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Norm estimates for the Bernardi integral transforms of functions defined by subordination

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ABSTRACT. In this paper, we obtain sharp norm estimates for the Bernardi integral transform of functions belonging to the class $\mathscr{K}(A,B)$, $-1 \le B < A \le 1$, which is a subclass of the well-known class of convex univalent functions. As a consequence, a number of open questions arise naturally, concerning $\mathscr{S}^*(A,B)$—a subclass of the well-known class of starlike univalent functions, and many other classes.

1. Introduction

Let $\mathscr{A}$ denote the class of functions f analytic in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$ and $\mathscr{LU}$ the subclass of $\mathscr{A}$ consisting of all *locally univalent* functions, namely, $\mathscr{LU} = \{f \in \mathscr{A} : f'(z) \neq 0, z \in \Delta\}$. In the sense of the Hornich operations ([6], see also [9]), we may regard $\mathscr{LU}$ as a vector space over $\mathbf{C}$ and we can define the norm of $f \in \mathscr{LU}$ by

$$\|f\| = \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|.$$

It is known that $\|f\| < \infty$ if and only if f is uniformly locally univalent, that is, there exists a constant $\rho = \rho(f) > 0$ such that f is univalent in each disk of hyperbolic radius ρ in Δ. Furthermore, $\|f\| \le 6$ if f is univalent in Δ and, conversely, f is univalent in Δ if $\|f\| \le 1$, and these bounds are sharp (Becker and Pommerenke [1]). For more geometric and analytic properties of f relating the norm, see [11]. Many authors have given norm estimates for classical subclasses of univalent functions (see [2, 8, 12, 15, 19, 20]).

In the sequel, $\mathscr{H}$ will stand for the class of functions f analytic in the unit disk Δ and $\mathscr{H}_a$ will denote the subclass $\{f \in \mathscr{H} : f(0) = a\}$, for $a \in \mathbf{C}$.

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We say that a function $\varphi \in \mathscr{H}$ is subordinate to $\psi \in \mathscr{H}$ and write $\varphi \prec \psi$ or $\varphi(z) \prec \psi(z)$ if there is a function $\omega \in \mathscr{H}_0$ with $|\omega| < 1$ satisfying $\varphi = \psi \circ \omega$. Note that the condition $\varphi \prec \psi$ is equivalent to the conditions $\varphi(\varDelta) \subset \psi(\varDelta)$ and $\varphi(0) = \psi(0)$ when ψ is univalent.

In this paper, we consider the subclasses $\mathscr{S}^*(A,B)$ and $\mathscr{K}(A,B)$ of $\mathscr{A}$ defined by (see Janowski [7])

$$\mathscr{S}^*(A,B) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathscr{K}(A,B) = \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

Here we assume that $-1 \le B < A \le 1$, but a relaxed restriction on A, B will be used in the last section. These classes are widely used in the literature. For $0 \le \alpha < 1$, we note that

$$\mathscr{S}^*(1-2\alpha, -1) = \mathscr{S}^*(\alpha) \qquad \text{and} \qquad \mathscr{K}(1-2\alpha, -1) = \mathscr{K}(\alpha)$$

are the classes of starlike functions of order α and convex functions of order α, respectively. We note that $f \in \mathscr{S}^*(A,B)$ if and only if $J[f] \in \mathscr{K}(A,B)$, where $J[f]$ denotes the well-known Alexander transform of f defined by

$$J[f](z) = \int_0^z \frac{f(t)}{t}\,dt = f(z) * (-\log(1-z)).$$

Here $*$ denotes the usual Hadamard product (or convolution). For $\gamma > -1$, the Bernardi integral transform $J_\gamma[f]$ of $f \in \mathscr{A}$ is defined by

$$J_\gamma[f](z) = zF(1, \gamma+1; \gamma+2; z) * f(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt, \tag{1}$$

where $F(a,b;c;z)$ denotes the Gaussian hypergeometric function and is defined by

$$F(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \qquad z \in \varDelta,$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol (here $(a)_0 = 1$) and c is not a non-positive integer. In this paper, we consider the Bernardi integral transform of functions in the class $\mathscr{K}(A,B)$. In order to state our result, we define the quantity $L(A,B,\gamma)$

$$L(A,B,\gamma) = (A-B)\left(\frac{\gamma+1}{\gamma+2}\right) \sup_{0 \le x < 1} (1-x^2) \frac{F(2-A/B, \gamma+2; \gamma+3; |B|x)}{F(1-A/B, \gamma+1; \gamma+2; |B|x)},$$

where A, B, γ are related by

$$-1 \le B < A \le \min\{1, B+1\}, \qquad B \ne 0,\ -1 < \gamma \text{ and } -2 \le -A/B \le \gamma + 1. \tag{2}$$

In a recent paper, the following result was proved (see [3, Theorem 1]).

Theorem A. *Let α be a constant with $0 \le \alpha < 1$. For every function $f \in \mathscr{K}(\alpha)$, the Alexander transform $J[f]$ of f satisfies the inequality $\|J[f]\| \le L(\alpha)$. The bound $L(\alpha)$ is sharp and satisfies $L(\alpha) \le 2(1-\alpha)$ for each α. Here, $L(\alpha) = L(1-2\alpha, -1, 0)$.*

The main aim of this paper is to extend Theorem A in the following form:

Theorem 1. *Let A, B, γ be real constants satisfying the condition* (2). *Then for every $f \in \mathscr{K}(A, B)$, the Bernardi transform $J_\gamma[f]$ of f satisfies the inequality $\|J_\gamma[f]\| \le L(A, B, \gamma)$. The bound $L(A, B, \gamma)$ is sharp and satisfies*

$$L(A, B, \gamma) \le \frac{(1+|B|)(A-B)(\gamma+1)}{\gamma+2}.$$

Here we remark that Theorem 1 reduces to Theorem A if one chooses $A = 1 - 2\alpha$, $B = -1$ and $\gamma = 0$. For the special case $B = -A$, Theorem 1 yields the following simple result:

Corollary 1. *Let $0 < A \le 1$ and $\gamma \ge 0$. We have then*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1-Az} \Rightarrow \|J_\gamma[f]\| \le L(A, -A, \gamma).$$

The bound $L(A, -A, \gamma)$ is sharp and satisfies

$$L(A, -A, \gamma) \le \frac{2A(1+A)(\gamma+1)}{\gamma+2}.$$

2. Preparatory results

For convenience sake, we will use the terminology "starlike" and "convex" in a broader sense in what follows. A function $f \in \mathscr{H}$ is called starlike (respectively convex) if f is univalent and if the image $f(\Delta)$ is starlike with respect to $f(0)$ (respectively convex). As is well known, f is starlike (respectively convex) if and only if $zf'(z)/(f(z) - f(0))$ (respectively $1 + zf''(z)/f'(z)$) has a positive real part. In particular, $f \in \mathscr{H}$ is convex if and only if $zf'(z)$ is starlike (with respect to the origin).

The following result is due to Ma and Minda [14, Theorem 1] (see also [12]).

Lemma 1. *Let $\psi \in \mathscr{H}_1$ be starlike and suppose that $g \in \mathscr{A}$ satisfies the equation*

$$1 + \frac{zg''(z)}{g'(z)} = \psi(z), \qquad z \in \Delta.$$

For $f \in \mathscr{A}$, the condition $1 + zf''(z)/f'(z) \prec \psi(z)$ then implies $f'(z) \prec g'(z)$.

We also need the following result due to Hallenbeck and Ruscheweyh [5].

Lemma 2. *Let $p(z)$ and $q(z)$ be analytic functions in the unit disk Δ with $p(0) = 1 = q(0)$. For $\alpha > 0$ suppose that the function $h(z) = q(z) + \alpha zq'(z)$ is convex. Then the condition $p(z) + \alpha zp'(z) \prec h(z)$ implies $p(z) \prec q(z)$.*

Combining Lemmas 1 and 2, we obtain the following result:

Proposition 1. *Let $\gamma > -1$ be given. Suppose that the function $\psi(z) = 1 + zg''(z)/g'(z)$ is starlike and that the function $g'(z)$ is convex for a given function $g \in \mathscr{A}$. If a function $f \in \mathscr{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z), \qquad z \in \Delta$$

then the inequalities $\|f\| \le \|g\|$ and $\|J_\gamma[f]\| \le \|J_\gamma[g]\|$ hold.

Proof. First, by Lemma 1, the hypothesis implies that $f'(z) \prec g'(z)$. Then we can see from [3, Proposition 5], that $\|f\| \le \|g\|$. Now we proceed to prove the inequality $\|J_\gamma[f]\| \le \|J_\gamma[g]\|$. It is enough to prove that $(J_\gamma[f])'(z) \prec (J_\gamma[g])'(z)$. It is easy to see that the Bernardi transform $J_\gamma[g]$ of g defined by (1) satisfies the equation

$$z(J_\gamma[g])'(z) + \gamma J_\gamma[g](z) = (\gamma + 1)g(z)$$

and so,

$$z(J_\gamma[g])''(z) + (\gamma + 1)(J_\gamma[g])'(z) = (\gamma + 1)g'(z).$$

In a similar fashion, we have

$$z(J_\gamma[f])''(z) + (\gamma + 1)(J_\gamma[f])'(z) = (\gamma + 1)f'(z).$$

Set $p(z) = (J_\gamma[f])'(z)$ and $q(z) = (J_\gamma[g])'(z)$. Then, the condition $f'(z) \prec g'(z)$ is equivalent to

$$zp'(z) + (\gamma + 1)p(z) = (\gamma + 1)f'(z) \prec (\gamma + 1)g'(z) = zq'(z) + (\gamma + 1)q(z).$$

This shows that

$$\frac{zp'(z)}{\gamma + 1} + p(z) \prec \frac{zq'(z)}{\gamma + 1} + q(z), \qquad z \in \Delta.$$

Since $g'(z)$ is convex, by Lemma 2, we get

$$(J_\gamma[f])'(z) = p(z) \prec q(z) = (J_\gamma[g])'(z)$$

for $\gamma > -1$. We thus proved the required inequality.

The following result is due to Küstner [13, Theorem 1.5] (see also [3, Lemma 7]).

Lemma 3. *Suppose that* $a, b, c \in \mathbf{R}$ *satisfy* $-1 \le a \le c$ *and* $0 < b \le c$. *Then there exists a Borel probability measure* μ *on the interval* $[0,1]$ *such that*

$$\frac{F(a+1, b+1; c+1; z)}{F(a, b; c; z)} = \int_0^1 \frac{d\mu(t)}{1 - tz}, \qquad z \in \Delta.$$

3. Proof of Theorem 1

Recall that

$$f \in \mathscr{K}(A, B) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} = \phi_{A,B}(z), \qquad z \in \Delta,$$

where $\phi_{A,B}$ is known to be a convex function and therefore starlike. Define $g \in \mathscr{A}$ by the relation

$$1 + \frac{zg''(z)}{g'(z)} = \frac{1 + Az}{1 + Bz}, \qquad z \in \Delta. \tag{3}$$

A simple computation shows that

$$g'(z) = \begin{cases} (1 + Bz)^{A/B-1} & \text{if } B \neq 0, \\ e^{Az} & \text{if } B = 0. \end{cases}$$

Then by Proposition 1 it suffices to check the convexity of $g'(z)$, to establish the inequality $\|J_\gamma[f]\| \le \|J_\gamma[g]\|$.

Clearly, $g'(z)$ is convex whenever $0 < |A| \le 1$ and $B = 0$. Next we consider the case when $B \neq 0$. Set $h = g'$. By the defining relation (3) we then have

$$\frac{h'(z)}{h(z)} = \frac{A - B}{1 + Bz}.$$

Taking the logarithmic derivative of both sides and multiplying with z, we obtain

$$\frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)} = -\frac{Bz}{1 + Bz}.$$

Therefore,

$$1+\frac{zh''(z)}{h'(z)}=1+\frac{zh'(z)}{h(z)}-\frac{Bz}{1+Bz}=\frac{1+Az}{1+Bz}-\frac{Bz}{1+Bz}=\frac{1+(A-B)z}{1+Bz}.$$

We write

$$S(z)=\frac{1+(A-B)z}{1+Bz}, \qquad z\in\Delta.$$

Since the Möbius transformation $S(z)$ has no pole in the unit disk Δ, the image $S(\Delta)$ is the disk centered at $\frac{1-B(A-B)}{1-B^2}$ and radius $\frac{A-2B}{1-B^2}$. Clearly the points $S(-1)$ and $S(1)$ are diametrically opposite points to this disk. Therefore, $h(z)$ is convex (equivalently, $S(z)=1+zh''(z)/h'(z)$ has a positive real part) if and only if $S(-1)\geq 0$ and $S(1)\geq 0$. The last condition is equivalent to $A\leq B+1$. For the case $B\neq 0$ this shows that $g'(z)$ is convex provided $A\leq B+1$.

We next compute the value of $\|J_\gamma[g]\|$. For $B\neq 0$, we see that

$$g'(z)=(1+Bz)^{A/B-1}=F(1,1-A/B;1;-Bz). \tag{4}$$

Then it follows easily that

$$\begin{aligned}(J_\gamma[g])'(z)&=F(1,\gamma+1;\gamma+2;z)*g'(z)\\&=F(1,\gamma+1;\gamma+2;z)*F(1,1-A/B;1;-Bz)\\&=F(1-A/B,\gamma+1;\gamma+2;-Bz).\end{aligned}$$

Thus,

$$\frac{(J_\gamma[g])''(z)}{(J_\gamma[g])'(z)}=(A-B)\left(\frac{\gamma+1}{\gamma+2}\right)\frac{F(2-A/B,\gamma+2;\gamma+3;-Bz)}{F(1-A/B,\gamma+1;\gamma+2;-Bz)}.$$

If $-1\leq B<0$, then by Lemma 3 the inequality

$$\left|\frac{(J_\gamma[g])''(z)}{(J_\gamma[g])'(z)}\right|\leq\frac{(J_\gamma[g])''(|z|)}{(J_\gamma[g])'(|z|)}, \qquad z\in\Delta$$

holds for $B<A$ and $-2\leq -A/B\leq\gamma+1$. Now we have

$$\begin{aligned}\|J_\gamma[g]\|&=\sup_{z\in\Delta}(1-|z|^2)\left|\frac{(J_\gamma[g])''(z)}{(J_\gamma[g])'(z)}\right|\\&=\sup_{0\leq x<1}(1-x^2)\frac{(J_\gamma[g])''(x)}{(J_\gamma[g])'(x)}\\&=(A-B)\left(\frac{\gamma+1}{\gamma+2}\right)\sup_{0\leq x<1}(1-x^2)\frac{F(2-A/B,\gamma+2;\gamma+3;-Bx)}{F(1-A/B,\gamma+1;\gamma+2;-Bx)}.\end{aligned}$$

If $0 < B \le 1$, similarly we have

$$\|J_\gamma[g]\| = (A-B)\left(\frac{\gamma+1}{\gamma+2}\right)\sup_{0\le x<1}(1-x^2)\frac{F(2-A/B,\gamma+2;\gamma+3;Bx)}{F(1-A/B,\gamma+1;\gamma+2;Bx)}.$$

So, for $0 < |B| \le 1$, $B < A$ and $-2 \le -A/B \le \gamma+1$ we obtain

$$\begin{aligned}\|J_\gamma[g]\| &= (A-B)\left(\frac{\gamma+1}{\gamma+2}\right)\sup_{0\le x<1}(1-x^2)\frac{F(2-A/B,\gamma+2;\gamma+3;|B|x)}{F(1-A/B,\gamma+1;\gamma+2;|B|x)}\\ &= L(A,B,\gamma).\end{aligned}$$

The sharpness is clear, as $L(A,B,\gamma) = \|J_\gamma[g]\|$ for $g \in \mathscr{K}(A,B)$ defined by (4). Next, to establish the bound for $L(A,B,\gamma)$, that is to prove

$$L(A,B,\gamma) \le (1+|B|)(A-B)\left(\frac{\gamma+1}{\gamma+2}\right),$$

it is enough to show, for $0 \le x < 1$, the inequality

$$(1-x^2)\frac{F(2-A/B,\gamma+2;\gamma+3;|B|x)}{F(1-A/B,\gamma+1;\gamma+2;|B|x)} \le 1+|B|x < 1+|B|.$$

Now by Lemma 3, we can write

$$(1-x^2)\frac{F(2-A/B,\gamma+2;\gamma+3;|B|x)}{F(1-A/B,\gamma+1;\gamma+2;|B|x)} = \int_0^1 \frac{1-x^2}{1-t|B|x}\,d\mu(t)$$

for a Borel probability measure μ on the interval $[0,1]$. Since

$$\frac{1-x^2}{1-t|B|x} \le \frac{1-|B|^2x^2}{1-|B|x} = 1+|B|x < 1+|B| \qquad \text{for } 0 \le t \le 1,$$

the desired inequality follows. □

4. Concluding remarks

Let β, γ, A and B be real numbers and suppose that $\beta > 0$, $\beta+\gamma > 0$, $-1 \le B < 1$ and $B < A \le 1 + \gamma(1-B)\beta^{-1}$. For $f \in \mathscr{S}^*(A,B)$, we consider $g = J_{\beta,\gamma}[f]$ defined by

$$g(z) = J_{\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f^\beta(t)dt\right]^{1/\beta}, \qquad z \in \varDelta. \tag{5}$$

Moreover, we define the order of (univalent) starlikeness of the class $J_{\beta,\gamma}[\mathscr{S}^*(A,B)]$ by the largest number $\delta = \delta(A,B;\beta,\gamma)$ such that

$$J_{\beta,\gamma}[\mathscr{S}^*(A,B)] \subset \mathscr{S}^*(\delta).$$

Before we propose a general problem, we recall a special case of a result from [16].

Lemma 4. *Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator defined by* (5).

(a) *If $-1 \le B < 1$ and $B < A \le 1 + \gamma(1 - B)\beta^{-1}$, then the order of (univalent) starlikeness of $J_{\beta,\gamma}[\mathscr{S}^*(A,B)]$ is given by*

$$\delta(A,B;\beta,\gamma) = \inf_{|z|<1} \operatorname{Re} q(z),$$

where q is given by

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}$$

with

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1 + Bzt}{1 + Bz}\right)^{\beta((A-B)/B)} t^{\beta+\gamma-1}\, dt & \text{if } B \ne 0, \\ \displaystyle\int_0^1 t^{\beta+\gamma-1} \exp(\beta Az(t-1))dt & \text{if } B = 0 \end{cases}$$

and

$$q(z) = \frac{\beta - \gamma Bz}{\beta(1 + Bz)} \qquad \text{when } A = -\frac{(\gamma+1)B}{\beta},\ B \ne 0.$$

(b) *Moreover, if $-1 \le B < 0$, $B < A \le \min\{1 + \gamma(1 - B)\beta^{-1}, -(\gamma + 1)B\beta^{-1}\}$, then*

$$\delta(A,B;\beta,\gamma) = q(-1) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{F\left(1, \beta\left(\frac{B-A}{B}\right); \beta+\gamma+1; \frac{-B}{1-B}\right)} - \gamma\right]. \quad (6)$$

(c) *Furthermore, if $0 < B < 1$, $B < A \le \min\{1 + \gamma(1 - B)\beta^{-1}, (2\beta + \gamma + 1)B\beta^{-1}\}$, then*

$$\delta(A,B;\beta,\gamma) = q(1) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{F\left(1, \beta\left(\frac{A-B}{B}\right); \beta+\gamma+1; \frac{B}{1+B}\right)} - \gamma\right]. \quad (7)$$

Under the hypotheses of Lemma 4, when $f \in \mathscr{S}^*(A,B)$, we get by [20, Theorem 2]

$$\|J_{\beta,\gamma}[f]\| \le 6 - 4\delta,$$

where δ is given either by (6) or (7) with the corresponding conditions.

As a special case, we mention the following: if $f \in \mathcal{S}^*(\alpha)$ and β, γ are real numbers such that $\beta > 0$, $\beta + \gamma > 0$ and

$$\max\left\{0, -\frac{\gamma}{\beta}, \frac{\beta - \gamma - 1}{2\beta}\right\} \leq \alpha < 1,$$

then $J_{\beta,\gamma}[f]$ defined by (5) is in $\mathcal{S}^*(\delta)$, where

$$\delta = \delta(\alpha, \beta, \gamma) = \frac{1}{\beta}\left[\frac{\beta + \gamma}{F(1, 2\beta(1-\alpha); \beta + \gamma + 1; 1/2)} - \gamma\right]. \tag{8}$$

Consequently, by [20, Theorem 2], we have the estimate

$$\|J_{\beta,\gamma}[f]\| \leq 6 - 4\delta,$$

where δ is given by (8).

In particular, for $f \in \mathcal{S}^*(\alpha)$ and $\max\{0, -\gamma\} \leq \alpha < 1$, we have $J_\gamma[f] \in \mathcal{S}^*(\delta(\alpha, \gamma))$, where

$$\delta = \delta(\alpha, \gamma) = \frac{\gamma + 1}{F(1, 2(1-\alpha); \gamma + 2; 1/2)} - \gamma. \tag{9}$$

Thus, we have

$$\|J_\gamma[f]\| \leq 6 - 4\delta,$$

where δ is given by (9). Consequently, the following result gives a norm estimate for the Bernardi integral transform of functions that are not necessarily univalent.

Corollary 2. *Let $\gamma > -1$ and $f \in \mathcal{S}^*(-\gamma)$. Then*

$$\|J_\gamma[f]\| \leq 6 - 4\left[\frac{\Gamma\left(\frac{3}{2} + \gamma\right)}{\sqrt{\pi}\,\Gamma(1+\gamma)} - \gamma\right].$$

Proof. Recall the well-known identity (see [18, p. 69])

$$F(2a, 2b; a + b + 1/2; 1/2) = \frac{\Gamma\left(a + b + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a + \frac{1}{2}\right)\Gamma\left(b + \frac{1}{2}\right)}.$$

Choose $a = 1/2$, $b = 1 - \alpha$ and $\alpha = -\gamma$. Then (9) yields

$$\delta(\gamma) = \delta(-\gamma, \gamma) = -\gamma + \frac{\Gamma\left(\frac{3}{2} + \gamma\right)}{\Gamma(1+\gamma)\Gamma\left(\frac{1}{2}\right)}$$

which may be written in terms of beta function given by

$$\delta(\gamma) = -\gamma + \frac{1}{B(1/2, 1+\gamma)}.$$

Thus, for $f \in \mathcal{S}^*(-\gamma)$ we notice that $J_\gamma[f] \in \mathcal{S}^*(\delta(\gamma))$. Therefore, we have

$$\|J_\gamma[f]\| \le 6 - 4\delta(\gamma)$$

and the conclusion follows.

Problem 1. *Find the sharp norm estimate for* $J_\gamma[f]$ *when* $f \in \mathcal{S}^*(-\gamma)$. *More generally, find a sharp norm estimate for* $J_{\beta,\gamma}[f]$ *whenever* $f \in \mathcal{S}^*(\alpha)$, $\alpha < 1$.

A number of problems of this type may be raised for various integral transforms. For example, there exist conditions on $\lambda(t)$ and subfamilies $\mathcal{F}$ of $\mathcal{A}$ such that the integral transform of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt \qquad (f \in \mathcal{F})$$

is close-to-convex or starlike or convex, respectively (see [4, 17, 10] for details). In view of this, one can ask for the norm estimate for $V_\lambda(f)$ when f runs over suitable subclasses $\mathcal{F}$ of $\mathcal{A}$. We remark that for the choice $\lambda(t) = (1+\gamma)t^\gamma$ $(\gamma > -1)$, $V_\lambda(f)(z)$ reduces to the Bernardi transform of f.

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References

[1] J. Becker and Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, J. Reine Angew. Math. **354** (1984), 74–94.

[2] Y. M. Chiang, Schwarzian derivative and second order differential equations, Ph.D. thesis, University of London, 1991.

[3] J. H. Choi, Y. C. Kim, S. Ponnusamy, and T. Sugawa, Norm estimates for the Alexander transforms of convex functions of order alpha, J. Math. Anal. Appl. **303** (2005), 661–668.

[4] R. Fournier and St. Ruscheweyh, On two extremal problems related to univalent functions, Rocky Mountain J. Math. **24**(2) (1994), 529–538.

[5] D. J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. **52** (1975), 191–195.

[6] H. Hornich, Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen, Monatsh. Math. **73** (1969), 36–45.

[7] W. Janowski, Some extremal problems for certain families of analytic functions I, Ann. Polon. Math. **23** (1973), 159–177.

[8] Y. C. Kim, S. Ponnusamy, and T. Sugawa, Geometric properties of nonlinear integral transforms of certain analytic functions, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), 57–60.

[9] ———, Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives, J. Math. Anal. Appl. **299** (2004), 433–447.
[10] Y. C. Kim and F. Rønning, Integral transforms of certain subclasses of analytic functions, J. Math. Anal. Appl. **258** (2001), 466–486.
[11] Y. C. Kim and T. Sugawa, Growth and coefficient estimates for uniformly locally univalent functions on the unit disk, Rocky Mountain J. Math. **32** (2002), 179–200.
[12] ———, Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions, Proc. Edinburgh Math. Soc. **49** (2006), 131–143.
[13] R. Küstner, Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α, Comput. Methods Funct. Theory **2** (2002), 597–610.
[14] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, Proceedings of the Conference on Complex Analysis (Z. Li, F. Ren, L. Yang, and S. Zhang, eds.), International Press Inc., 1992, pp. 157–169.
[15] Y. Okuyama, The norm estimates of pre-Schwarzian derivatives of spiral-like functions, Complex Variables Theory Appl. **42** (2000), 225–239.
[16] S. Ponnusamy, Integrals of certain n-valent functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **42** (1988), 105–113.
[17] S. Ponnusamy and F. Rønning, Duality for Hadamard products applied to certain integral transforms, Complex Variables Theory Appl. **32** (1997), 263–287.
[18] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.
[19] T. Sugawa, On the norm of the pre-Schwarzian derivatives of strongly starlike functions, Ann. Univ. Mariae Curie-Skłodowska, Sectio A **52** (1998), 149–157.
[20] S. Yamashita, Norm estimates for function starlike or convex of order alpha, Hokkaido Math. J. **28** (1999), 217–230.

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