\|_{H^{s_0-1}}d\tau \\
&\le C\mathscr{E}_0 M_\infty^{(0)}(t)\int_{t-1}^{t}(t-\tau)^{-(1-\varepsilon)}(1+\tau)^{-3(n-1)/4}(\log(1+\tau))^{1/2}d\tau \\
&\le C\mathscr{E}_0(1+t)^{-3(n-1)/4}(\log(1+t))^{1/2}M_\infty^{(0)}(t).
\end{aligned}$$

We thus conclude that if $\mathscr{E}_0$ is sufficiently small, then $M_\infty^{(0)}(t) \le C\mathscr{E}_0$. This completes the proof.

To obtain the decay estimate for $\|\partial_x w(t)\|_\infty$ we first show that $\|\partial_x w(t)\|_\infty$ decays in the order $t^{-(n-1)/2}$. We set

$$\tilde{M}_\infty^{(1)}(t) = \sup_{0\le\tau\le t}(1+\tau)^{(n-1)/2}\|\partial_x w(\tau)\|_\infty.$$

Proposition 3.8. *Let $s \ge s_0+2$. Then there exists a positive number ε_3 such that*

$$\|\partial_x w(t)\|_\infty \le C\mathscr{E}_0(1+t)^{-(n-1)/2},$$

provided that $\mathscr{E}_0 < \varepsilon_3$.

PROOF. Since $s \geq s_0 + 2$, we see from Lemmas 3.5 and 3.6 that

$$\begin{aligned}\|\operatorname{div} \mathcal{N}\|_{H^{s_0}} &\leq C\|w\|_\infty \|w\|_{H^s} \\ &\leq C\mathscr{E}_0(1+t)^{-3(n-1)/4}(\log(1+t))^{1/2} M_\infty^{(0)}(t).\end{aligned}$$

Similarly to the proof of Theorem 3.7, we can obtain the desired estimate. We omit the details. This completes the proof.

To prove $\|\partial_x w(t)\|_\infty = O(t^{-(n-1)/2-1/2} L(t)^{1/2})$, we next derive a decay estimate for $\|\partial_x^2 m(t)\|_2$. We set

$$M^{(2)}(t) = \sup_{0 \leq \tau \leq t} (1+\tau)^{(n-1)/4+1/2} L(\tau)^{-1/2} \{\|\partial_\tau w(\tau)\|_2 + \|\partial_x^2 m(\tau)\|_2\}.$$

Based on the decay estimates obtained above, it is now straightforward to obtain the following estimates for the nonlinearity.

LEMMA 3.9. *Let $s \geq s_0 + 2$. Assume that $\mathscr{E}_0 < \varepsilon_3$. Then the following inequalities hold.*

(i) $\|\partial_t \operatorname{div} \mathcal{N}\|_2 \leq C\mathscr{E}_0\{1 + M^{(2)}(t)\}(1+t)^{-(n-1)/4-1/2} L(t)^{1/2}$.

(ii) $\|\partial_x \operatorname{div} \mathcal{N}\|_2 \leq C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2} L(t)^{1/2}$.

PROPOSITION 3.10. *Let $s \geq s_0 + 2$. Then there exists a positive number ε_4 such that*

$$\|\partial_t w(t)\|_2 + \|\partial_x^2 m(t)\|_2 \leq C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2} L(t)^{1/2},$$

provided that $\mathscr{E}_0 < \varepsilon_4$.

PROOF. Since $\|\partial_t w(t)\|_2 + \|\partial_x^2 m(t)\|_2 \leq E_0^s(t) \leq C\mathscr{E}_0$ for all $t \geq 0$, we may assume that $t \geq 2$. As in the proof of Theorem 3.4, we write $w(t) = I_0(t) + I_1(t) + I_2(t)$. By Lemma 3.2 we have

$$\|\partial_t I_0(t)\|_2 \leq C\mathscr{E}_0 t^{-(n-1)/4-1/2}.$$

Since

$$\partial_t I_1(t) = \mathscr{U}(1) \operatorname{div} \mathcal{N}(t-1) + \int_0^{t-1} \partial_t \mathscr{U}(t-\tau) \operatorname{div} \mathcal{N}(\tau) d\tau,$$

we see from Lemmas 3.2 and 3.3 that

$$\begin{aligned}\|\partial_t I_1(t)\|_2 &\leq C\|\operatorname{div} \mathcal{N}(t-1)\|_2 + C\int_0^{t-1} (1+t-\tau)^{-(n-1)/4-1/2} \|\operatorname{div} \mathcal{N}(\tau)\|_1 d\tau \\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)} \|\operatorname{div} \mathcal{N}(\tau)\|_2 d\tau \\ &\leq C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}.\end{aligned}$$

By integration by parts, we have

$$\partial_t I_2(t) = \int_{t-1}^{t} \mathscr{U}(t-\tau)\partial_\tau \operatorname{div} \mathscr{N}(\tau)d\tau.$$

Applying Lemmas 3.1 and 3.9, we then find that

$$\begin{aligned}\|\partial_t I_2(t)\|_2 &\le C\int_{t-1}^{t} \|\partial_\tau \operatorname{div} \mathscr{N}(\tau)\|_2 d\tau \\ &\le C\mathscr{E}_0\{1+M^{(2)}(t)\}\int_{t-1}^{t}(1+\tau)^{-(n-1)/4-1/2}L(\tau)^{1/2}d\tau \\ &\le C\mathscr{E}_0\{1+M^{(2)}(t)\}(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.\end{aligned}$$

We thus obtain

$$\|\partial_t w(t)\|_2 \le C\mathscr{E}_0\{1+M^{(2)}(t)\}(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.$$

We next estimate $\|\partial_{x'}\partial_x m(t)\|_2$. In view of the proof of Lemma 3.2 ([10, 11]), one can see that

$$\|\partial_{x'}\partial_x \mathscr{U}^{(0)}(t)w_0\|_2 \le C(1+t)^{-(n-1)/4-1}\|w_0\|_1 \tag{3.10}$$

and

$$\|\partial_{x'}\partial_x \mathscr{U}^{(\infty)}(t)w_0\|_2 \le Ce^{-ct}\|\partial_{x'}w_0\|_{H^1\times L^2}. \tag{3.11}$$

We see from (3.10) and (3.11) that

$$\|\partial_{x'}\partial_x I_0(t)\|_2 \le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}.$$

By (3.10), (3.11) and Lemma 3.9, we have

$$\begin{aligned}\|\partial_{x'}\partial_x I_1(t)\|_2 &\le C\int_0^{t-1}(1+t-\tau)^{-(n-1)/4-1}\|\operatorname{div} \mathscr{N}(\tau)\|_1 d\tau \\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\partial_{x'} \operatorname{div} \mathscr{N}(\tau)\|_2 d\tau \\ &\le C\mathscr{E}_0\int_0^{t-1}(1+t-\tau)^{-(n-1)/4-1}(1+\tau)^{-(n-1)/4-1/2}d\tau \\ &\quad + C\mathscr{E}_0\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-(n-1)/4-1/2}L(\tau)^{1/2}d\tau \\ &\le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.\end{aligned}$$

Since $\partial_{x'}$ commutes with $\mathscr{U}(t)$, we similarly obtain

$$\|\partial_{x'}\partial_x I_2(t)\|_2 \le C\int_{t-1}^{t}(t-\tau)^{-1/2}\|\partial_{x'}\operatorname{div}\mathcal{N}(\tau)\|_2 d\tau$$
$$\le C\mathcal{E}_0\int_{t-1}^{t}(t-\tau)^{-1/2}(1+\tau)^{-(n-1)/4-1/2}L(\tau)^{1/2}d\tau$$
$$\le C\mathcal{E}_0(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.$$

We thus obtain

$$\|\partial_{x'}\partial_x w(t)\|_2 \le C\mathcal{E}_0(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.$$

It remains to estimate $\|\partial_{x_n}^2 m(t)\|_2$. From equation (3.1) we find that

$$\nu\partial_{x_n}^2 m' = \partial_t m' - \nu\Delta' m' - \tilde{\nu}\nabla'\operatorname{div} m + \gamma\nabla'\phi - (\operatorname{div}\mathcal{N})',$$
$$(\nu+\tilde{\nu})\partial_{x_n}^2 m^n = \partial_t m^n - \nu\Delta' m^n - \tilde{\nu}\partial_{x_n}\nabla'\cdot m' + \gamma\partial_{x_n}\phi - (\operatorname{div}\mathcal{N})^n,$$

where $\nabla' = (\partial_{x_1},\dots,\partial_{x_{n-1}})$ and $\operatorname{div}\mathcal{N} = ((\operatorname{div}\mathcal{N})',(\operatorname{div}\mathcal{N})^n)$. It follows that

$$\|\partial_{x_n}^2 m(t)\|_2 \le C\{\|\partial_t m(t)\|_2 + \|\partial_{x'}\partial_x m(t)\|_2 + \|\partial_x\phi(t)\|_2 + \|\operatorname{div}\mathcal{N}(t)\|_2\}$$
$$\le C\mathcal{E}_0\{1+M^{(2)}(t)\}(1+t)^{-(n-1)/4-1/2}L(t)^{1/2}.$$

Therefore, we arrive at $M^{(2)}(t) \le C\mathcal{E}_0\{1+M^{(2)}(t)\}$. The desired inequality now follows if $\mathcal{E}_0$ is assumed to be sufficiently small. This completes the proof.

The following inequalities immediately follow from Proposition 3.10.

Lemma 3.11. *Let $s \ge s_0+2$ and assume that $\mathcal{E}_0 < \varepsilon_4$. Then*

$$\|\operatorname{div}\mathcal{N}\|_1 \le C\mathcal{E}_0(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}.$$

To estimate $\|\partial_x u(t)\|_\infty$ we also use the following inequality.

Lemma 3.12. *Assume that $s \ge s_0+2$ when $n \ge 4$, $s \ge s_0+3$ when $n=3$ and $s \ge s_0+4$ when $n=2$. Assume also that $\mathcal{E}_0 < \varepsilon_4$. Then*

$$\|\operatorname{div}\mathcal{N}\|_{H^{s_0}} \le C\mathcal{E}_0\{1+M_\infty^{(1)}(t)\}(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}.$$

Proof. We write $\operatorname{div}\mathcal{N}$ as

$$\operatorname{div}\mathcal{N} = \{\nu\nabla\cdot(\nabla(F_1(\phi))m) + \nu\nabla\cdot(F_1(\phi)\nabla m)$$
$$+ \tilde{\nu}\nabla(\nabla(F_1(\phi))\cdot m) + \tilde{\nu}\nabla(F_1(\phi)\operatorname{div} m)\}$$
$$+ \{\nabla(F_2(\phi))\phi + F_2(\phi)\nabla\phi\} + \left\{-\frac{1}{\rho^2}(\nabla\phi\cdot m)m - \frac{1}{\rho}m\cdot\nabla m\right\}$$
$$\equiv J_1 + J_2 + J_3,$$

where $F_j(\phi)$, $j = 1, 2$, are some smooth functions. By Lemma 3.6, we see that

$$\|J_1\|_{H^{s_0}} \le C\{\|\phi\|_\infty \|\partial_x m\|_{H^{s_0+1}} + \|\partial_x m\|_\infty \|\phi\|_{H^{s_0+2}} + \|\partial_x \phi\|_\infty \|m\|_{H^{s_0+1}}\}.$$

We here used the inequality: $\|m\|_\infty \le C\|\partial_x m\|_\infty$, which follows from $m|_{x_n=0,a} = 0$.

Similarly we can obtain

$$\|J_2\|_{H^{s_0}} \le C\{\|\phi\|_\infty \|\partial_x \phi\|_{H^{s_0}} + \|\partial_x \phi\|_\infty \|\phi\|_{H^{s_0+1}}\},$$

$$\|J_3\|_{H^{s_0}} \le C\|\partial_x w\|_\infty \|w\|_{H^{s_0+1}}.$$

Consequently, we have

$$\|\mathrm{div}\, \mathcal{N}\|_{H^{s_0}} \le C\{\|w\|_\infty (\|\partial_x \phi\|_{H^{s_0}} + \|\partial_x m\|_{H^{s_0+1}}) + \|\partial_x w\|_\infty \|w\|_{H^{s_0+2}}\}. \tag{3.12}$$

Let us now consider the case $n \ge 4$. Since $\frac{n-1}{4} \ge \frac{3}{4} > \frac{1}{2}$ for $n \ge 4$, we see from (3.12), Lemma 3.5 and Theorem 3.7 that

$$\|\mathrm{div}\, \mathcal{N}\|_{H^{s_0}} \le C\mathscr{E}_0\{M_\infty(t) + M_\infty^{(1)}(t)\}(1+t)^{-(n-1)/2-1/2},$$

which yields the desired inequality for $n \ge 4$.

We next consider the case $n = 3$. Since $s_0 = 2$ when $n = 3$, we see from Theorem 3.4 and Lemma 3.5 that

$$\begin{aligned}\|\partial_x \phi(t)\|_{H^{s_0}} &\le C\|\partial_x \phi(t)\|_2^{1/3} \|\partial_x \phi(t)\|_{H^3}^{2/3} \\ &\le C\mathscr{E}_0(1+t)^{-2/3}(\log(1+t))^{1/3} \le C\mathscr{E}_0(1+t)^{-1/2}.\end{aligned}$$

We also obtain, by Theorem 3.4, Lemma 3.5 and Proposition 3.10,

$$\begin{aligned}\|\partial_x m(t)\|_{H^{s_0+1}} &\le \|\partial_x m(t)\|_2 + \|\partial_x^2 m(t)\|_{H^2} \\ &\le \|\partial_x m(t)\|_2 + C\|\partial_x^2 m(t)\|_2^{1/3} \|\partial_x^2 m(t)\|_{H^3}^{2/3} \\ &\le C\mathscr{E}_0\{(1+t)^{-1} + (1+t)^{-2/3}(\log(1+t))^{1/3}\} \\ &\le C\mathscr{E}_0(1+t)^{-1/2}.\end{aligned}$$

This, together with (3.12), implies the desired inequality for $n = 3$ as in the case $n \ge 4$.

We finally consider the case $n = 2$. In this case we also have $s_0 = 2$ but $\|w(t)\|_{H^s} \le C\mathscr{E}_0(1+t)^{-1/4}(\log(1+t))^{1/2}$. Therefore,

$$\|\partial_x \phi(t)\|_{H^{s_0}} \le C\|\partial_x \phi(t)\|_2^{1/2} \|\partial_x \phi(t)\|_{H^4}^{1/2} \le C\mathscr{E}_0(1+t)^{-1/2}(\log(1+t))^{1/4}$$

and

$$\begin{aligned}\|\partial_x m(t)\|_{H^{s_0+1}} &\le \|\partial_x m(t)\|_2 + \|\partial_x^2 m(t)\|_{H^2}\\ &\le \|\partial_x m(t)\|_2 + C\|\partial_x^2 m(t)\|_2^{1/2}\|\partial_x^2 m(t)\|_{H^4}^{1/2}\\ &\le C\mathscr{E}_0\{(1+t)^{-3/4} + (1+t)^{-1/2}(\log(1+t))^{1/2}\}\\ &\le C\mathscr{E}_0(1+t)^{-1/2}(\log(1+t))^{1/2}.\end{aligned}$$

This, together with (3.12), implies the desired inequality for $n = 2$ as in the case $n \ge 4$. This completes the proof.

We now establish the estimate for $\|\partial_x w(t)\|_\infty$.

Theorem 3.13. *Assume that $s \ge s_0 + 2$ when $n \ge 4$, $s \ge s_0 + 3$ when $n = 3$ and $s \ge s_0 + 4$ when $n = 2$. Then there exists a positive number ε_5 such that*

$$\|\partial_x w(t)\|_\infty \le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}L(t)^{1/2},$$

provided that $\mathscr{E}_0 < \varepsilon_5$.

Proof. As in the proof of Theorem 3.4, we write $w(t) = I_0(t) + I_1(t) + I_2(t)$. By Lemma 3.2, we have

$$\|\partial_x I_0(t)\|_\infty \le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}.$$

We also see from Lemmas 3.2, 3.11 and 3.12 that

$$\begin{aligned}\|\partial_x I_1(t)\|_\infty &\le C\int_0^{t-1}(1+t-\tau)^{-(n-1)/2-1}\|\mathrm{div}\,\mathscr{N}(\tau)\|_1 d\tau\\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\,\mathscr{N}(\tau)\|_{H^{s_0}} d\tau\\ &\le C\mathscr{E}_0\int_0^{t-1}(1+t-\tau)^{-(n-1)/2-1}(1+\tau)^{-(n-1)/2-1/2}L(\tau)^{1/2}d\tau\\ &\quad + C\mathscr{E}_0\{1+M_\infty^{(1)}(t)\}\int_0^{t-1}e^{-c(t-\tau)}(1+\tau)^{-(n-1)/2-1/2}L(\tau)^{1/2}d\tau\\ &\le C\mathscr{E}_0\{1+M_\infty^{(1)}(t)\}(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}.\end{aligned}$$

As for $I_2(t)$, we apply Lemmas 3.1 and 3.12 to obtain

$$\begin{aligned}\|\partial_x I_2(t)\|_\infty &\le C\int_{t-1}^{t}(t-\tau)^{-(1-\varepsilon)}\|\mathrm{div}\,\mathscr{N}(\tau)\|_{H^{s_0}} d\tau\\ &\le C\mathscr{E}_0\{1+M_\infty^{(1)}(t)\}\int_{t-1}^{t}(t-\tau)^{-(1-\varepsilon)}(1+\tau)^{-(n-1)/2-1/2}L(\tau)^{1/2}d\tau\\ &\le C\mathscr{E}_0\{1+M_\infty^{(1)}(t)\}(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}.\end{aligned}$$

We thus conclude that $M_{\infty}^{(0)}(t) \le C\mathscr{E}_0(1 + M_{\infty}^{(1)}(t))$, from which the desired inequality follows if $\mathscr{E}_0$ is sufficiently small. This completes the proof.

We next prove the asymptotic behavior in L^{∞} space.

THEOREM 3.14. *Under the same assumption of Theorem* 3.13, *it holds*

$$\|w(t) - u^{(0)}(t)\|_{\infty} \le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}L(t).$$

PROOF. We write $\mathscr{N} = \mathscr{N}^{(1)} + \mathscr{N}^{(2)}$ as in Lemma 3.3 (vi). We see from (3.12), Lemma 3.5 and Theorems 3.7 and 3.13 that

$$\|\mathrm{div}\, \mathscr{N}\|_{H^{s_0-1}} \le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}. \tag{3.13}$$

This, together with Lemmas 3.2 and 3.3, implies that

$$\begin{aligned}
\|I_1(t)\|_{\infty} &\le C\int_0^{t-1}(1+t-\tau)^{-(n-1)/2-1/2}(\|\mathscr{N}^{(1)}(\tau)\|_1 + \|\mathrm{div}\, \mathscr{N}^{(2)}(\tau)\|_1)d\tau \\
&\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\, \mathscr{N}(\tau)\|_{H^{s_0-1}}d\tau \\
&\le C\mathscr{E}_0\int_0^{t-1}(1+t-\tau)^{-(n-1)/2-1/2}(1+\tau)^{-(n-1)/2-1/2}d\tau \\
&\quad + C\mathscr{E}_0\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-(n-1)/2-1/2}L(\tau)^{1/2}d\tau \\
&\le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}L(t).
\end{aligned}$$

Also, by Lemma 3.1 and (3.13), we have

$$\begin{aligned}
\|I_2(t)\|_{\infty} &\le C\int_{t-1}^{t}(t-\tau)^{-(1-\varepsilon)}\|\mathrm{div}\, \mathscr{N}(\tau)\|_{H^{s_0-1}}d\tau \\
&\le C\mathscr{E}_0\int_0^{t-1}(t-\tau)^{-(1-\varepsilon)}(1+\tau)^{-(n-1)/2-1/2}L(\tau)^{1/2}d\tau \\
&\le C\mathscr{E}_0(1+t)^{-(n-1)/2-1/2}L(t)^{1/2}.
\end{aligned}$$

This completes the proof.

We finally consider the estimates in L^1 norm.

THEOREM 3.15. *In addition to the assumption of Theorem* 3.4, *assume also that* $w_0 \in W^{1,1} \times L^1$. *Then the following estimates hold.*

(i) $$\|w(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{1,1}\times L^1}\}.$$

(ii) $$\|w(t) - u^{(0)}(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{1,1}\times L^1}\}(1+t)^{-1/2}L(t).$$

PROOF. By Lemmas 3.1 and 3.3, we have

$$\|w(t)\|_1 \le C\|w_0\|_{W^{1,1}\times L^1} + C\int_0^t \|\mathrm{div}\, \mathcal{N}(\tau)\|_1 d\tau$$
$$\le C\{\mathscr{E}_0 + \|w_0\|_{W^{1,1}\times L^1}\}$$

for $0 \le t \le 2$.

Assume that $t \ge 2$. As in the proof of Theorem 3.4, we write $w(t) = I_0(t) + I_1(t) + I_2(t)$. By Lemma 3.2 we have

$$\|I_0(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{1,1}\times L^1}\}.$$

By Lemmas 3.2 and 3.3, we have

$$\|I_1(t)\|_1 \le C\int_0^{t-1} (1+t-\tau)^{-1/2}(\|\mathcal{N}^{(1)}(\tau)\|_1 + \|\mathrm{div}\, \mathcal{N}^{(2)}(\tau)\|_1)d\tau$$
$$+ C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\, \mathcal{N}(\tau)\|_1 d\tau$$
$$\le C\mathscr{E}_0\int_0^{t-1} (1+t-\tau)^{-1/2}(1+\tau)^{-(n-1)/2-1/2} d\tau$$
$$+ C\mathscr{E}_0\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-3/4}(\|\partial_x v(\tau)\|_{H^3}^{1/3} + 1)d\tau$$
$$\le C\mathscr{E}_0\left\{(1+t)^{-1/2}L(t) + \int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-3/4}\|\partial_x v(\tau)\|_{H^3}^{1/3} d\tau\right\}.$$

As for the last term on the right, we see from Hölder's inequality that

$$\int_0^{t-1} e^{-c(t-\tau)}(1+\tau)^{-3/4}\|\partial_x v(\tau)\|_{H^3}^{1/3} d\tau \le C(1+t)^{-3/4} D_0^s(t)^{1/3}.$$

We thus obtain

$$\|I_1(t)\|_1 \le C\mathscr{E}_0(1+t)^{-1/2}L(t).$$

Similarly,

$$\|I_2(t)\|_1 \le C\int_{t-1}^t \|\mathrm{div}\, \mathcal{N}(\tau)\|_1 d\tau \le C\mathscr{E}_0\int_{t-1}^t (1+\tau)^{-3/4}(\|\partial_x v(\tau)\|_{H^3}^{1/3} + 1)d\tau$$
$$\le C\mathscr{E}_0(1+t)^{-3/4}.$$

We thus obtain the inequality (i).

Furthermore, by Lemma 3.2, we have

$$\|I_0(t) - u^{(0)}(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{1,1}\times L^1}\}(1+t)^{-1/2}.$$

Combining this with the estimates for $\|I_1(t)\|_1$ and $\|I_2(t)\|_1$ obtained above, we arrive at the inequality (ii). This completes the proof.

To estimate $\|\partial_x w(t)\|_1$ we make use of the following inequality.

LEMMA 3.16. *Assume that* $s \ge s_0 + 1$ *when* $n \ge 3$ *and* $s \ge s_0 + 2$ *when* $n = 2$. *Assume also that* $\mathscr{E}_0 < \varepsilon_1$. *Then*

$$\|\mathrm{div}\,\mathscr{N}\|_{W^{1,1}} \le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}(\log(1+t))^{1/2}.$$

PROOF. By Lemma 3.3, we have

$$\|\mathrm{div}\,\mathscr{N}\|_1 \le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}.$$

Here we used the fact that $\|\partial_x v(t)\|_{H^3} \le E_0^s(t) \le C\mathscr{E}_0$ when $n = 2$ since $s \ge s_0 + 2 = 4$ for $n = 2$.

A direct computation, together with Lemma 3.5, yields the inequality

$$\|\partial_x\,\mathrm{div}\,\mathscr{N}\|_1 \le C\mathscr{E}_0(1+t)^{-(n-1)/4-1/2}(\log(1+t))^{1/2}.$$

We omit the details. This completes the proof.

We now establish the decay estimate for $\|\partial_x w(t)\|_1$.

THEOREM 3.17. *Assume that* $s \ge s_0 + 1$ *when* $n \ge 3$ *and* $s \ge s_0 + 2$ *when* $n = 2$. *Assume also that* $w_0 \in H^s \cap (W^{2,1} \times W^{1,1})$. *Then*

$$\|\partial_x w(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{2,1}\times W^{1,1}}\}(1+t)^{-1/2},$$

provided that $\mathscr{E}_0 < \varepsilon_1$.

PROOF. We first note that $m_0|_{x_n=0,a} = 0$ since u_0 satisfies the compatibility condition. Therefore, for $0 \le t \le 2$, we see from Lemmas 3.1 and 3.16 that

$$\begin{aligned}\|\partial_x w(t)\|_1 &\le C\|w_0\|_{W^{2,1}\times W^{1,1}} + C\int_0^t (t-\tau)^{-1/2}\|\mathrm{div}\,\mathscr{N}(\tau)\|_{W^{1,1}}d\tau \\ &\le C\|w_0\|_{W^{2,1}\times W^{1,1}} + C\mathscr{E}_0\int_0^t (t-\tau)^{-1/2}d\tau \\ &\le C\{\mathscr{E}_0 + \|w_0\|_{W^{2,1}\times W^{1,1}}\}.\end{aligned}$$

We next consider the estimate for $t \ge 2$. As in the proof of Theorem 3.4, we write $w(t) = I_0(t) + I_1(t) + I_2(t)$. By Lemma 3.2, we have

$$\|\partial_x I_0(t)\|_1 \le C\{\mathscr{E}_0 + \|w_0\|_{W^{2,1}\times W^{1,1}}\}(1+t)^{-1/2}.$$

We also see from Lemmas 3.2 and 3.16 that

$$\begin{aligned}\|\partial_x I_1(t)\|_1 &\le C\int_0^{t-1}(1+t-\tau)^{-1}\|\mathrm{div}\,\mathcal{N}(\tau)\|_1 d\tau \\ &\quad + C\int_0^{t-1} e^{-c(t-\tau)}\|\mathrm{div}\,\mathcal{N}(\tau)\|_{W^{1,1}}d\tau \\ &\le C\mathscr{E}_0\int_0^{t-1}(1+t-\tau)^{-1}(1+\tau)^{-(n-1)/4-1/2}(\log(1+\tau))^{1/2}d\tau \\ &\le C\mathscr{E}_0(1+t)^{-1/2},\end{aligned}$$

and, by Lemmas 3.1 and 3.16,

$$\begin{aligned}\|\partial_x I_2(t)\|_1 &\le C\int_{t-1}^{t}(t-\tau)^{-1/2}\|\mathrm{div}\,\mathcal{N}(\tau)\|_{W^{1,1}}d\tau \\ &\le C\mathscr{E}_0\int_{t-1}^{t}(t-\tau)^{-1/2}(1+\tau)^{-(n-1)/4-1/2}(\log(1+\tau))^{1/2}d\tau \\ &\le C\mathscr{E}_0(1+t)^{-1/2}.\end{aligned}$$

We thus obtain the desired estimate. This completes the proof.

Appendix: Proof of (3.7)

In this section we give a proof of the estimate (3.7) for the Stokes system. The argument is similar to that in the proof of [25, Theorem III.1.5.1]. We begin with

Lemma A.1. *Let $\mathscr{D} = \mathbf{R}^n$ or $\mathscr{D} = \mathbf{R}^n_+ = \{x = (x', x_n); x_n > 0\}$ and let k be a nonnegative integer. Assume that $v \in H^{k+2}(\mathscr{D})$, $p \in H^{k+1}(\mathscr{D})$ satisfy*

$$\begin{aligned}\mathrm{div}\, v = f \qquad &\text{in } \mathscr{D}, \\ -\mu\Delta v + P'(\rho_*)\nabla p = g \qquad &\text{in } \mathscr{D}, \\ v = 0 \qquad &\text{on } \{x_n = 0\} \text{ in case } \mathscr{D} = \mathbf{R}^n_+.\end{aligned}$$

Then

$$\|\partial_x^{k+2} v\|_{L^2(\mathscr{D})} + \|\partial_x^{k+1} p\|_{L^2(\mathscr{D})} \le C\{\|\partial_x^{k+1} f\|_{L^2(\mathscr{D})} + \|\partial_x^k g\|_{L^2(\mathscr{D})}\}.$$

Proof. See, e.g., [6].

In what follows we assume that $v \in H^{k+2}(\Omega)$, $p \in H^{k+1}(\Omega)$ satisfy

$$\begin{aligned}\mathrm{div}\, v = f \qquad &\text{in } \Omega, \\ -\mu\Delta v + P'(\rho_*)\nabla p = g \qquad &\text{in } \Omega, \\ v = 0 \qquad &\text{on } \{x_n = 0, a\}.\end{aligned} \tag{A.1}$$

We take a family of open cubes $\{Q_j\}_{j=1}^{\infty}$ that has the properties: (i) $\bar{\Omega} \subset \bigcup_{j=1}^{\infty} Q_j$, (ii) $\Omega_j \equiv \Omega \cap Q_j \neq \emptyset$, (iii) Q_j's are congruent with each other, and (iv) $\{Q_j\}_{j=1}^{\infty}$ has the finite intersection property.

LEMMA A.2. *Set* $\bar{p}_j = \frac{1}{|\Omega_j|}\int_{\Omega_j} p(x)dx$. *Then it holds*

$$\|p - \bar{p}_j\|_{L^2(\Omega_j)} \leq C\{\|g\|_{L^2(\Omega_j)} + \|\partial_x v\|_{L^2(\Omega_j)}\}.$$

Here C *is a positive constant independent of* j.

PROOF. We see from (A.1) that

$$-\mu\Delta v + P'(\rho_*)\nabla(p - \bar{p}_j) = g \qquad \text{a.e. } x.$$

For any $\varphi \in C_0^{\infty}(\Omega_j)$, we have

$$\begin{aligned} |P'(\rho_*)(\nabla(p - \bar{p}_j), \varphi)_{L^2(\Omega_j)}| &= |(g, \varphi)_{L^2(\Omega_j)} + \mu(\nabla v, \nabla\varphi)_{L^2(\Omega_j)}| \\ &\leq \|g\|_{L^2(\Omega_j)}\|\varphi\|_{L^2(\Omega_j)} + \mu\|\nabla v\|_{L^2(\Omega_j)}\|\nabla\varphi\|_{L^2(\Omega_j)} \\ &\leq C\{\|g\|_{L^2(\Omega_j)} + \|\nabla v\|_{L^2(\Omega_j)}\}\|\nabla\varphi\|_{L^2(\Omega_j)}. \end{aligned}$$

Here we used the Poincaré inequality: $\|\varphi\|_{L^2(\Omega_j)} \leq C\|\nabla\varphi\|_{L^2(\Omega_j)}$. We thus obtain

$$\|\nabla(p - \bar{p}_j)\|_{H^{-1}(\Omega_j)} \leq C\{\|g\|_{L^2(\Omega_j)} + \|\partial_x v\|_{L^2(\Omega_j)}\}.$$

Since $\|p - \bar{p}_j\|_{L^2(\Omega_j)} \leq C\|\nabla(p - \bar{p}_j)\|_{H^{-1}(\Omega_j)}$ (See, e.g., [25, Lemma II.1.5.4]), we have

$$\|p - \bar{p}_j\|_{L^2(\Omega_j)} \leq C\{\|g\|_{L^2(\Omega_j)} + \|\partial_x v\|_{L^2(\Omega_j)}\}.$$

This completes the proof.

In the following we take a family of smooth functions $\{\chi_j\}_{j=1}^{\infty}$ that satisfies $\operatorname{supp}\chi_j \subset Q_j$ and $\sum_{j=1}^{\infty}\chi_j^2 \equiv 1$.

PROOF OF (3.7). We set $v_j = \chi_j v$ and $p_j = \chi_j(p - \bar{p}_j)$. Then we see from (A.1) that

$$\begin{cases} \operatorname{div} v_j = F_j, \\ -\mu\Delta v_j + P'(\rho_*)\nabla p_j = G_j. \end{cases}$$

Here

$$\begin{aligned} F_j &= \chi_j f + v \cdot \nabla\chi_j, \\ G_j &= \chi_j g - 2\mu\nabla\chi_j \cdot \nabla v - \mu\Delta\chi_j v + \gamma\nabla\chi_j(p - \bar{p}_j). \end{aligned}$$

By Lemma A.1 we have

$$\|\partial_x^2 v_j\|_2 + \|\partial_x p_j\|_2 \le C\{\|\partial_x F_j\|_2 + \|G_j\|_2\}. \tag{A.2}$$

By Lemma A.2 we have

$$\begin{aligned} \|G_j\|_2 &\le C\{\|g\|_{L^2(\Omega_j)} + \|v\|_{H^1(\Omega_j)} + \|p - \bar{p}_j\|_{L^2(\Omega_j)}\} \\ &\le C\{\|g\|_{L^2(\Omega_j)} + \|v\|_{H^1(\Omega_j)}\}. \end{aligned} \tag{A.3}$$

We also have

$$\|\partial_x F_j\|_2 \le C\{\|f\|_{H^1(\Omega_j)} + \|v\|_{H^1(\Omega_j)}\}. \tag{A.4}$$

Therefore, we see from (A.2)–(A.4) that

$$\|\partial_x^2 v_j\|_2 + \|\partial_x p_j\|_2 \le C\{\|f\|_{H^1(\Omega_j)} + \|g\|_{L^2(\Omega_j)} + \|v\|_{H^1(\Omega_j)}\}. \tag{A.5}$$

Furthermore, since

$$\chi_j \partial_x^2 v = \partial_x^2 v_j + [\chi_j, \partial_x^2] v,$$

$$\chi_j \partial_x p = \chi_j \partial_x (p - \bar{p}_j) = \partial_x p_j + [\chi_j, \partial_x](p - \bar{p}_j),$$

we see that

$$\|\chi_j \partial_x^2 v\|_2 \le \|\partial_x^2 v_j\|_2 + C\|v\|_{H^1(\Omega_j)} \tag{A.6}$$

and, by Lemma A.2,

$$\begin{aligned} \|\chi_j \partial_x p\|_2 &= \|\chi_j \partial_x (p - \bar{p}_j)\|_2 \\ &\le \|\partial_x p_j\|_2 + C\{\|g\|_{L^2(\Omega_j)} + \|\partial_x v\|_{L^2(\Omega_j)}\}. \end{aligned} \tag{A.7}$$

It then follows from (A.5)–(A.7) that

$$\begin{aligned} \|\partial_x^2 v\|_2^2 + \|\partial_x p\|_2^2 &= \sum_{j=1}^{\infty} \|\chi_j \partial_x^2 v\|_2^2 + \|\chi_j \partial_x p\|_2^2 \\ &\le C \sum_{j=1}^{\infty} \{\|f\|_{H^1(\Omega_j)}^2 + \|g\|_{L^2(\Omega_j)}^2 + \|v\|_{H^1(\Omega_j)}^2\} \\ &\le C\{\|f\|_{H^1}^2 + \|g\|_2^2 + \|v\|_{H^1}^2\} \\ &\le C\{\|f\|_{H^1}^2 + \|g\|_2^2 + \|\partial_x v\|_2^2\}. \end{aligned}$$

Here we used the Poincaré inequality for v. The estimate (3.7) is thus obtained for $k = 0$.

The case $k \ge 1$ can be shown by induction on k. We have already seen that (3.7) holds for $k = 0$. Suppose that (3.7) holds for all $k \le \ell$. We will prove (3.7) to hold for $k = \ell + 1$. We apply Lemma A.1 to obtain

(A.8) $$\|\partial_x^{\ell+3} v_j\|_2 + \|\partial_x^{\ell+2} p_j\|_2 \le C\{\|\partial_x^{\ell+2} F_j\|_2 + \|\partial_x^{\ell+1} G_j\|_2\}.$$

By Lemma A.2 we have

(A.9) $$\begin{aligned}\|\partial_x^{\ell+1} G_j\|_2 &\le C\{\|g\|_{H^{\ell+1}(\Omega_j)} + \|v\|_{H^{\ell+2}(\Omega_j)} + \|p - \bar{p}_j\|_{H^{\ell+1}(\Omega_j)}\} \\ &\le C\{\|g\|_{H^{\ell+1}(\Omega_j)} + \|v\|_{H^{\ell+2}(\Omega_j)} + \|\partial_x p\|_{H^{\ell}(\Omega_j)}\}.\end{aligned}$$

We also have

(A.10) $$\|\partial_x^{\ell+2} F_j\|_2 \le C\{\|f\|_{H^{\ell+2}(\Omega_j)} + \|v\|_{H^{\ell+2}(\Omega_j)}\}.$$

Therefore, we obtain

(A.11) $$\begin{aligned}&\|\partial_x^{\ell+3} v_j\|_2 + \|\partial_x^{\ell+2} p_j\|_2 \\ &\quad \le C\{\|f\|_{H^{\ell+2}(\Omega_j)} + \|g\|_{H^{\ell+1}(\Omega_j)} + \|v\|_{H^{\ell+2}(\Omega_j)} + \|\partial_x p\|_{H^{\ell}(\Omega_j)}\}.\end{aligned}$$

Since

$$\chi_j \partial_x^{\ell+3} v = \partial_x^{\ell+3} v_j + [\chi_j, \partial_x^{\ell+3}] v,$$
$$\chi_j \partial_x^{\ell+2} p = \chi_j \partial_x^{\ell+2} (p - \bar{p}_j) = \partial_x^{\ell+2} p_j + [\chi_j, \partial_x^{\ell+2}](p - \bar{p}_j)$$

we see that

(A.12) $$\|\chi_j \partial_x^{\ell+3} v\|_2 \le C\{\|\partial_x^{\ell+3} v_j\|_2 + \|v\|_{H^{\ell+2}(\Omega_j)}\}$$

and, by Lemma A.2,

(A.13) $$\|\chi_j \partial_x^{\ell+2} p\|_2 \le C\{\|\partial_x^{\ell+2} p_j\|_2 + \|\partial_x p\|_{H^{\ell}(\Omega_j)} + \|g\|_{L^2(\Omega_j)} + \|\partial_x v\|_{L^2(\Omega_j)}\}.$$

It then follows from (A.11)–(A.13) that

$$\begin{aligned}&\|\partial_x^{\ell+3} v\|_2^2 + \|\partial_x^{\ell+2} p\|_2^2 \\ &\quad = \sum_{j=1}^{\infty} \|\chi_j \partial_x^{\ell+3} v\|_2^2 + \|\chi_j \partial_x^{\ell+2} p\|_2^2 \\ &\quad \le C \sum_{j=1}^{\infty} \{\|f\|_{H^{\ell+2}(\Omega_j)}^2 + \|g\|_{H^{\ell+1}(\Omega_j)}^2 + \|\partial_x^2 v\|_{H^{\ell}(\Omega_j)}^2 + \|\partial_x p\|_{H^{\ell}(\Omega_j)}^2 + \|v\|_{H^1(\Omega_j)}^2\} \\ &\quad \le C\{\|f\|_{H^{\ell+2}(\Omega)}^2 + \|g\|_{H^{\ell+1}(\Omega)}^2 + \|\partial_x^2 v\|_{H^{\ell}(\Omega)}^2 + \|\partial_x p\|_{H^{\ell}(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2\}.\end{aligned}$$

By the inductive assumption and the Poincaré inequality we obtain

$$\|\partial_x^{\ell+3} v\|_2^2 + \|\partial_x^{\ell+2} p\|_2^2 \le C\{\|f\|_{H^{\ell+2}(\Omega)}^2 + \|g\|_{H^{\ell+1}(\Omega)}^2 + \|\partial_x v\|_2^2\}.$$

Therefore, the estimate (3.7) holds for $k = \ell + 1$. This completes the proof.

Acknowledgement

The author is very grateful to the referee for valuable suggestions and comments.

References

[1] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, J. Math. Soc. Japan, **55** (2003), 469–497.

[2] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, Part 2, $\lambda = 0$ case, J. Math. Fluid Mech., **5** (2003), 245–274.

[3] H. Abels and M. Wiegner, Resolvent estimates for the Stokes operator on an infinite layer, Diff. Int. Eq., **18** (2005), 1081–1110.

[4] J. T. Beale and T. Nishida, Large-time behavior of viscous surface waves, Lecture Notes in Num. Appl. Anal., **8** (1985), 1–14, Recent Topics in Nonlinear PDE II, Sendai, 1985.

[5] R. Benabidallah, Résolubilité globale de l'équation d'un gaz visqueux isotherme dans un domaine non borne compris entre deux plans, Ricerche Mat., **46** (1997), 389–423.

[6] G. P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. 1, Springer-Verlag, New York (1994).

[7] D. Hoff and K. Zumbrun, Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, Indiana Univ. Math. J., **44** (1995), 604–676.

[8] D. Hoff and K. Zumbrun, Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves, Z. Angew. Math. Phys., **48** (1997), 597–614.

[9] T. Hosono, Asymptotic behavior of solutions for nonlinear partial differential equation with dissipation, Doctoral Thesis, Kyushu University (2006).

[10] Y. Kagei, Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer, Funkcial. Ekvac., **50** (2007), 287–337.

[11] Y. Kagei, Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer, Publ. Res. Inst. Math. Sci., **43** (2007), 763–794.

[12] Y. Kagei and S. Kawashima, Local solvability of initial boundary value problem for a quasilinear hyperbolic-parabolic system, J. Hyperbolic Differential Equations, **3** (2006), 195–232.

[13] Y. Kagei and S. Kawashima, Stability of planar stationary solutions to the compressible Navier-Stokes equation on the half space, Comm. Math. Phys., **266** (2006), 401–430.

[14] Y. Kagei and T. Kobayashi, On large time behavior of solutions to the compressible Navier-Stokes equations in the half space in $\mathbf{R}^3$, Arch. Rational Mech. Anal., **165** (2002), 89–159.

[15] Y. Kagei and T. Kobayashi, Asymptotic behavior of solutions to the compressible Navier-Stokes equations on the half space, Arch. Rational Mech. Anal., **177** (2005), 231–330.

[16] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, Doctoral Thesis, Kyoto University (1983).

[17] S. Kawashima, A. Matsumura and T. Nishida, On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation, Comm. Math. Phys., **70** (1979), 97–124.

[18] T. Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in $\mathbf{R}^3$, J. Differential Equations, **184** (2002), 587–619.

[19] T. Kobayashi and Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbf{R}^3$, Comm. Math. Phys., **200** (1999), 621–659.

[20] T. Kobayashi and Y. Shibata, Remarks on the rate of decay of solutions to linearized compressible Navier-Stokes equations, Pacific J. Math., **207** (2002), 199–234.

[21] A. Matsumura, An energy method for the equations of motion of compressible viscous and heat-conductive fluids, University of Wisconsin-Madison, MRC Technical Summary Report #2194 (1981), 1–16.

[22] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A, **55** (1979), 337–342.

[23] A. Matsumura and T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ., **20** (1980), 67–104.

[24] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Comm. Math. Phys., **89** (1983), 445–464.

[25] H. Sohr, The Navier-Stokes Equations: an elementary functional analytic approach, Birkhäuser, Basel (2001).

Yoshiyuki Kagei
Faculty of Mathematics
Kyushu University
Fukuoka 812-8581, *Japan*
E-mail: kagei@math.kyushu-u.ac.jp