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Kiguradze classes for radial entire solutions of higher order quasilinear elliptic equations

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0. Introduction

We consider higher order quasilinear elliptic partial differential equations of the form

$$(0.1) \qquad (-1)^m \Delta^m u + \sigma f(|x|, u, \Delta u, \dots, \Delta^{m-1} u) = 0, \qquad x \in \boldsymbol{R}^N,$$

where $m \geqq 2$, $N \geqq 3$, $\sigma = +1$ or $\sigma = -1$, $|x|$ is the Euclidean length of $x = (x_1, \dots, x_N)$, and Δ^i denotes the i-th iterate of the N-dimensional Laplacian $\Delta = \sum_{k=1}^N \partial^2/\partial x_k^2$. It is always assumed that the function f in (0.1) is continuous on $[0, \infty) \times \boldsymbol{R}^m$ and satisfies the sign condition

$$(0.2) \quad u_0 f(t, u_0, u_1, \dots, u_{m-1}) \geqq 0, \not\equiv 0 \qquad \text{for } (t, u_0, u_1, \dots, u_{m-1}) \in [0, \infty) \times \boldsymbol{R}^m.$$

A prototype of (0.1) satisfying (0.2) is the multi-dimensional generalized Emden-Fowler equation

$$(0.3) \qquad (-1)^m \Delta^m u + \sigma p(|x|) |u|^\gamma \operatorname{sgn} u = 0, \qquad x \in \boldsymbol{R}^N,$$

where $\gamma > 0$ and p is continuous on $[0, \infty)$ and $p(t) \geqq 0, \not\equiv 0$ for $t \geqq 0$.

We are concerned with the problem of existence (and nonexistence) of radial entire solutions of (0.1) which have no zero in $\boldsymbol{R}^N$. By a radial entire solution of (0.1) we mean a radially symmetric function $u(|x|) \in C^{2m}(\boldsymbol{R}^N)$ which satisfies (0.1) at every point of $\boldsymbol{R}^N$. The study of this problem was initiated by Walter [20, 21] and followed by Walter and Rhee [22], Kusano and Swanson [13], Kusano, Naito and Swanson [10–12], and Usami [19]. In particular, it is shown in [11] that the equation

$$(0.4) \qquad (-1)^m \Delta^m u + \sigma f(|x|, u) = 0, \qquad x \in \boldsymbol{R}^N,$$

generalizing (0.3), may possess a variety of positive or negative radial entire solutions with different types of asymptotic behavior as $|x| \to \infty$.

The purpose of this paper is to provide a theory which unifies and furthers basic theories developed in [10–13] and which enables us to obtain detailed information about the structure of radial entire solutions with no zero of equation (0.1). Our theory is based on the fact (Theorem 2.1 below) that a radial entire solution $u(|x|)$ of (0.1) which is either positive or negative throughout $\boldsymbol{R}^N$ satisfies the inequalities

$$(0.5)_j \qquad \begin{cases} u(|x|)(\Delta^i u)(|x|) > 0 & \text{for all large } |x|, \quad 0 \leqq i \leqq j-1, \\ (-1)^{i-j} u(|x|)(\Delta^i u)(|x|) \geqq 0 & \text{for all } x, \qquad j \leqq i \leqq m, \end{cases}$$

for some integer $j \in \{0, 1, 2, \dots, m\}$ such that

$$(0.6) \qquad j \text{ is odd in case } \sigma = +1, \text{ and } j \text{ is even in case } \sigma = -1,$$

or, in short, $(-1)^{j+1}\sigma = 1$. It should be observed that this fact is quite similar to the following theorem, known as Kiguradze's lemma [4, 5], regarding the ordinary differential equation

$$(0.7) \qquad (-1)^m y^{(m)} + \sigma f(t, y, y', \dots, y^{(m-1)}) = 0, \qquad t > 0,$$

where m, σ and f are as in (0.1):

THEOREM 0.1. *If $y(t)$ is a nonoscillatory solution of* (0.7), *then there exists an integer $j \in \{0, 1, 2, \dots, m\}$ such that* (0.6) *holds and*

$$(0.8)_j \qquad \begin{cases} y(t) y^{(i)}(t) > 0, & 0 \leqq i \leqq j-1, \\ (-1)^{i-j} y(t) y^{(i)}(t) \geqq 0, & j \leqq i \leqq m, \end{cases}$$

for all sufficiently large t.

In view of $(0.5)_j$ and $(0.8)_j$ it is natural to conjecture that the structure of radial entire solutions having no zero of (0.1) is similar to the structure of nonoscillatory solutions of (0.7), and that, with suitable modifications, known basic results for (0.7) can be carried over to (0.1). In this paper, efforts will be made to verify the truth of this conjecture.

For this purpose let us review the basic results for nonoscillatory solutions of (0.7). Denote by $\mathcal{N}$ the set of all functions $y \in C^m$ that are defined and have no zero on some half-line $[T_y, \infty)$ and satisfy $(-1)^m \sigma y(t) y^{(m)}(t) \leqq 0$ for all large t. A nonoscillatory solution $y(t)$ of (0.7) clearly belongs to $\mathcal{N}$. For an integer $j \in \{0, 1, 2, \dots, m\}$ satisfying (0.6), let $\mathcal{N}_j$ denote the set of all $y \in \mathcal{N}$ that satisfy $(0.8)_j$ for all large t. The set $\mathcal{N}_j$ is often referred to as the Kiguradze class of degree j. In view of $(0.8)_j$ it is easily seen that if $y \in \mathcal{N}_j$ for $1 \leqq j \leqq m-1$, then there exist positive constants c_1, c_2 and T such that

$$(0.9) \qquad c_1 t^{j-1} \leqq |y(t)| \leqq c_2 t^j \qquad \text{for } t \geqq T,$$

and $y(t)$ has the integral representation

$$(0.10) \qquad y(t) = \sum_{k=0}^{j-1} \frac{y^{(k)}(T)}{k!}(t-T)^k + \frac{y^{(j)}(\infty)}{j!}(t-T)^j$$

$$+(-1)^{m-j}\int_T^t \frac{(t-s)^{j-1}}{(j-1)!}\int_s^\infty \frac{(r-s)^{m-j-1}}{(m-j-1)!}y^{(m)}(r)\,drds$$

for $t \geqq T$, where $y^{(j)}(\infty) = \lim y^{(j)}(t)$ as $t \to \infty$. From (0.9) and (0.10) it follows that exactly one of the next three cases holds for each $y \in \mathcal{N}_j$, $1 \leqq j \leqq m-1$:

(0.11) $\quad \lim_{t\to\infty} \dfrac{y(t)}{t^j}$ exists and is a nonzero finite value;

(0.12) $\quad \lim_{t\to\infty} \dfrac{y(t)}{t^j} = 0$ and $\lim_{t\to\infty} \dfrac{y(t)}{t^{j-1}} = \pm\infty$;

(0.13) $\quad \lim_{t\to\infty} \dfrac{y(t)}{t^{j-1}}$ exists and is a nonzero finite value.

This admits a further classification of $\mathcal{N}_j$ for $1 \leqq j \leqq m-1$:

$$\mathcal{N}_j = \mathcal{N}_j[\max] \cup \mathcal{N}_j[\mathrm{int}] \cup \mathcal{N}_j[\min], \tag{0.14}$$

where $\mathcal{N}_j[\max]$, $\mathcal{N}_j[\mathrm{int}]$ and $\mathcal{N}_j[\min]$ denote the sets of all $y \in \mathcal{N}_j$ satisfying (0.11), (0.12) and (0.13), respectively. The study of the existence (and nonexistence) of nonoscillatory solutions in the Kiguradze classes $\mathcal{N}_j$ and the three subclasses of $\mathcal{N}_j$ appearing in (0.14) has been one of the central problems in the qualitative theory of ordinary differential equations of the form (0.7); see, e.g., the papers [2, 8, 9, 16, 17, 18].

Consider the generalized Emden-Fowler equation

$$(-1)^m y^{(m)} + \sigma p(t)|y|^\gamma \operatorname{sgn} y = 0, \qquad t > 0, \tag{0.15}$$

where $\gamma > 0$ and p is continuous on $[0, \infty)$ and $p(t) \geqq 0$, $\not\equiv 0$ for $t > 0$. Equation (0.15) corresponds to (0.3). Fundamental and important results for (0.15) are the following theorems.

THEOREM 0.2. *Let j be an integer such that $(-1)^{j+1}\sigma = 1$ and $1 \leqq j \leqq m-1$.*

(i) *Equation* (0.15) *has a nonoscillatory solution of class* $\mathcal{N}_j[\max]$ *if and only if*

$$\int_0^\infty t^{m-j-1+\gamma j} p(t)\,dt < \infty. \tag{0.16}$$

(ii) *Equation* (0.15) *has a nonoscillatory solution of class* $\mathcal{N}_j[\min]$ *if and only if*

$$\int_0^\infty t^{m-j+\gamma(j-1)} p(t)\,dt < \infty. \tag{0.17}$$

THEOREM 0.3. *Let j be an integer such that $(-1)^{j+1}\sigma = 1$ and $1 \leqq j \leqq m-1$.*

(i) *If* (0.15) *is strictly superlinear, i.e., $\gamma > 1$, then a necessary and sufficient condition for* (0.15) *to have a nonoscillatory solution of class $\mathcal{N}_j$ is that* (0.17) *is satisfied.*

(ii) *If* (0.15) *is strictly sublinear, i.e., $0 < \gamma < 1$, then a necessary and sufficient condition for* (0.15) *to have a nonoscillatory solution of class $\mathcal{N}_j$ is that* (0.16) *is satisfied.*

The problem of characterizing the solutions of class $\mathcal{N}_j[\text{int}]$ has been settled for the strictly sublinear case of (0.15), whereas it remains open for the strictly superlinear case.

THEOREM 0.4. *Let j be such that $(-1)^{j+1}\sigma = 1$ and $1 \leqq j \leqq m-1$. Assume that equation* (0.15) *is strictly sublinear, i.e., $0 < \gamma < 1$. Then, equation* (0.15) *has a nonoscillatory solution of class $\mathcal{N}_j[\text{int}]$ if and only if*

$$(0.18) \qquad \begin{cases} \displaystyle\int_0^\infty t^{m-j-1+\gamma j} p(t)\,dt < \infty \quad \text{and} \\ \displaystyle\int_0^\infty t^{m-j+\gamma(j-1)} p(t)\,dt = \infty. \end{cases}$$

Surprisingly, all the corresponding results also hold, with slight modifications, for radial entire solutions of (0.1) without zero in $\boldsymbol{R}^N$. Let $\mathcal{K}$ denote the set of all radial entire functions $u(|x|) \in C^{2m}(\boldsymbol{R}^N)$ that have no zero in $\boldsymbol{R}^N$ and satisfy $(-1)^m \sigma u(|x|)(\Delta^m u)(|x|) \leqq 0$ for $x \in \boldsymbol{R}^N$. For an integer $j \in \{0, 1, 2, \dots, m\}$ with $(-1)^{j+1}\sigma = 1$, denote by $\mathcal{K}_j$ the set of all $u(|x|) \in \mathcal{K}$ satisfying $(0.5)_j$. It can be shown that if $u \in \mathcal{K}_j$ for $1 \leqq j \leqq m-1$, then there exist positive constants c_1 and c_2 such that

$$(0.19) \qquad c_1|x|^{2(j-1)} \leqq |u(|x|)| \leqq c_2|x|^{2j} \quad \text{for all large } |x|,$$

and $u(|x|)$ is expressed as

$$(0.20) \qquad u(|x|) = \sum_{k=0}^{j-1} \rho_N(k)(\Delta^k u)(0)|x|^{2k} + \rho_N(j)(\Delta^j u)(\infty)|x|^{2j} + (-1)^{m-j}\Phi_N^j \Psi_N^{m-j}(\Delta^m u)(|x|), \qquad x \in \boldsymbol{R}^N,$$

where

$$(0.21) \quad \rho_N(0) = 1, \quad \rho_N(k) = 1/[2^k k!\, N(N+2)\cdots(N+2k-2)] \quad \text{for } k = 1, 2, \dots,$$

and Φ_N^j and Ψ_N^{m-j} denote, respectively, the j-th iterate and $(m-j)$-th iterate of the integral operators Φ_N and Ψ_N defined by

(0.22) $$(\Phi_N h)(t) = t^{-N+2}\int_0^t s^{N-3}\int_0^s rh(r)\,dr\,ds, \qquad t \geqq 0,$$

(0.23) $$(\Psi_N h)(t) = t^{-N+2}\int_0^t s^{N-3}\int_s^\infty rh(r)\,dr\,ds, \qquad t \geqq 0.$$

The corresponding decomposition of $\mathscr{K}_j$, $1 \leqq j \leqq m-1$, is formulated as

(0.24) $$\mathscr{K}_j = \mathscr{K}_j[\max] \cup \mathscr{K}_j[\mathrm{int}] \cup \mathscr{K}_j[\min],$$

where $\mathscr{K}_j[\max]$, $\mathscr{K}_j[\mathrm{int}]$ and $\mathscr{K}_j[\min]$ denote the sets of all $u \in \mathscr{K}_j$ such that

(0.25) $$\lim_{|x|\to\infty} \frac{u(|x|)}{|x|^{2j}} \quad \text{exists and is a nonzero finite value,}$$

(0.26) $$\lim_{|x|\to\infty} \frac{u(|x|)}{|x|^{2j}} = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \frac{u(|x|)}{|x|^{2(j-1)}} = \pm\infty,$$

(0.27) $$\lim_{|x|\to\infty} \frac{u(|x|)}{|x|^{2(j-1)}} \quad \text{exists and is a nonzero finite value,}$$

respectively. Moreover, the following Theorems 0.5–0.7 for the multi-dimensional generalized Emden-Fowler equation (0.3), which correspond to Theorems 0.2–0.4, can be proved.

THEOREM 0.5. *Let j be an integer such that $(-1)^{j+1}\sigma = 1$ and $1 \leqq j \leqq m-1$.*

(i) *Equation* (0.3) *has a radial entire solution of class $\mathscr{K}_j[\max]$ if and only if $N \geqq 2(m-j)+1$ and*

(0.28) $$\int_0^\infty t^{2(m-j)-1+2\gamma j} p(t)\,dt < \infty.$$

(ii) *Equation* (0.3) *has a radial entire solution of class $\mathscr{K}_j[\min]$ if and only if $N \geqq 2(m-j)+3$ and*

(0.29) $$\int_0^\infty t^{2(m-j)+1+2\gamma(j-1)} p(t)\,dt < \infty.$$

THEOREM 0.6. *Let j be an integer such that $(-1)^{j+1}\sigma = 1$ and $1 \leqq j \leqq m-1$.*

(i) *Let $N \geqq 2(m-j)+3$. Suppose that* (0.3) *is strictly superlinear, i.e., $\gamma > 1$. Then a necessary and sufficient condition for* (0.3) *to have a radial entire solution of class $\mathscr{K}_j$ is that* (0.29) *holds.*

(ii) *Supppose that* (0.3) *is strictly sublinear, i.e., $0 < \gamma < 1$. Then a necessary and sufficient condition for* (0.3) *to have a radial entire solution of* *class* $\mathscr{K}_j$ *is that* $N \geqq 2(m-j)+1$ *and* (0.28) *holds.*

THEOREM 0.7. *Let j be an integer such that* $(-1)^{j+1}\sigma = 1$, $1 \leqq j \leqq m-1$ *and let* $N \geqq 2(m-j)+3$. *Supppose that* (0.3) *is strictly sublinear* $(0<\gamma<1)$. *Then, equation* (0.3) *has a radial entire solution of class* $\mathscr{K}_j[\mathrm{int}]$ *if and only if*

$$(0.30) \qquad \begin{cases} \displaystyle\int_0^\infty t^{2(m-j)-1+2\gamma j} p(t)\,dt < \infty & \textit{and} \\ \displaystyle\int_0^\infty t^{2(m-j)+1+2\gamma(j-1)} p(t)\,dt = \infty. \end{cases}$$

These results show that the structure of the radial entire solutions of (0.3) having no zero in $\boldsymbol{R}^N$ has a striking similarity to that of the nonoscillatory solutions of (0.15), but not to that of the nonoscillatory solutions of the equation

$$(-1)^m y^{(2m)} + \sigma p(t)|y|^\gamma \operatorname{sgn} y = 0, \qquad t>0.$$

It should be noticed here that the restriction on the dimension N is essential for the existence of radial entire solutions of (0.3) having specific asymptotic properties on $\boldsymbol{R}^N$ (see, e.g., Theorem 0.5). This delicate relation between the order $2m$ of the equation and the space dimension N is a remarkable feature which is not shared by the one-dimensional differential equation (0.15).

The complete analysis of solutions in the extreme classes $\mathscr{N}_0$ and $\mathscr{N}_m$ is difficult even for the simple ordinary differential equations of the type (0.15). For the detailed discussions and related results the reader is referred to [3–7, 14]. It turns out, however, that the situation is different for the class $\mathscr{K}_0$ of solutions of the elliptic equation (0.3) in $\boldsymbol{R}^N$. In fact, the class $\mathscr{K}_0$ can also be decomposed into the three subclasses $\mathscr{K}_0[\max]$, $\mathscr{K}_0[\mathrm{int}]$ and $\mathscr{K}_0[\min]$, and necessary and sufficient conditions can be obtained for the strictly sublinear equation (0.3) to have solutions of classes $\mathscr{K}_0$, $\mathscr{K}_0[\max]$, $\mathscr{K}_0[\mathrm{int}]$ and $\mathscr{K}_0[\min]$, respectively.

It is the properties of the iterated integral operators Φ_N and Ψ_N defined by (0.22) and (0.23) that play an important part throughout the paper. These properties are stated and proved in Section 1. Section 2 contains the classification into the Kiguradze classes $\mathscr{K}_j$ $(0 \leqq j \leqq m)$ of radial entire functions $u(|x|) \in C^{2m}(\boldsymbol{R}^N)$ satisfying $u(|x|) \neq 0$ and $(-1)^m \sigma u(|x|)(\Delta^m u)(|x|) \leqq 0$ for $x \in \boldsymbol{R}^N$ as well as further classification of $\mathscr{K}_j$ according to the possible asymptotic behavior of members of $\mathscr{K}_j$ as $|x| \to \infty$. The integral representations for $u \in \mathscr{K}_j$, which are formed from iterates of Φ_N and Ψ_N, are also given in Section 2. In Section 3, elliptic equations of the form (0.1) and certain elliptic differential inequalities are considered and comparison theorems for the existence of solutions in the classes $\mathcal{K}_j$ are established. Detailed discussions of the existence of solutions of (0.1) in the classes $\mathcal{K}_j$, $\mathcal{K}_j[\max]$, $\mathcal{K}_j[\mathrm{int}]$ and $\mathcal{K}_j[\min]$ are presented in Sections 4 and 5; Section 4 concerns the case $1 \leqq j \leqq m$ and Section 5 concerns the case $j = 0$. Finally, in Section 6, we state important consequences of the theorems given in Sections 4 and 5. It is shown, for example, that if m is even, $\sigma = 1$, and $N \geqq 2m+1$, then a necessary and sufficient condition for the existence of a radial entire solution of (0.3) with no zero in $\boldsymbol{R}^N$ is that

$$\int_0^\infty t^{2m-1} p(t)\,dt < \infty \qquad \text{for the case } \gamma > 1, \text{ and}$$

$$\int_0^\infty t^{1+2\gamma(m-1)} p(t)\,dt < \infty \qquad \text{for the case } 0 < \gamma < 1.$$

This result may be considered as a higher-dimensional version of a well known theorem of Kiguradze [4, 5] and Ličko and Švec [15].

1. Preliminary lemmas

We begin by stating and proving some preparatory results which will be needed in the proofs of our theorems.

Let $N \geqq 3$ be an integer. The N-dimensional Laplacian Δ acting on radial C^2 functions is written in the polar form

$$\Delta = t^{-N+1}\frac{d}{dt}t^{N-1}\frac{d}{dt} = t^{-1}\frac{d}{dt}t^{-N+3}\frac{d}{dt}t^{N-2}, \qquad t = |x|. \tag{1.1}$$

For an integer $i = 1, 2, \ldots$, we denote by $\mathscr{D}^i[0, \infty)$ the set of all functions $h\colon [0, \infty) \to \boldsymbol{R}$ such that $\Delta^k h(|x|)$, $0 \leqq k \leqq i$, are well defined and continuous on $\boldsymbol{R}^N$, where Δ^k is the k-th iterate of Δ. Note that if $\Delta^k h(|x|)$ is defined for $|x| \geqq t_0 (\geqq 0)$, then $\Delta^k h(|x|)$ is clearly radial for $|x| \geqq t_0$. It should be also noticed that $h \in \mathscr{D}^1[0, \infty)$ if and only if $h \in C^2[0, \infty)$ and $h'(0) = 0$. Therefore, if $h \in \mathscr{D}^i[0, \infty)$, then $\Delta^k h(t)$, $0 \leqq k \leqq i$, are continuous on $[0, \infty)$ and $(\Delta^k h)'(0) = 0$, $0 \leqq k \leqq i-1$.

LEMMA 1.1. *If* $h \in \mathscr{D}^1[0, \infty)$ *and* $\Delta h(t) \geqq 0$ [*resp.* $\leqq 0$] *for* $t \geqq 0$, *then* $h'(t) \geqq 0$ [*resp.* $\leqq 0$] *for* $t \geqq 0$.

This elementary lemma is an immediate consequence of (1.1).

LEMMA 1.2. (i) *If* $h \in C^2[t_0, \infty)$, $t_0 > 0$, *satisfies* $\Delta h(t) \geqq c_0 t^p$, $t \geqq t_0$, *for some constants* $c_0 > 0$ *and* $p > -2$, *then*

$$h'(t) \geqq c_1 t^{p+1}, \quad h(t) \geqq c_2 t^{p+2}, \quad t \geqq t_1,$$

for some constants $c_1 > 0$, $c_2 > 0$ *and* $t_1 \geqq t_0$.

(ii) *If* $h \in C^2[t_0, \infty)$, $t_0 > 0$, *satisfies* $\Delta h(t) \leqq c_0 t^p$, $t \geqq t_0$, *for some constants* $c_0 > 0$ *and* $p > -2$, *then*

$$h'(t) \leqq c_1 t^{p+1}, \quad h(t) \leqq c_2 t^{p+2}, \quad t \geqq t_1,$$

for some constants $c_1 > 0$, $c_2 > 0$ *and* $t_1 \geqq t_0$.

PROOF. It suffices to prove part (i). Integrating the inequality $(t^{N-1}h'(t))' \geqq c_0 t^{N+p-1}$ over $[t_0, t]$, we have

$$t^{N-1}h'(t) \geqq t_0^{N-1}h'(t_0) + \frac{c_0}{N+p}(t^{N+p} - t_0^{N+p}), \qquad t \geqq t_0,$$

which implies the existence of $c_1 > 0$ and $t_0' \geqq t_0$ such that $t^{N-1}h'(t) \geqq c_1 t^{N+p}$ or $h'(t) \geqq c_1 t^{p+1}$ for $t \geqq t_0'$. Integration of the last inequality yields

$$h(t) \geqq h(t_0') + \frac{c_1}{p+2}(t^{p+2} - t_0'^{p+2}), \qquad t \geqq t_0',$$

which shows that $h(t) \geqq c_2 t^{p+2}$, $t \geqq t_1$, for some $c_2 > 0$ and $t_1 \geqq t_0'$.

Let $L_\lambda^1(0, \infty)$, $\lambda \geqq 0$, denote the set of all real-valued measurable functions h on $(0, \infty)$ such that

$$\int_0^\infty t^\lambda |h(t)|\, dt < \infty.$$

Define the integral operators Φ_N: $C[0, \infty) \to C^2[0, \infty)$ and Ψ_N: $C[0, \infty) \cap L_1^1(0, \infty) \to C^2[0, \infty)$ by (0.22) and (0.23), respectively. It is sometimes useful to note that Φ_N and Ψ_N can be rewritten as

$$(1.2) \qquad (\Phi_N h)(t) = \frac{1}{N-2}\left\{-\int_0^t \left(\frac{s}{t}\right)^{N-2} sh(s)\, ds + \int_0^t sh(s)\, ds\right\}, \qquad t \geqq 0,$$

$$(1.3) \qquad (\Psi_N h)(t) = \frac{1}{N-2}\left\{\int_0^t \left(\frac{s}{t}\right)^{N-2} sh(s)\, ds + \int_t^\infty sh(s)\, ds\right\}, \qquad t \geqq 0.$$

The operator Φ_N satisfies $(\Phi_N h)(0) = (\Phi_N h)'(0) = 0$ and $(\Phi_N h)(t)$ is a nondecreasing function on $[0, \infty)$ for any $h \in C[0, \infty)$ with $h(t) \geqq 0$, $t \geqq 0$. If $h \in C[0, \infty)$ and

$$(1.4) \qquad \int_0^\infty sh(s)\, ds = \lim_{t \to \infty} \int_0^t sh(s)\, ds$$

exists in the extended real line $\bar{\boldsymbol{R}}$, then

$$\lim_{t\to\infty} (\Phi_N h)(t) = \frac{1}{N-2}\int_0^\infty sh(s)\,ds.$$

Likewise it is shown that $\lim_{t\to\infty}(\Psi_N h)(t) = (\Psi_N h)'(0) = 0$ and

$$(\Psi_N h)(0) = \frac{1}{N-2}\int_0^\infty sh(s)\,ds$$

for $h\in C[0, \infty)\cap L_1^1(0, \infty)$, and that $(\Psi_N h)(t)$ is nonincreasing on $[0, \infty)$ for all nonnegative $h\in C[0, \infty)\cap L_1^1(0, \infty)$.

An easy calculation by means of (1.1) shows that

(1.5) $$\Delta(\Phi_N h)(t) = h(t),\ t\geqq 0, \qquad \text{for } h\in C[0, \infty),$$

(1.6) $$\Delta(\Psi_N h)(t) = -h(t),\ t\geqq 0, \qquad \text{for } h\in C[0, \infty)\cap L_1^1(0, \infty).$$

LEMMA 1.3. *Let $i\geqq 1$ be an integer. If $h\in\mathscr{D}^i[0, \infty)$, then*

(1.7) $$h(t) = \sum_{k=0}^{i-1}\rho_N(k)(\Delta^k h)(0)\,t^{2k} + \Phi_N^i(\Delta^i h)(t), \qquad t\geqq 0,$$

where Φ_N^i denotes the i-th iterate of Φ_N, and $\rho_N(k)$ is defined by (0.21).

LEMMA 1.4. *Let $i\geqq 1$ be an integer. If $h\in C[0, \infty)$ and $h(t)\geqq 0$ for $t\geqq 0$, then*

(1.8) $$0\leqq(\Phi_N^i h)(t)\leqq a_N(i)\,t^{2i-2}\int_0^t sh(s)\,ds, \qquad t\geqq 0,$$

where $a_N(i) = 1/[2^{i-1}(i-1)!(N-2)^i]$. If in addition the improper integral (1.4) *exists in the extended real line $\bar{\boldsymbol{R}}$, then*

(1.9) $$\lim_{t\to\infty}\frac{(\Phi_N^i h)(t)}{t^{2i-2}} = b_N(i)\int_0^\infty sh(s)\,ds,$$

where $b_N(i) = 1/[2^{i-1}(i-1)!(N-2)N\cdots(N+2i-4)]$.

The proofs of Lemmas 1.3 and 1.4 are given in [12, Lemma 2.4] and [10, Lemma 1], respectively.

LEMMA 1.5. *If h is a $C[0, \infty)$ function such that $\lim_{t\to\infty}h(t) = h(\infty)$ exists in $\bar{\boldsymbol{R}}$, then*

(1.10) $$\lim_{t\to\infty}\frac{(\Phi_N^i h)(t)}{t^{2i}} = \rho_N(i)h(\infty), \qquad i = 1, 2,\ldots.$$

PROOF. The conclusion (1.10) is true for $i = 1$, since repeated application of L'Hospital's rule shows that

$$\lim_{t\to\infty}\frac{(\Phi_N h)(t)}{t^2} = \lim_{t\to\infty}\frac{\displaystyle\int_0^t s^{N-3}\int_0^s rh(r)\,dr\,ds}{t^N}$$

$$= \lim_{t\to\infty}\frac{t^{N-3}\displaystyle\int_0^t rh(r)\,dr}{Nt^{N-1}} = \lim_{t\to\infty}\frac{\displaystyle\int_0^t rh(r)\,dr}{Nt^2}$$

$$= \lim_{t\to\infty}\frac{th(t)}{2Nt} = \frac{h(\infty)}{2N}.$$

If we assume the truth of (1.10) for some $i \geqq 1$, we obtain

$$\lim_{t\to\infty}\frac{(\Phi_N^{i+1} h)(t)}{t^{2i+2}} = \lim_{t\to\infty}\frac{\displaystyle\int_0^t s^{N-3}\int_0^s r(\Phi_N^i h)(r)\,dr\,ds}{t^{N+2i}}$$

$$= \lim_{t\to\infty}\frac{t^{N-3}\displaystyle\int_0^t r(\Phi_N^i h)(r)\,dr}{(N+2i)t^{N+2i-1}} = \lim_{t\to\infty}\frac{\displaystyle\int_0^t r(\Phi_N^i h)(r)\,dr}{(N+2i)t^{2i+2}}$$

$$= \lim_{t\to\infty}\frac{t(\Phi_N^i h)(t)}{(N+2i)(2i+2)t^{2i+1}} = \frac{1}{2(i+1)(N+2i)}\lim_{t\to\infty}\frac{(\Phi_N^i h)(t)}{t^{2i}}$$

$$= \rho_N(i+1)h(\infty).$$

Thus Lemma 1.5 follows by induction.

The following lemma is contained in [12, Lemma 2.8].

LEMMA 1.6. *Suppose that* $h \in \mathscr{D}^1[0, \infty)$ *and* $\Delta h(t) \geqq 0$ *or* $\leqq 0$ *for* $t \geqq 0$. *Then,* $\lim_{t\to\infty} h(t) = h(\infty)$ *exists and is finite if and only if* $\Delta h \in L_1^1(0, \infty)$, *in which case*

$$h(t) = h(\infty) - \Psi_N(\Delta h)(t), \qquad t \geqq 0. \tag{1.11}$$

LEMMA 1.7. *Let* $i \geqq 1$ *be an integer. Suppose that* $h \in \mathscr{D}^i[0, \infty)$ *and*

$$(-1)^k(\Delta^k h)(t) \geqq 0, \quad t \geqq 0, \quad k = 1, 2, \ldots, i. \tag{1.12}$$

If $\lim_{t\to\infty} h(t) = h(\infty)$ *exists and is finite, then* $\Psi_N^i(\Delta^i h)$ *is well defined on* $[0, \infty)$ *and*

$$h(t) = h(\infty) + (-1)^i \Psi_N^i(\Delta^i h)(t), \qquad t \geqq 0. \tag{1.13}$$

PROOF. From the hypothesis it follows that $(-1)^k(\Delta^k h)(t)$, $k=1, 2,\dots, i-1$, are nonnegative and nonincreasing on $[0, \infty)$, and hence $(-1)^k(\Delta^k h)(t)$, $k=0, 1,\dots, i-1$, have the finite limits $(-1)^k(\Delta^k h)(\infty)$ as $t \to \infty$. By Lemma 1.6 we have $\Delta^k h \in L_1^1(0, \infty)$ for $k=1, 2,\dots, i$. Since $(\Delta^k h)(\infty)$, $k=1, 2,\dots, i-1$, are finite, this means that $(\Delta^k h)(\infty)=0$ for $k=1, 2,\dots, i-1$. Thus, by Lemma 1.6 again,

$$(1.14)\qquad (\Delta^k h)(t) = -\Psi_N(\Delta^{k+1} h)(t), \quad t \geqq 0, \quad k=1, 2,\dots, i-1; \text{ and}$$

$$(1.15)\qquad h(t) = h(\infty) - \Psi_N(\Delta h)(t), \qquad t \geqq 0.$$

Combining (1.14) with (1.15), we see that $\Psi_N^k(\Delta^i h)$ is well defined for $k=1, 2,\dots, i$ and that (1.13) holds.

LEMMA 1.8. *Let $i \geqq 1$ be an integer and let $h \in C[0, \infty)$ be such that $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$. Then, $\Psi_N^i h$ is well defined on $[0, \infty)$ if and only if*

$$(1.16)\qquad N \geqq 2i+1 \quad \textit{and} \quad h \in L_{2i-1}^1(0, \infty).$$

If (1.16) *holds, then*

$$(1.17)\qquad c_N(i)\left(t^{-N+2i}\int_0^t s^{N-1}h(s)\,ds + \int_t^\infty s^{2i-1}h(s)\,ds\right)$$
$$\leqq (\Psi_N^i h)(t) \leqq d_N(i)\left(t^{-N+2i}\int_0^t s^{N-1}h(s)\,ds + \int_t^\infty s^{2i-1}h(s)\,ds\right)$$

for $t \geqq 0$, where $c_N(i)$ and $d_N(i)$ are positive constants defined by

$$(1.18)\qquad c_N(i) = 1/[(N-2)^i(N-4)\cdots(N-2i)],$$

$$(1.19)\qquad d_N(i) = 1/[2^{i-1}(N-2)(N-4)\cdots(N-2i)].$$

For the proof of Lemma 1.8, see Kusano, Naito and Swanson [12, Lemma 2.7]. The inequality (1.17) shows, in particular, that if $\Psi_N^i h$ is well defined on $[0, \infty)$ for $h \in C[0, \infty)$, $h(t) \geqq 0$ on $[0, \infty)$, then

$$(1.20)\qquad 0 \leqq (\Psi_N^i h)(t) \leqq d_N(i)\int_0^\infty s^{2i-1}h(s)\,ds < \infty, \qquad t \geqq 0,$$

and furthermore this inequality (1.20) together with (1.8) implies

$$(1.21)\qquad 0 \leqq \Phi_N^i \Psi_N^k h(t) \leqq A_N(i, k)\,t^{2i}\int_0^\infty s^{2k-1}h(s)\,ds, \qquad t \geqq 0,$$

for $i=0, 1, 2,\dots$ and $k=1, 2,\dots$, where $A_N(i, k) = d_N(k)$ in the case of $i=0$ and $A_N(i, k) = a_N(i)d_N(k)/2$ in the case of $i \geqq 1$.

LEMMA 1.9. *Let $i \geqq 1$ be an integer. Suppose that $h \in C[0, \infty)$, $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$ and $(\Psi_N^i h)(t)$ is well defined on $[0, \infty)$. Then, the function $t^{N-2i}(\Psi_N^i h)(t)$ is nondecreasing in $t \in [0, \infty)$ and*

$$\lim_{t\to\infty} t^{N-2i}(\Psi_N^i h)(t) = e_N(i) \int_0^\infty s^{N-1} h(s)\, ds \in (0, \infty], \tag{1.22}$$

where $e_N(i) = 1/[2^{i-1}(i-1)!(N-2)(N-4)\cdots(N-2i)]$.

PROOF. The proof is done by induction. Let $i = 1$. Then, since

$$\frac{d}{dt}[t^{N-2}(\Psi_N h)(t)] = t^{N-3} \int_t^\infty sh(s)\, ds \geqq 0, \qquad t \geqq 0,$$

the nondecreasing property of $t^{N-2}(\Psi_N h)(t)$ on $[0, \infty)$ is clear. Moreover it is easy to see that

$$\lim_{t\to\infty} t^{N-2}(\Psi_N h)(t) = \lim_{t\to\infty}\left(\frac{1}{N-2}\int_0^t s^{N-1}h(s)\, ds + \frac{t^{N-2}}{N-2}\int_t^\infty sh(s)\, ds\right)$$
$$= \frac{1}{N-2}\int_0^\infty s^{N-1} h(s)\, ds.$$

Thus the assertion holds for $i = 1$. Assume that the assertion is true for some i, $i \geqq 1$. Suppose that $\Psi_N^{i+1}h$ is well defined on $[0, \infty)$ for an $h \in C[0, \infty)$ with $h(t) \geqq 0$, $\not\equiv 0$, $t \geqq 0$. Notice that we have $N \geqq 2i + 3$ by Lemma 1.8. Using the definition of $\Psi_N^{i+1}h = \Psi_N \Psi_N^i h$ and the nondecreasing property of $t^{N-2i}(\Psi_N^i h)(t)$ on $[0, \infty)$, we find that

$$\frac{d}{dt}[t^{N-2i-2}(\Psi_N^{i+1}h)(t)]$$
$$= -t^{N-2i-2} \cdot t^{-N+1}\int_0^t s^{N-1}(\Psi_N^i h)(s)\, ds + (N-2i-2)t^{N-2i-3}(\Psi_N^{i+1}h)(t)$$
$$= -\frac{2i}{N-2} t^{-2i-1}\int_0^t s^{2i-1} \cdot s^{N-2i}(\Psi_N^i h)(s)\, ds$$
$$+ \frac{N-2i-2}{N-2} t^{N-2i-3}\int_t^\infty s^{-N+2i+1} s^{N-2i}(\Psi_N^i h)(s)\, ds$$
$$\geqq -\frac{2i}{N-2} t^{-2i-1} \cdot t^{N-2i}(\Psi_N^i h)(t)\int_0^t s^{2i-1}\, ds$$
$$+ \frac{N-2i-2}{N-2} t^{N-2i-3} \cdot t^{N-2i}(\Psi_N^i h)(t)\int_t^\infty s^{-N+2i+1}\, ds$$
$$= 0 \qquad \text{for } t \geqq 0,$$

which implies that $t^{N-2i-2}(\Psi_N^{i+1}h)(t)$ is nondecreasing on $[0, \infty)$. Since (1.22) is assumed to hold, application of L'Hospital's rule shows that

$$\begin{aligned}
&\lim_{t\to\infty} t^{N-2i-2}(\Psi_N^{i+1}h)(t)\\
&= \frac{1}{N-2}\lim_{t\to\infty}\left(\frac{1}{t^{2i}}\int_0^t s^{N-1}(\Psi_N^i h)(s)\,ds + \frac{1}{t^{-N+2i+2}}\int_t^\infty s(\Psi_N^i h)(s)\,ds\right)\\
&= \frac{1}{N-2}\lim_{t\to\infty}\left(\frac{t^{N-1}(\Psi_N^i h)(t)}{2it^{2i-1}} - \frac{t(\Psi_N^i h)(t)}{(-N+2i+2)t^{-N+2i+1}}\right)\\
&= \frac{1}{2i(N-2i-2)}\lim_{t\to\infty} t^{N-2i}(\Psi_N^i h)(t)\\
&= e_N(i+1)\int_0^\infty s^{N-1}h(s)\,ds,
\end{aligned}$$

which proves (1.22) with i replaced by $i+1$. This completes the proof of Lemma 1.9.

LEMMA 1.10. *Let $i \geqq 1$ be an integer, and suppose that $h \in C[0, \infty)$, $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$ and that $(\Psi_N^i h)(t)$ is well defined on $[0, \infty)$. Define the function $q_{N,i}(t)$ by*

$$q_{N,i}(t) = \min\{1, t^{-N+2i}\}, \qquad t \geqq 0. \tag{1.23}$$

(i) *Then,*

$$(\Psi_N^i h)(t) \geqq c_N(i)\int_0^\infty \min\{s^{2i-1}, s^{N-1}\}\,h(s)\,ds\cdot q_{N,i}(t), \qquad t \geqq 0. \tag{1.24}$$

(ii) *If in addition $h \in L^1_{N-1}(0, \infty)$, then*

$$(\Psi_N^i h)(t) \leqq d_N(i)\int_0^\infty \max\{s^{2i-1}, s^{N-1}\}\,h(s)\,ds\cdot q_{N,i}(t), \qquad t \geqq 0. \tag{1.25}$$

Here $c_N(i)$ and $d_N(i)$ are positive constants defined by (1.18) *and* (1.19), *respectively.*

PROOF. (i) By Lemma 1.9 the function $t^{N-2i}(\Psi_N^i h)(t)$ is nondecreasing on $[0, \infty)$, and hence, in particular, $t^{N-2i}(\Psi_N^i h)(t) \geqq (\Psi_N^i h)(1)$ for $t \in [1, \infty)$. On the other hand, the nonincreasing property of $(\Psi_N^i h)(t)$ implies that $(\Psi_N^i h)(t) \geqq (\Psi_N^i h)(1)$ for $t \in [0, 1]$. Therefore we have

$$(\Psi_N^i h)(t) \geqq (\Psi_N^i h)(1)\cdot q_{N,i}(t), \qquad t \geqq 0. \tag{1.26}$$

From inequality (1.17) it follows that

$$(\Psi_N^i h)(1) \geqq c_N(i)\left(\int_0^1 s^{N-1} h(s)\,ds + \int_1^\infty s^{2i-1} h(s)\,ds\right)$$

or equivalently

$$(\Psi_N^i h)(1) \geqq c_N(i)\int_0^\infty \min\{s^{2i-1}, s^{N-1}\}\, h(s)\,ds,$$

implying, together with (1.26), that the desired inequality (1.24) holds.

(ii) Since $t^{N-2i}(\Psi_N^i h)(t)$ tends nondecreasingly to $e_N(i)\int_0^\infty s^{N-1}h(s)\,ds$ as $t \to \infty$, we have

$$t^{N-2i}(\Psi_N^i h)(t) \leqq e_N(i)\int_0^\infty s^{N-1} h(s)\,ds, \qquad t \geqq 0,$$

and in particular

$$t^{N-2i}(\Psi_N^i h)(t) \leqq d_N(i)\int_0^\infty s^{N-1} h(s)\,ds, \qquad t \geqq 1.$$

By the nonincreasing property of $\Psi_N^i h$ and inequality (1.17) we have

$$(\Psi_N^i h)(t) \leqq (\Psi_N^i h)(0) \leqq d_N(i)\int_0^\infty s^{2i-1} h(s)\,ds$$

for $t \geqq 0$ and in particular for $0 \leqq t \leqq 1$. Then we easily see that (1.25) holds.

LEMMA 1.11. *Let i and k be integers, $i, k \geqq 1$. Suppose that $h \in C[0, \infty)$, $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$, and that $\Psi_N^k h$ is well defined for $t \geqq 0$ (that is, $N \geqq 2k+1$ and $h \in L^1_{2k-1}(0, \infty)$). Then,*

(i) *there exists a positive constant $B_N(i, k)$ such that*

$$(1.27) \qquad (\Phi_N^i \Psi_N^k h)(t) \geqq B_N(i, k)\left(t^{2i}\int_t^\infty s^{2k-1} h(s)\,ds + t^{2i-2}\int_0^t s^{2k+1} h(s)\,ds\right), \qquad t \geqq 0;$$

(ii) *if in addition $N \geqq 2k+3$, there exists a positive constant $C_N(i, k)$ such that*

$$(1.28) \qquad (\Phi_N^i \Psi_N^k h)(t) \leqq C_N(i, k)\left(t^{2i}\int_t^\infty s^{2k-1} h(s)\,ds + t^{2i-2}\int_0^t s^{2k+1} h(s)\,ds\right), \qquad t \geqq 0.$$

To prove (i) of Lemma 1.11, the following lemma is needed.

LEMMA 1.12. *Let i be an integer, $i \geqq 1$. Suppose that $h \in C[0, \infty)$ is a nonnegative nonincreasing function on $[0, \infty)$. Then,*

$$(1.29) \qquad (\Phi_N^i h)(t) \geqq 2\rho_N(i) t^{2i-2} \int_0^t sh(s)\,ds, \qquad t \geqq 0.$$

PROOF. Fix t in $(0, \infty)$. We may assume that $\int_0^t sh(s)\,ds > 0$. From the Cauchy generalized mean value theorem it follows that

$$(1.30) \qquad \frac{(\Phi_N h)(t)}{\int_0^t sh(s)\,ds} = \frac{\int_0^t s^{N-3} \int_0^s rh(r)\,dr\,ds}{t^{N-2} \int_0^t sh(s)\,ds}$$

$$= \frac{\xi^{N-3} \int_0^\xi rh(r)\,dr}{\xi^{N-2} \cdot \xi h(\xi) + (N-2)\xi^{N-3} \int_0^\xi sh(s)\,ds}$$

$$= \frac{\int_0^\xi rh(r)\,dr}{\xi^2 h(\xi) + (N-2) \int_0^\xi sh(s)\,ds}$$

for some $\xi \in (0, t)$. Note that

$$(1.31) \qquad \xi^2 h(\xi) \leqq 2 \int_0^\xi sh(s)\,ds, \qquad \xi \geqq 0,$$

for any nonnegative nonincreasing function $h \in C[0, \infty)$. Then (1.30) and (1.31) lead us to

$$(\Phi_N h)(t) \geqq \frac{1}{N} \int_0^t sh(s)\,ds, \qquad t \geqq 0,$$

which implies that (1.29) holds for $i = 1$. Assume that (1.29) is satisfied for some $i \geqq 1$. Using the Cauchy mean value theorem and (1.31) again, we find that

$$\frac{(\Phi_N^{i+1}h)(t)}{t^{2i}\int_0^t sh(s)\,ds} = \frac{\int_0^t s^{N-3}\int_0^s r(\Phi_N^i h)(r)\,dr\,ds}{t^{N+2i-2}\int_0^t sh(s)\,ds} \tag{1.32}$$

$$= \frac{\xi^{N-3}\int_0^\xi r(\Phi_N^i h)(r)\,dr}{\xi^{N+2i-2}\cdot\xi h(\xi) + (N+2i-2)\xi^{N+2i-3}\int_0^\xi sh(s)\,ds}$$

$$\geqq \frac{\int_0^\xi r(\Phi_N^i h)(r)\,dr}{(N+2i)\xi^{2i}\int_0^\xi sh(s)\,ds}$$

for some $\xi \in (0, t)$. Exactly as in the above, the last term of (1.32) can be estimated as follows:

$$\frac{\int_0^\xi r(\Phi_N^i h)(r)\,dr}{\xi^{2i}\int_0^\xi sh(s)\,ds} = \frac{\eta(\Phi_N^i h)(\eta)}{\eta^{2i}\cdot\eta h(\eta) + 2i\eta^{2i-1}\int_0^\eta sh(s)\,ds} \tag{1.33}$$

$$\geqq \frac{(\Phi_N^i h)(\eta)}{2(i+1)\eta^{2i-2}\int_0^\eta sh(s)\,ds}$$

for some $\eta \in (0, \xi)$ $(\subset (0, t))$. From (1.29), (1.32) and (1.33) it follows that

$$\frac{(\Phi_N^{i+1}h)(t)}{t^{2i}\int_0^t sh(s)\,ds} \geqq \frac{2\rho_N(i)}{2(i+1)(N+2i)} = 2\rho_N(i+1),$$

which proves (1.29) with i replaced by $i+1$. The inductive proof of Lemma 1.12 is complete.

PROOF OF LEMMA 1.11. (i) Let $N \geqq 2k+1$ and $h \in C[0, \infty) \cap L^1_{2k-1}(0, \infty)$, $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$. By (1.17) in Lemma 1.8 we have

$$(\Psi_N^k h)(t) \geqq c_N(k)\int_t^\infty s^{2k-1}h(s)\,ds, \qquad t \geqq 0.$$

Since $(\Psi_N^k h)(t)$ is nonnegative and nonincreasing on $[0, \infty)$, it follows from Lemma 1.12 that

$$(\Phi_N^i \Psi_N^k h)(t) \geqq 2\rho_N(i) t^{2i-2} \int_0^t s(\Psi_N^k h)(s)\,ds, \qquad t \geqq 0.$$

Consequently we have

$$\begin{aligned}
&(\Phi_N^i \Psi_N^k h)(t) \\
&\geqq 2\rho_N(i) c_N(k) t^{2i-2} \int_0^t s\left(\int_s^\infty r^{2k-1} h(r)\,dr\right) ds \\
&= \rho_N(i)\, c_N(k) \left(t^{2i} \int_t^\infty s^{2k-1} h(s)\,ds + t^{2i-2} \int_0^t s^{2k+1} h(s)\,ds \right), \qquad t \geqq 0,
\end{aligned}$$

which implies that (1.27) holds for $B_N(i, k) = \rho_N(i) c_N(k)$.

(ii) Let $N \geqq 2k + 3$ and $h \in C[0, \infty) \cap L^1_{2k-1}(0, \infty)$, $h(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$. By Lemma 1.4 we have

$$(\Phi_N^i \Psi_N^k h)(t) \leqq a_N(i) t^{2i-2} \int_0^t s(\Psi_N^k h)(s)\,ds, \qquad t \geqq 0.$$

Using the upper estimate for $\Psi_N^k h$ given in (1.17), we obtain

$$\begin{aligned}
&(\Phi_N^i \Psi_N^k h)(t) \\
&\leqq a_N(i)\, d_N(k)\, t^{2i-2} \left(\int_0^t s^{-N+2k+1} \int_0^s r^{N-1} h(r)\,dr\,ds + \int_0^t s \int_s^\infty r^{2k-1} h(r)\,dr\,ds \right) \\
&= a_N(i) d_N(k) t^{2i-2} \left(\frac{t^{-N+2k+2}}{-N+2k+2} \int_0^t s^{N-1} h(s)\,ds \right. \\
&\quad \left. + \frac{N-2k}{2(N-2k-2)} \int_0^t s^{2k+1} h(s)\,ds + \frac{1}{2} t^2 \int_t^\infty s^{2k-1} h(s)\,ds \right), \qquad t \geqq 0.
\end{aligned}$$

In view of the assumption $-N + 2k + 2 < 0$, we see that (1.28) is satisfied for

$$C_N(i, k) = a_N(i) d_N(k) \max\left\{ \frac{N-2k}{2(N-2k-2)}, \frac{1}{2} \right\}.$$

This completes the proof of Lemma 1.11.

2. Kiguradze's classes

THEOREM 2.1. *Let $m \geqq 2$ be an integer and $\sigma = +1$ or $\sigma = -1$. Suppose that $u \in \mathscr{D}^m[0, \infty)$ has no zero in $[0, \infty)$ and satisfies*

$$(2.1) \qquad (-1)^m \sigma u(t) \Delta^m u(t) \leqq 0, \qquad t \geqq 0.$$

Then there exist an integer $j \in \{0, 1, \dots, m\}$ and $T_u \geqq 0$ such that

(2.2) j *is odd if* $\sigma = +1$; j *is even if* $\sigma = -1$

and

$$(2.3)_j \qquad \begin{cases} u(t)\Delta^i u(t) > 0, & 0 \leqq i \leqq j-1, \quad \text{for } t \geqq T_u, \\ (-1)^{i-j}u(t)\Delta^i u(t) \geqq 0, & j \leqq i \leqq m, \quad \text{for } t \geqq 0. \end{cases}$$

REMARK 2.1. (i) Condition (2.2) may be rewritten as $(-1)^{j+1}\sigma = 1$.
(ii) If $j = 0$, then $(2.3)_j$ reduces to

$$(2.3)_0 \qquad u(t) \neq 0 \quad \text{and} \quad (-1)^i u(t)\Delta^i u(t) \geqq 0, \quad 1 \leqq i \leqq m, \qquad \text{for } t \geqq 0.$$

PROOF OF THEOREM 2.1. To prove the existence of j and T_u satisfying (2.2) and $(2.3)_j$, it is convenient to distinguish the two cases: $(-1)^m\sigma = -1$ and $(-1)^m\sigma = +1$.

(I) *The case of* $(-1)^m\sigma = -1$. Without loss of generality we may assume that $u(t) > 0$ for $t \geqq 0$. The hypothesis (2.1) then implies $\Delta^m u(t) \geqq 0$ for $t \geqq 0$. By Lemma 1.1 applied to $h = \Delta^{m-1}u$, $\Delta^{m-1}u(t)$ is nondecreasing on $[0, \infty)$. There are two possibilities for $\Delta^{m-1}u(t)$:

(a_{m-1}) $\qquad \Delta^{m-1}u(t_{m-1}) > 0$ for some $t_{m-1} > 0$; or

(b_{m-1}) $\qquad \Delta^{m-1}u(t) \leqq 0$ for every $t \geqq 0$.

Suppose that the case (a_{m-1}) occurs. Since $\Delta^{m-1}u(t) \geqq \Delta^{m-1}u(t_{m-1}) > 0$ for $t \geqq t_{m-1}$, successive application of Lemma 1.2 shows, in particular, that $\Delta^i u(t) > 0$ $(0 \leqq i \leqq m-1)$ for all sufficiently large t. Then (2.2) and $(2.3)_j$ hold for $j = m$. Suppose that the case (b_{m-1}) occurs. Then $\Delta^{m-2}u(t)$ is nonnegative for $t \geqq 0$. Indeed, if $\Delta^{m-2}u$ takes a negative value at some $t_{m-2} > 0$, then $\Delta^{m-2}u(t) \leqq \Delta^{m-2}u(t_{m-2}) < 0$ for $t \geqq t_{m-2}$ since $\Delta^{m-2}u$ is nonincreasing on $[0, \infty)$. Repeated application of Lemma 1.2 shows that $\Delta^i u(t) < 0$ $(0 \leqq i \leqq m-2)$ for all large t, which is a contradiction to the assumption that $u(t) > 0$ on $[0, \infty)$. Thus $\Delta^{m-2}u(t) \geqq 0$ for $t \geqq 0$. Arguing exactly as in the above discussions starting from the fact that $\Delta^m u(t) \geqq 0$ for $t \geqq 0$, we have the two possibilities:

(a_{m-3}) $\qquad \Delta^{m-3}u(t_{m-3}) > 0$ for some $t_{m-3} > 0$; or

(b_{m-3}) $\qquad \Delta^{m-3}u(t) \leqq 0$ for every $t \geqq 0$.

If (a_{m-3}) occurs, then (2.2) and $(2.3)_j$ hold for $j = m-2$. If (b_{m-3}) occurs, then $\Delta^{m-4}u(t) \geqq 0$ on $[0, \infty)$. In this case, repeating the above procedure, we can conclude without difficulty that $(2.3)_j$ holds for some $j \in \{0, 1, \dots, m\}$, which is even or odd according as m is even $(\sigma = -1)$ or m is odd $(\sigma = +1)$.

(II) *The case of* $(-1)^m\sigma = +1$. We may assume that $u(t) > 0$ for $t \geqq 0$. By Lemma 1.1, $\Delta^{m-1}u$ is nonincreasing on $[0, \infty)$. If $\Delta^{m-1}u(t_{m-1}) < 0$

for some $t_{m-1} > 0$, then Lemma 1.2 implies that $\Delta^i u(t) < 0$ $(0 \leqq i \leqq m-1)$ for all large t, contradicting the positivity of u. Thus $\Delta^{m-1} u(t) \geqq 0$ for $t \geqq 0$. Therefore, discussing as in the case of (I), we can show that there exist an integer $j \in \{0, 1, \dots, m-1\}$ and T_u satisfying (2.2) and $(2.3)_j$ with m replaced by $m-1$. It is clear that $(-1)^{m-j}\Delta^m u(t) \geqq 0$ for $t \geqq 0$. This completes the proof of Theorem 2.1.

Let $m \geqq 2$ be an integer and let $\sigma = +1$ or $\sigma = -1$. Let $\mathscr{K}$ denote the set of all functions $u \in \mathscr{D}^m[0, \infty)$ that have no zero in $[0, \infty)$ and satisfy $(-1)^m \sigma u(t) \Delta^m u(t) \leqq 0$ for $t \geqq 0$. For an integer $j \in \{0, 1, \dots, m\}$, denote by $\mathscr{K}_j$ the set of all functions $u \in \mathscr{K}$ satisfying $(2.3)_j$. The set $\mathscr{K}_j$ will be called the *Kiguradze class* of degree j. Theorem 2.1 means that $\mathscr{K}$ has the decomposition

$$\begin{aligned}
\mathscr{K} &= \mathscr{K}_1 \cup \mathscr{K}_3 \cup \cdots \cup \mathscr{K}_{m-1} && \text{for } m \text{ even}, \quad \sigma = +1,\\
\mathscr{K} &= \mathscr{K}_0 \cup \mathscr{K}_2 \cup \cdots \cup \mathscr{K}_{m-1} && \text{for } m \text{ odd}, \quad \sigma = -1,\\
\mathscr{K} &= \mathscr{K}_0 \cup \mathscr{K}_2 \cup \cdots \cup \mathscr{K}_{m} && \text{for } m \text{ even}, \quad \sigma = -1,\\
\mathscr{K} &= \mathscr{K}_1 \cup \mathscr{K}_3 \cup \cdots \cup \mathscr{K}_{m} && \text{for } m \text{ odd}, \quad \sigma = +1.
\end{aligned}$$

THEOREM 2.2. (i) *Let* $u \in \mathscr{K}_m$. *Then*

$$u(t) = \sum_{k=0}^{m-1} \rho_N(k)(\Delta^k u)(0) t^{2k} + \Phi_N^m(\Delta^m u)(t), \qquad t \geqq 0. \tag{2.4}$$

(ii) *Let* $u \in \mathscr{K}_j$ *for* $1 \leqq j \leqq m-1$. *Then* $\Psi_N^{m-j}(\Delta^m u)$ *is well defined on* $[0, \infty)$, $\Delta^j u(\infty) = \lim_{t\to\infty}(\Delta^j u)(t)$ *exists and is finite and*

$$\begin{aligned}
u(t) = \sum_{k=0}^{j-1} \rho_N(k)(\Delta^k u)(0) t^{2k} &+ \rho_N(j)(\Delta^j u)(\infty) t^{2j}\\
&+ (-1)^{m-j}\Phi_N^j \Psi_N^{m-j}(\Delta^m u)(t), \qquad t \geqq 0.
\end{aligned} \tag{2.5}$$

(iii) *Let* $u \in \mathscr{K}_0$. *Then* $\Psi_N^m(\Delta^m u)$ *is well defined on* $[0, \infty)$, $u(\infty) = \lim_{t\to\infty} u(t)$ *exists and is finite and*

$$u(t) = u(\infty) + (-1)^m \Psi_N^m(\Delta^m u)(t), \qquad t \geqq 0. \tag{2.6}$$

PROOF. Suppose that $u \in \mathscr{K}_j$ $(0 \leqq j \leqq m)$. Let $1 \leqq j \leqq m$. Lemma 1.3 with $i = j$ and $h = u$ shows that

$$u(t) = \sum_{k=0}^{j-1} \rho_N(k)(\Delta^k u)(0) t^{2k} + \Phi_N^j(\Delta^j u)(t), \qquad t \geqq 0. \tag{2.7}$$

If $j = m$, then (2.7) becomes (2.4).

Let $0 \leqq j \leqq m-1$ and let $u(t) > 0$ for $t \geqq 0$. Then, by $(2.3)_j$, $\Delta^j u(t)$ is nonnegative and nonincreasing on $[0, \infty)$. The limit $\Delta^j u(\infty) = \lim_{t\to\infty}(\Delta^j u)(t)$

exists and is finite. From Lemma 1.7 applied to the case $i = m - j$ and $h = \Delta^j u$, we see that $\Psi_N^{m-j}(\Delta^m u)$ is well defined on $[0, \infty)$ and

$$\Delta^j u(t) = \Delta^j u(\infty) + (-1)^{m-j} \Psi_N^{m-j}(\Delta^m u)(t), \qquad t \geqq 0. \tag{2.8}$$

If $j = 0$, then (2.8) reduces to (2.6). If $1 \leqq j \leqq m - 1$, then, in view of the identity

$$\Phi_N^i(1)(t) = \rho_N(i) t^{2i}, \qquad t \geqq 0, \quad i = 1, 2, \dots, \tag{2.9}$$

(2.7) and (2.8) together yield (2.5). The proof of Theorem 2.2 is complete.

Let us study the asymptotic behavior as $t \to \infty$ of u in the Kiguradze class $\mathscr{K}_j$ $(0 \leqq j \leqq m)$. First suppose that $u \in \mathscr{K}_m$. Since

$$\Delta^i t^{2k} = \begin{cases} \dfrac{\rho_N(k-i)}{\rho_N(k)} t^{2k-2i} & \text{for } i \leqq k \\ 0 & \text{for } i > k, \end{cases} \tag{2.10}$$

application of the operators Δ^i to (2.4) yields

$$\Delta^i u(t) = \sum_{k=i}^{m-1} \rho_N(k-i)(\Delta^k u)(0) t^{2k-2i} + \Phi_N^{m-i}(\Delta^m u)(t)$$

for $t \geqq 0$, $0 \leqq i \leqq m - 1$, and hence by Lemma 1.4 we have

$$\begin{aligned} \lim_{t \to \infty} \frac{\Delta^i u(t)}{t^{2(m-i-1)}} &= \rho_N(m-i-1)(\Delta^{m-1} u)(0) + b_N(m-i) \int_0^\infty s(\Delta^m u)(s)\, ds \\ &= \rho_N(m-i-1)\left[(\Delta^{m-1} u)(0) + \frac{1}{N-2} \int_0^\infty s(\Delta^m u)(s)\, ds\right] \end{aligned}$$

for $0 \leqq i \leqq m - 1$. In particular,

$$\lim_{t \to \infty} \Delta^{m-1} u(t) = (\Delta^{m-1} u)(0) + \frac{1}{N-2} \int_0^\infty s(\Delta^m u)(s)\, ds,$$

which does not vanish because $\Delta^{m-1} u(t)$ is eventually positive [resp. negative] and nondecreasing [resp. nonincreasing] on $[0, \infty)$ if $u \in \mathscr{K}_m$ is positive [resp. negative]. Therefore we can conclude that exactly one of the following two cases holds for $u \in \mathscr{K}_m$:

$$\text{(2.11)}_m \quad \begin{cases} \lim\limits_{t \to \infty} \dfrac{\Delta^i u(t)}{t^{2(m-i-1)}}, \quad 0 \leqq i \leqq m - 1, \quad \text{are equal to} \\ \text{either } +\infty \text{ or } -\infty; \end{cases}$$

$$(2.12)_m \qquad \begin{cases} \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(m-i-1)}}, \quad 0 \leqq i \leqq m-1, \quad \text{exist and are either} \\ \text{positive finite values or negative finite values.} \end{cases}$$

The set of all $u \in \mathscr{K}_m$ satisfying $(2.11)_m$ or $(2.12)_m$ is denoted, respectively, by $\mathscr{K}_m[\text{inc}]$ or $\mathscr{K}_m[\text{min}]$.

Let $u \in \mathscr{K}_j$ for $1 \leqq j \leqq m-1$. In this case, it follows from (2.5) that

$$(2.13) \qquad \Delta^i u(t) = \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0)\, t^{2k-2i} + \rho_N(j-i)(\Delta^j u)(\infty)\, t^{2j-2i} + (-1)^{m-j} \Phi_N^{j-i} \Psi_N^{m-j}(\Delta^m u)(t), \qquad t \geqq 0,$$

for $0 \leqq i \leqq j-1$ and

$$(2.14) \qquad (\Delta^j u)(t) = (\Delta^j u)(\infty) + (-1)^{m-j} \Psi_N^{m-j}(\Delta^m u)(t), \qquad t \geqq 0.$$

Using Lemma 1.5 and noticing that $\lim_{t\to\infty}(\Psi_N h)(t) = 0$ for any $h \in C[0, \infty) \cap L_1^1(0, \infty)$, we obtain

$$\lim_{t\to\infty} \frac{(\Delta^i u)(t)}{t^{2(j-i)}} = \rho_N(j-i)(\Delta^j u)(\infty) + (-1)^{m-j} \rho_N(j-i) \lim_{t\to\infty} \Psi_N^{m-j}(\Delta^m u)(t)$$
$$= \rho_N(j-i)(\Delta^j u)(\infty)$$

for $0 \leqq i \leqq j$. If $(\Delta^j u)(\infty) = 0$, then Lemma 1.4 shows that

$$\lim_{t\to\infty} \frac{(\Delta^i u)(t)}{t^{2(j-i-1)}}$$
$$= \rho_N(j-i-1)(\Delta^{j-1} u)(0) + (-1)^{m-j} b_N(j-i) \int_0^\infty s \Psi_N^{m-j}(\Delta^m u)(s)\, ds$$
$$= \rho_N(j-i-1)\left[(\Delta^{j-1} u)(0) + \frac{(-1)^{m-j}}{N-2} \int_0^\infty s \Psi_N^{m-j}(\Delta^m u)(s)\, ds\right]$$

for $0 \leqq i \leqq j-1$. Note that

$$\lim_{t\to\infty} (\Delta^{j-1} u)(t) = (\Delta^{j-1} u)(0) + \frac{(-1)^{m-j}}{N-2} \int_0^\infty s \Psi_N^{m-j}(\Delta^m u)(s)\, ds$$

is not zero, because $(\Delta^{j-1} u)(t)$ is eventually positive [resp. negative] and nondecreasing [resp. nonincreasing] on $[0, \infty)$ if $u \in \mathscr{K}_j$ is positive [resp. negative]. Consequently one of the following three cases can occur for $u \in \mathscr{K}_j$, $1 \leqq j \leqq m-1$:

$$(2.15)_j \quad \begin{cases} \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(j-i)}}, \quad 0 \leqq i \leqq j, \quad \text{exist and are either} \\ \text{positive finite values or negative finite values;} \end{cases}$$

$$(2.16)_j \quad \begin{cases} \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(j-i)}} = 0, \quad 0 \leqq i \leqq j, \text{ and} \\ \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(j-i-1)}}, \quad 0 \leqq i \leqq j-1, \quad \text{are equal to either } +\infty \text{ or } -\infty; \end{cases}$$

$$(2.17)_j \quad \begin{cases} \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(j-i)}} = 0, \quad 0 \leqq i \leqq j, \text{ and} \\ \lim_{t\to\infty} \dfrac{\Delta^i u(t)}{t^{2(j-i-1)}}, \quad 0 \leqq i \leqq j-1, \text{ exist and are positive finite values} \\ \text{or negative finite values.} \end{cases}$$

Denote by $\mathscr{K}_j[\max]$, $\mathscr{K}_j[\mathrm{int}]$ and $\mathscr{K}_j[\min]$ the sets of all functions u in $\mathscr{K}_j$ satisfying $(2.15)_j$, $(2.16)_j$ and $(2.17)_j$, respectively.

Finally suppose that $u \in \mathscr{K}_0$. We have (2.6). If $\Delta^m u(t) \equiv 0$ on $[0, \infty)$, then $u(t)$ is a nonzero constant function. If $\Delta^m u(t) \not\equiv 0$ on $[0, \infty)$, then Lemma 1.8 implies in particular that $N \geqq 2m+1$, because $\Psi_N^m(\Delta^m u)$ is well defined on $[0, \infty)$. If in addition $u(\infty) = 0$, then, using Lemma 1.9, we see from (2.6) that

$$\lim_{t\to\infty} t^{N-2m} u(t) = (-1)^m e_N(m) \int_0^\infty s^{N-1} (\Delta^m u)(s)\, ds.$$

Therefore we conclude that exactly one of the following three cases holds for each non-constant function $u \in \mathscr{K}_0$:

$$(2.18)_0 \qquad \lim_{t\to\infty} u(t) \text{ exists and is nonzero finite value;}$$

$$(2.19)_0 \qquad \lim_{t\to\infty} u(t) = 0 \text{ and } \lim_{t\to\infty} t^{N-2m} u(t) = +\infty \text{ or } -\infty;$$

$$(2.20)_0 \qquad \lim_{t\to\infty} t^{N-2m} u(t) \text{ exists and is a nonzero finite value.}$$

We denote by $\mathscr{K}_0[\max]$, $\mathscr{K}_0[\mathrm{int}]$ and $\mathscr{K}_0[\min]$ the sets of all functions u in $\mathscr{K}_0$ satisfying $(2.18)_0$, $(2.19)_0$ and $(2.20)_0$, respectively.

From the above observation we obtain the following result.

THEOREM 2.3. (i) *If $u \in \mathscr{K}_m$, then there are constants $c_* > 0$ and $T \geqq 1$ such that*

(2.21) $$|\Delta^i u(t)| \geqq c_* t^{2(m-i-1)}, \quad t \geqq T, \quad 0 \leqq i \leqq m-1.$$

(ii) *If* $u \in \mathscr{K}_j$ *for* $1 \leqq j \leqq m-1$, *then there are constants* $c_* > 0$, $c^* > 0$ *and* $T \geqq 1$ *such that*

(2.22) $$\begin{cases} c_* t^{2(j-i-1)} \leqq |\Delta^i u(t)| \leqq c^* t^{2(j-i)}, & t \geqq T, \quad 0 \leqq i \leqq j-1, \\ |\Delta^j u(t)| \leqq c^*, & t \geqq T. \end{cases}$$

(iii) *If* $u \in \mathscr{K}_0$, *then there are constants* $c_* > 0$, $c^* > 0$ *and* $T \geqq 1$ *such that*

(2.23) $$c_* t^{-N+2m} \leqq |u(t)| \leqq c^*, \qquad t \geqq T.$$

By definition, the Kiguradze classes $\mathscr{K}_j$, $0 \leqq j \leqq m$, are decomposed as follows:

$$\mathscr{K}_m = \mathscr{K}_m[\text{inc}] \cup \mathscr{K}_m[\text{min}],$$
$$\mathscr{K}_j = \mathscr{K}_j[\text{max}] \cup \mathscr{K}_j[\text{int}] \cup \mathscr{K}_j[\text{min}] \qquad \text{for} \quad 0 \leqq j \leqq m-1.$$

Sometimes it is useful to note that

$$\begin{aligned} \mathscr{K}_m[\text{inc}] &= \{u \in \mathscr{K}_m \colon \lim_{t\to\infty} \Delta^{m-1} u(t) = +\infty \text{ or } -\infty\} \\ &= \{u \in \mathscr{K}_m \colon \lim_{t\to\infty} u(t)/t^{2(m-1)} = +\infty \text{ or } -\infty\}; \end{aligned}$$

$$\begin{aligned} \mathscr{K}_m[\text{min}] &= \{u \in \mathscr{K}_m \colon \lim_{t\to\infty} \Delta^{m-1} u(t) \text{ exists in } \boldsymbol{R} - \{0\}\} \\ &= \{u \in \mathscr{K}_m \colon \lim_{t\to\infty} u(t)/t^{2(m-1)} \text{ exists in } \boldsymbol{R} - \{0\}\}; \end{aligned}$$

and for $1 \leqq j \leqq m-1$

$$\begin{aligned} \mathscr{K}_j[\text{max}] &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} \Delta^j u(t) \text{ exists in } \boldsymbol{R} - \{0\}\} \\ &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} u(t)/t^{2j} \text{ exists in } \boldsymbol{R} - \{0\}\}; \end{aligned}$$

$$\begin{aligned} \mathscr{K}_j[\text{int}] &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} \Delta^j u(t) = 0,\ \lim_{t\to\infty} \Delta^{j-1} u(t) = +\infty \text{ or } -\infty\} \\ &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} u(t)/t^{2j} = 0,\ \lim_{t\to\infty} u(t)/t^{2(j-1)} = +\infty \text{ or } -\infty\}; \end{aligned}$$

$$\begin{aligned} \mathscr{K}_j[\text{min}] &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} \Delta^{j-1} u(t) \text{ exists in } \boldsymbol{R} - \{0\}\} \\ &= \{u \in \mathscr{K}_j \colon \lim_{t\to\infty} u(t)/t^{2(j-1)} \text{ exists in } \boldsymbol{R} - \{0\}\}. \end{aligned}$$

If $u \in \mathscr{K}_j[\text{min}]$ for $1 \leqq j \leqq m$, then the integral representation (2.4) or (2.5) for u can be refined as follows:

THEOREM 2.4. *Let* $u \in \mathscr{K}_j[\min]$ *for* $1 \leqq j \leqq m$. *Then* $\Psi_N^{m-j+1}(\Delta^m u)$ *is well defined on* $[0, \infty)$ *and*

$$u(t) = \sum_{k=0}^{j-2} \rho_N(k)(\Delta^k u)(0)\,t^{2k} + \rho_N(j-1)(\Delta^{j-1}u)(\infty)\,t^{2(j-1)} \tag{2.24}$$

$$+ (-1)^{m-j+1}\Phi_N^{j-1}\Psi_N^{m-j+1}(\Delta^m u)(t), \qquad t \geqq 0,$$

where $(\Delta^{j-1}u)(\infty) = \lim_{t\to\infty}(\Delta^{j-1}u)(t)$ *and for* $j = 1$ *the sum* $\sum_{k=0}^{j-2}$ *must be interpreted as* 0.

PROOF. Since $\Delta^{j-1}u(\infty)$ exists and is finite, Lemma 1.6 shows that $\Psi_N(\Delta^j u)$ is well defined on $[0, \infty)$ and

$$\Delta^{j-1}u(t) = \Delta^{j-1}u(\infty) - \Psi_N(\Delta^j u)(t), \qquad t \geqq 0. \tag{2.25}$$

On the other hand, we have

$$\Delta^j u(t) = (-1)^{m-j}\Psi_N^{m-j}(\Delta^m u)(t), \qquad t \geqq 0, \tag{2.26}$$

for $u \in \mathscr{K}_j[\min]$ $(1 \leqq j \leqq m)$. Indeed, if $j = m$, then (2.26) is trivial. If $1 \leqq j \leqq m-1$, then (2.8) in the proof of Theorem 2.2 reduces to (2.26) because of $\Delta^j u(\infty) = 0$. By (2.25) and (2.26) we see that $\Psi_N^{m-j+1}(\Delta^m u)$ is well defined on $[0, \infty)$ and

$$\Delta^{j-1}u(t) = \Delta^{j-1}u(\infty) + (-1)^{m-j+1}\Psi_N^{m-j+1}(\Delta^m u)(t), \qquad t \geqq 0.$$

Then the desired equality (2.24) follows from Lemma 1.3 and (2.9).

For $j \in \{0, 1, \dots, m\}$ and $(-1)^{j+1}\sigma = 1$, the set of all functions $u \in \mathscr{K}$ satisfying $(2.3)_j$ with $T_u = 0$, i.e.,

$$\begin{cases} u(t)\Delta^i u(t) > 0, & 0 \leqq i \leqq j-1, \quad \text{for } t \geqq 0, \\ (-1)^{i-j}u(t)\Delta^i u(t) \geqq 0, & j \leqq i \leqq m, \quad \text{for } t \geqq 0, \end{cases} \tag{$2.3)_j^*$}$$

is an important subset of $\mathscr{K}_j$. This subset of $\mathscr{K}_j$ is denoted by $\mathscr{K}_j^*$, and the sets $\mathscr{K}_j^* \cap \mathscr{K}_j[\max]$, $\mathscr{K}_j^* \cap \mathscr{K}_j[\mathrm{int}]$ and $\mathscr{K}_j^* \cap \mathscr{K}_j[\min]$ are denoted by $\mathscr{K}_j^*[\max]$, $\mathscr{K}_j^*[\mathrm{int}]$ and $\mathscr{K}_j^*[\min]$, respectively. For example, $\mathscr{K}_j^*[\max]$, $1 \leqq j \leqq m-1$, is the set of all $u \in \mathscr{K}$ such that u satisfies $(2.3)_j^*$ and $\lim_{t\to\infty} u(t)/t^{2j}$ exists in $\boldsymbol{R} - \{0\}$. If $j = 0$, then the sets $\mathscr{K}_0^*$, $\mathscr{K}_0^*[\max]$, $\mathscr{K}_0^*[\mathrm{int}]$ and $\mathscr{K}_0^*[\min]$ are identical to the sets without the asterisk, respectively.

3. Comparison theorems

In this section elliptic differential equations of the form (0.1) are considered under the assumption that $m \geqq 2$, $N \geqq 3$, $\sigma = +1$ or $\sigma = -1$, and f is continuous on $[0, \infty) \times \boldsymbol{R}^m$ and satisfies (0.2). From Theorem 2.1 it follows that a radial entire solution $u(t)$, $t = |x|$, of (0.1) with no zero in $\boldsymbol{R}^N$, if exists, falls into one and only one Kiguradze class $\mathscr{K}_j$ such that $(-1)^{j+1}\sigma = 1$, $0 \leqq j \leqq m$. The existence of a radial entire solution u in a given class $\mathscr{K}_j$ of equation (0.1) is guaranteed by the existence of a radial entire function $v(t)$, $t = |x|$, in the same class $\mathscr{K}_j$ satisfying the differential inequality

$$(3.1) \qquad \{\sigma(-1)^m \Delta^m v + g(|x|, v, \Delta v, \ldots, \Delta^{m-1} v)\} \operatorname{sgn} v \leqq 0, \qquad x \in \boldsymbol{R}^N.$$

Here the function g is continuous on $[0, \infty) \times \boldsymbol{R}^m$ and satisfies the sign condition

$$(3.2) \quad v_0 g(t, v_0, v_1, \ldots, v_{m-1}) \geqq 0, \ \not\equiv 0 \qquad \text{for } (t, v_0, v_1, \ldots, v_{m-1}) \in [0, \infty) \times \boldsymbol{R}^m.$$

Two comparison theorems illustrating such a situation are presented in this section. They will be crucial in proving our main existence theorems given in Sections 4 and 5.

THEOREM 3.1. *Let j be an integer such that $(-1)^{j+1}\sigma = 1$ and $0 \leqq j \leqq m$. Let g in* (3.1) *be continuous on $[0, \infty) \times \boldsymbol{R}^m$ and satisfy* (3.2). *Suppose that the following inequality holds*:

$$(3.3) \qquad f(t, u_0, u_1, \ldots, u_{m-1}) \operatorname{sgn} u_0 \leqq g(t, v_0, v_1, \ldots, v_{m-1}) \operatorname{sgn} v_0$$

for all $(t, u_0, u_1, \ldots, u_{m-1})$ and $(t, v_0, v_1, \ldots, v_{m-1})$ such that

$$(3.4)_j \qquad \begin{cases} t \geqq 0, \quad u_0 v_0 > 0, \quad 0 < u_i \operatorname{sgn} u_0 \leqq v_i \operatorname{sgn} v_0, & 0 \leqq i \leqq j-1, \\ 0 \leqq (-1)^{i-j} u_i \operatorname{sgn} u_0 \leqq (-1)^{i-j} v_i \operatorname{sgn} v_0, & j \leqq i \leqq m-1. \end{cases}$$

(i) *Let $j = m$. If there exists a radial entire function v of class $\mathscr{K}_m^*$ satisfying* (3.1), *then equation* (0.1) *has a radial entire solution u of the same class $\mathscr{K}_m^*$. Furthermore, if the function v is of class $\mathscr{K}_m^*$*[min], *then* (0.1) *has a radial entire solution u of class $\mathscr{K}_m^*$*[min].

(ii) *Let $1 \leqq j \leqq m-1$. If there exists a radial entire function v of class $\mathscr{K}_j^*$ satisfying* (3.1), *then equation* (0.1) *has a radial entire solution u of the same class $\mathscr{K}_j^*$. Furthermore, if the function v is of class $\mathscr{K}_j^*$*[max] [*resp.* $\mathscr{K}_j^*$[int] $\cup$ $\mathscr{K}_j^*$[min], $\mathscr{K}_j^*$[min]], *then* (0.1) *has a radial entire solution u of class $\mathscr{K}_j^*$*[max] [*resp.* $\mathscr{K}_j^*$[int] $\cup$ $\mathscr{K}_j^*$[min], $\mathscr{K}_j^*$[min]].

(iii) *Let $j = 0$. If there exists a radial entire function v of class $\mathscr{K}_0$*[max] *satisfying* (3.1), *then equation* (0.1) *has a radial entire solution u of class $\mathscr{K}_0$*[max].

As before, let $\mathscr{D}^{m-1}[0, \infty)$ denote the set of all functions u on $[0, \infty)$ such that $\Delta^i u(|x|)$, $0 \leqq i \leqq m-1$, are defined and continuous on $\boldsymbol{R}^N$. The $\mathscr{D}^{m-1}[0, \infty)$ becomes a Fréchet space with the topology induced by the family of seminorms

$$\|u\|_n = \sum_{k=0}^{m-1} \sup\{|\Delta^k u(t)| : 0 \leqq t \leqq n\}, \qquad n = 1, 2, \ldots.$$

For simplicity we use the notation

$$\bar{u}(t) = (u(t), \Delta u(t), \ldots, \Delta^{m-1}u(t)) \qquad \text{for } u \in \mathscr{D}^{m-1}[0, \infty). \tag{3.5}$$

PROOF OF THEOREM 3.1. *The proof of part* (ii). We first prove (ii). Let $1 \leqq j \leqq m-1$ and $v \in \mathscr{K}_j^*$. We may suppose that $v(t) > 0$ for $t \geqq 0$. Then we have $\Delta^i v(t) > 0$ $(0 \leqq i \leqq j-1)$ and $(-1)^{i-j}\Delta^i v(t) \geqq 0$ $(j \leqq i \leqq m-1)$ for $t \geqq 0$. From (ii) of Theorem 2.2 it follows that

$$v(t) = \sum_{k=0}^{j-1} \rho_N(k)(\Delta^k v)(0)t^{2k} + \rho_N(j)(\Delta^j v)(\infty)t^{2j}$$
$$+ (-1)^{m-j}\Phi_N^j \Psi_N^{m-j}(\Delta^m v)(t), \qquad t \geqq 0,$$

and so

$$\Delta^i v(t) = \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k v)(0)t^{2k-2i} + \rho_N(j-i)(\Delta^j v)(\infty)t^{2j-2i}$$
$$+ (-1)^{m-j}\Phi_N^{j-i}\Psi_N^{m-j}(\Delta^m v)(t), \qquad t \geqq 0,\ 0 \leqq i < j;$$
$$\Delta^j v(t) = \Delta^j v(\infty) + (-1)^{m-j}\Psi_N^{m-j}(\Delta^m v)(t), \qquad t \geqq 0; \text{ and}$$
$$(-1)^{i-j}\Delta^i v(t) = (-1)^{m-j}\Psi_N^{m-i}(\Delta^m v)(t), \qquad t \geqq 0,\ j < i \leqq m-1,$$

where $\Delta^j v(\infty) = \lim_{t\to\infty}\Delta^j v(t) \in [0, \infty)$. Therefore, by (3.1), we see that

$$\Delta^i v(t) \geqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k v)(0)t^{2k-2i} + \rho_N(j-i)(\Delta^j v)(\infty)t^{2j-2i} \tag{3.6}$$
$$+ \Phi_N^{j-i}\Psi_N^{m-j}g(\cdot, \bar{v})(t), \qquad t \geqq 0,\ 0 \leqq i < j;$$

$$\Delta^j v(t) \geqq \Delta^j v(\infty) + \Psi_N^{m-j}g(\cdot, \bar{v})(t), \qquad t \geqq 0; \text{ and} \tag{3.7}$$

$$(-1)^{i-j}\Delta^i v(t) \geqq \Psi_N^{m-i}g(\cdot, \bar{v})(t), \qquad t \geqq 0,\ j < i \leqq m-1. \tag{3.8}$$

Consider the set U of all $u \in \mathscr{D}^{m-1}[0, \infty)$ such that

$$\sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k v)(0)t^{2k-2i} + \rho_N(j-i)(\Delta^j v)(\infty)t^{2j-2i}$$
$$\leqq \Delta^i u(t) \leqq \Delta^i v(t), \qquad t \geqq 0,\ 0 \leqq i < j;$$
$$(\Delta^j v)(\infty) \leqq \Delta^j u(t) \leqq \Delta^j v(t), \qquad t \geqq 0; \text{ and}$$
$$0 \leqq (-1)^{i-j}\Delta^i u(t) \leqq (-1)^{i-j}\Delta^i v(t), \qquad t \geqq 0,\ j < i \leqq m-1.$$

It is clear that U is a closed convex subset of $\mathscr{D}^{m-1}[0, \infty)$. Define the mapping M on U as follows:

$$(3.9)\qquad \begin{cases} (Mu)(t) = \sum_{k=0}^{j-1} \rho_N(k)(\Delta^k v)(0)\, t^{2k} + \rho_N(j)(\Delta^j v)(\infty)\, t^{2j} \\ \qquad + \Phi_N^j \Psi_N^{m-j} f(\cdot, \bar{u})(t), \qquad t \geqq 0. \end{cases}$$

Since condition (3.3) holds for all $(t, u_0, \ldots, u_{m-1})$ and $(t, v_0, \ldots, v_{m-1})$ satisfying $(3.4)_j$, we have

$$(3.10)\qquad 0 \leqq f(t, \bar{u}(t)) \leqq g(t, \bar{v}(t)), \qquad t \geqq 0,\ u \in U,$$

and hence M is well defined on U and maps U into $\mathscr{D}^{m-1}[0, \infty)$. It will be shown by the aid of the Schauder-Tychonoff theorem (see, e.g., [1, p. 161]) that M has a fixed point u in U.

(a) *M maps U into itself.* Let $u \in U$. Then, taking account of the lower estimates (3.6)–(3.8) for $\Delta^i v$ and inequality (3.10), we easily see that $Mu \in U$. Thus M maps U into U.

(b) *M is continuous on U.* It is sufficient to verify that if $u, u_\nu \in U$ $(\nu = 1, 2, \ldots)$ and $(\Delta^i u_\nu)(t) \to (\Delta^i u)(t)$ as $\nu \to \infty$, $0 \leqq i \leqq m-1$, uniformly on every compact subinterval of $[0, \infty)$, then $\Delta^i(Mu_\nu)(t) \to \Delta^i(Mu)(t)$ as $\nu \to \infty$, $0 \leqq i \leqq m-1$, uniformly on every compact subinterval of $[0, \infty)$. By the definition of M we have

$$|\Delta^i(Mu_\nu)(t) - \Delta^i(Mu)(t)| \leqq \Phi_N^{j-i} \Psi_N^{m-j} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t),$$
$$t \geqq 0,\ 0 \leqq i \leqq j-1,$$

$$|\Delta^i(Mu_\nu)(t) - \Delta^i(Mu)(t)| \leqq \Psi_N^{m-i} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t),$$
$$t \geqq 0,\ j \leqq i \leqq m-1.$$

Furthermore, with the aid of (1.20) and (1.21) we see that

$$\Phi_N^{j-i} \Psi_N^{m-j} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t)$$
$$\leqq A_N(j-i, m-j)\, t^{2(j-i)} \int_0^\infty s^{2(m-j)-1} |f(s, \bar{u}_\nu(s)) - f(s, \bar{u}(s))|\, ds,$$
$$t \geqq 0,\ 0 \leqq i \leqq j-1,$$

and

$$\Psi_N^{m-i} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t)$$
$$\leqq d_N(m-i) \int_0^\infty s^{2(m-i)-1} |f(s, \bar{u}_\nu(s)) - f(s, \bar{u}(s))|\, ds,$$
$$t \geqq 0,\ j \leqq i \leqq m-1.$$

Note that (3.10) implies

$$|f(s, \bar{u}_\nu(s)) - f(s, \bar{u}(s))| \leqq 2g(s, \bar{v}(s)), \qquad s \geqq 0,$$

and the well-definedness of $\Psi_N^{m-i} g(\cdot, \bar{v})$, $j \leqq i \leqq m-1$, implies

$$\int_0^\infty s^{2(m-i)-1} g(s, \bar{v}(s))\, ds < \infty, \qquad j \leqq i \leqq m-1.$$

Then the Lebesgue dominated convergence theorem shows that

$$\Phi_N^{j-i} \Psi_N^{m-j} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t) \to 0 \quad \text{as } \nu \to \infty, \qquad 0 \leqq i \leqq j-1,$$

$$\Psi_N^{m-i} |f(\cdot, \bar{u}_\nu) - f(\cdot, \bar{u})|(t) \to 0 \qquad \text{as } \nu \to \infty, \qquad j \leqq i \leqq m-1,$$

uniformly on every compact subinterval of $[0, \infty)$. Consequently $\Delta^i(Mu_\nu)(t)$, $0 \leqq i \leqq m-1$, converge to $\Delta^i(Mu)(t)$ as $\nu \to \infty$ uniformly on compact subintervals of $[0, \infty)$. This proves the continuity of M.

(c) *$M(U)$ is relatively compact.* It suffices to verify that $\{\Delta^i(Mu)(t) : u \in U\}$, $0 \leqq i \leqq m-1$, are uniformly bounded and equicontinuous at every point of $[0, \infty)$. Since $|\Delta^i(Mu)(t)| \leqq |\Delta^i v(t)|$, $t \geqq 0$, $0 \leqq i \leqq m-1$, for all $u \in U$, the uniform boundedness is obvious. Notice that

$$\frac{d}{dt} \Delta^i(Mu)(t) = \sum_{k=i+1}^{j-1} 2(k-i) \rho_N(k-i) (\Delta^k v)(0) t^{2k-2i-1}$$

$$+ 2(j-i)\rho_N(j-i)(\Delta^j v)(\infty) t^{2j-2i-1} + t^{-N+1} \int_0^t s^{N-1} \Phi_N^{j-i-1} \Psi_N^{m-j} f(\cdot, \bar{u})(s)\, ds,$$

$$t \geqq 0,\ 0 \leqq i \leqq j-1,$$

$$(-1)^{i-j} \frac{d}{dt} \Delta^i(Mu)(t) = -t^{-N+1} \int_0^t s^{N-1} \Psi_N^{m-i-1} f(\cdot, \bar{u})(s)\, ds,$$

$$t \geqq 0,\ j \leqq i \leqq m-1,$$

where the sum $\sum_{k=i+1}^{j-1}$ must be interpreted as 0 when $i+1 > j-1$. Then, by means of (3.10), we see that $|(d/dt)\Delta^i(Mu)(t)|$, $0 \leqq i \leqq m-1$, are majorized on $[0, \infty)$ by certain positive functions which are independent of $u \in U$. This proves the equicontinuity of $\{\Delta^i(Mu)(t) : u \in U\}$, $0 \leqq i \leqq m-1$.

In view of (a)–(c) we can apply the Schauder-Tychonoff fixed point theorem to conclude that M has a fixed point $u \in U$. Evidently this fixed point u is a member of class $\mathscr{K}_j^*$ and $u(|x|)$ is an entire solution of (0.1).

Suppose in addition that $v \in \mathscr{K}_j^*[\max]$ and $v(t) > 0$ for $t \geqq 0$, that is, $v \in \mathscr{K}_j^*$ and $\Delta^j v(\infty)$ is a positive finite value. Then, for the fixed point $u \in U$ of M obtained above, $\Delta^j u(\infty)$ exists and is equal to $\Delta^j v(\infty)$ (see the definition of U). This implies that if $v \in \mathscr{K}_j^*[\max]$, then $u \in \mathscr{K}_j^*[\max]$.

If $v \in \mathscr{K}_j^*[\min]$ and $v(t) > 0$ for $t \geqq 0$, that is, $v \in \mathscr{K}_j^*$, $\Delta^j v(\infty) = 0$ and $\Delta^{j-1} v(\infty)$ is a positive finite value, then the fixed point $u \in U$ of M has the properties that $\Delta^j u(\infty) = 0$ and $\Delta^{j-1} u(\infty)$ exists in the interval $[\Delta^{j-1} v(0), \Delta^{j-1} v(\infty)] \subset (0, \infty)$. This means that if $v \in \mathscr{K}_j^*[\min]$, then $u \in \mathscr{K}_j^*[\min]$. Likewise it is seen that if $\Delta^j v(\infty) = 0$, then $\Delta^j u(\infty) = 0$; and hence $v \in \mathscr{K}_j^*[\text{int}] \cup \mathscr{K}_j^*[\min]$ implies $u \in \mathscr{K}_j^*[\text{int}] \cup \mathscr{K}_j^*[\min]$. This completes the proof of (ii) of Theorem 3.1.

The proof of part (i). Let $v \in \mathscr{K}_m^*$ and $v(t) > 0$ for $t \geqq 0$. Using (2.4) and (3.1), we see that

$$\Delta^i v(t) \geqq \sum_{k=i}^{m-1} \rho_N(k-i)(\Delta^k v)(0)\, t^{2k-2i} + \Phi_N^{m-i} g(\cdot, \bar{v})(t),$$
$$t \geqq 0,\ 0 \leqq i \leqq m-1.$$

Consider the set U of all functions $u \in \mathscr{D}^{m-1}[0, \infty)$ such that

$$\sum_{k=i}^{m-1} \rho_N(k-i)(\Delta^k v)(0)\, t^{2k-2i} \leqq \Delta^i u(t) \leqq \Delta^i v(t), \qquad t \geqq 0,\ 0 \leqq i \leqq m-1,$$

and define the mapping $M: U \to \mathscr{D}^{m-1}[0, \infty)$ by

$$(Mu)(t) = \sum_{k=0}^{m-1} \rho_N(k)(\Delta^k v)(0)\, t^{2k} + \Phi_N^m f(\cdot, \bar{u})(t), \qquad t \geqq 0.$$

Then, arguing as above, we can show that (a) M maps U into itself, (b) M is continuous on U, and (c) $M(U)$ is relatively compact. The Schauder-Tychonoff fixed point theorem guarantees that M has a fixed point u in U. It is clear that this $u \in U$ belongs to $\mathscr{K}_m^*$ and that $u(|x|)$ is a solution of (0.1). If $v \in \mathscr{K}_m^*$ and $\Delta^{m-1} v(\infty)$ is finite and positive, then the fixed point $u \in U$ of M has the property that $\lim_{t\to\infty} \Delta^{m-1} u(t) = \Delta^{m-1} u(\infty)$ exists in the interval $[\Delta^{m-1} v(0), \Delta^{m-1} v(\infty)]$. This means that if $v \in \mathscr{K}_m^*[\min]$, then $u \in \mathscr{K}_m^*[\min]$.

The proof of part (iii). Let $v \in \mathscr{K}_0[\max]$ and $v(t) > 0$ for $t \geqq 0$. The limit $v(\infty)$ is finite and positive. From (2.6) and (3.1) it follows that

$$v(t) \geqq v(\infty) + \Psi_N^m g(\cdot, \bar{v})(t), \qquad t \geqq 0, \text{ and}$$
$$(-1)^i \Delta^i v(t) \geqq \Psi_N^{m-i} g(\cdot, \bar{v})(t), \qquad t \geqq 0,\ 1 \leqq i \leqq m-1.$$

Denote by U the set of all $u \in \mathscr{D}^{m-1}[0, \infty)$ such that

$$v(\infty) \leqq u(t) \leqq v(t), \qquad t \geqq 0, \text{ and}$$
$$0 \leqq (-1)^i \Delta^i u(t) \leqq (-1)^i \Delta^i v(t), \qquad t \geqq 0,\ 1 \leqq i \leqq m-1,$$

and define the mapping $M: U \to \mathscr{D}^{m-1}[0, \infty)$ by

$$(Mu)(t) = v(\infty) + \Psi_N^m f(\cdot, \bar{u})(t), \qquad t \geqq 0.$$

It can be shown, via the Schauder-Tychonoff theorem, that M has a fixed element u in U, which gives rise to a positive entire solution $u(|x|)$ of (0.1) such that $\lim_{t\to\infty} u(t) = v(\infty)$. The proof of (iii) of Theorem 3.1 is complete.

The next theorem concerns the existence of a solution of (0.1) in $\mathscr{K}_0[\text{int}]$ or $\mathscr{K}_0[\min]$. These classes can be nonempty only for $\sigma = -1$.

THEOREM 3.2. *Let* $\sigma = -1$ *in* (0.1) *and* (3.1). *Suppose that* g *in* (3.1) *is continuous on* $[0, \infty) \times \boldsymbol{R}^m$ *and satisfies* (3.2), *and that*

$$f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \leqq g(t, v_0, v_1, \dots, v_{m-1}) \operatorname{sgn} v_0 \tag{3.3}$$

for all $(t, u_0, u_1, \dots, u_{m-1})$ *and* $(t, v_0, v_1, \dots, v_{m-1})$ *satisfying*

$$\begin{cases} t \geqq 0, \quad u_0 v_0 > 0, \quad 0 < u_0 \operatorname{sgn} u_0 \leqq v_0 \operatorname{sgn} v_0, \\ 0 \leqq (-1)^i u_i \operatorname{sgn} u_0 \leqq (-1)^i v_i \operatorname{sgn} v_0, \qquad 1 \leqq i \leqq m-1. \end{cases} \tag{3.4_0}$$

Suppose that there is a continuous function $h(t, u_0)$ *on* $[0, \infty) \times \boldsymbol{R}$ *such that*

$$u_0 h(t, u_0) \geqq 0, \ \not\equiv 0 \qquad \text{for } (t, u_0) \in [0, \infty) \times \boldsymbol{R}, \tag{3.11}$$

$$h(t, u_0) \operatorname{sgn} u_0 \leqq f(t, v_0, v_1, \dots, v_{m-1}) \operatorname{sgn} v_0 \tag{3.12}$$

for all (t, u_0) *and* $(t, v_0, v_1, \dots, v_{m-1})$ *satisfying* $(3.4)_0$, *and such that, for each* $t \geqq 0$, $h(t, u_0)/u_0$ *is nonincreasing in* $u_0 \in (0, \infty)$ *and nondecreasing in* $u_0 \in (-\infty, 0)$, *and*

$$\lim_{u_0 \to \pm 0} \frac{h(t, u_0)}{u_0} = +\infty. \tag{3.13}$$

If there exists a radial entire function v *satisfying* (3.1) *and such that* $v(|x|) \neq 0$, $x \in \boldsymbol{R}^N$,

$$g(|x|, v(|x|), \Delta v(|x|), \dots, \Delta^{m-1} v(|x|)) \geqq 0, \ \not\equiv 0, \qquad x \in \boldsymbol{R}^N,$$

and $v(|x|) \to 0$ *as* $|x| \to \infty$, *then equation* (0.1) *has a radial entire solution* u *such that* $u(|x|) \neq 0$ *for* $x \in \boldsymbol{R}^N$ *and* $u(|x|) \to 0$ *as* $|x| \to \infty$. *Furthermore, if the function* v *is of class* $\mathscr{K}_0[\min]$, *then* (0.1) *has a radial entire solution* u *of class* $\mathscr{K}_0[\min]$.

PROOF. Let $v(|x|)$ be a radial entire function satisfying the conditions mentioned above. We may suppose that $v(t) > 0$ for $t \geqq 0$. Clearly v belongs to $\mathscr{K}_0[\text{int}] \cup \mathscr{K}_0[\min]$. From (iii) of Theorem 2.2 we have

$$(-1)^i \Delta^i v(t) = (-1)^m \Psi_N^{m-i}(\Delta^m v)(t), \qquad t \geqq 0,\ 0 \leqq i \leqq m-1,$$

which implies by (3.1) that

$$(3.14) \qquad (-1)^i \Delta^i v(t) \geqq \Psi_N^{m-i} g(\cdot, \bar{v})(t), \qquad t \geqq 0,\ 0 \leqq i \leqq m-1.$$

Since $\Psi_N^m g(\cdot, \bar{v})$ is well defined on $[0, \infty)$ and $g(t, \bar{v}(t)) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$, it follows from (i) of Lemma 1.10 that

$$(3.15) \qquad v(t) \geqq c_N(m) \int_0^\infty \min\{s^{2m-1}, s^{N-1}\} g(s, \bar{v}(s))\, ds \cdot q_{N,m}(t), \qquad t \geqq 0,$$

where $q_{N,m}(t) = \min\{1, t^{-N+2m}\}$, $t \geqq 0$. Let c be a number such that

$$(3.16) \qquad 0 < c < c_N(m) \int_0^\infty \min\{s^{2m-1}, s^{N-1}\} g(s, \bar{v}(s))\, ds.$$

Then, by (3.15),

$$(3.17) \qquad c q_{N,m}(t) \leqq v(t), \qquad t \geqq 0.$$

Noting that $0 \leqq h(t, c q_{N,m}(t)) \leqq f(t, \bar{v}(t)) \leqq g(t, \bar{v}(t))$, $t \geqq 0$, because of (3.11), (3.12) and (3.3), we find that

$$\int_0^\infty \min\{s^{2m-1}, s^{N-1}\} h(s, c q_{N,m}(s))\, ds < \infty.$$

On the other hand, the nonincreasing property of $h(t, u_0)/u_0$ implies

$$\int_0^\infty \min\{s^{2m-1}, s^{N-1}\} h(s, c q_{N,m}(s))\, ds$$
$$\geqq c \int_0^\infty \min\{s^{2m-1}, s^{N-1}\} q_{N,m}(s) \frac{h(s, c)}{c}\, ds.$$

Take a positive number c sufficiently small so that

$$(3.18) \qquad c_N(m) \int_0^\infty \min\{s^{2m-1}, s^{N-1}\} q_{N,m}(s) \frac{h(s, c)}{c}\, ds \geqq 1.$$

Such a choice of c is possible, since the left-hend side of (3.18) diverges to $+\infty$ as $c \to +0$. To see the divergence it suffices to apply the Lebesgue monotone convergence theorem by taking account of the nonincreasing property of $h(t, u_0)/u_0$ and condition (3.13).

Now, for a positive constant c satisfying (3.16) and (3.18), consider the set U of all $u \in \mathscr{D}^{m-1}[0, \infty)$ such that

$$c q_{N,m}(t) \leqq u(t) \leqq v(t), \qquad t \geqq 0,$$

$$0 \leqq (-1)^i \Delta^i u(t) \leqq (-1)^i \Delta^i v(t), \qquad t \geqq 0,\ 1 \leqq i \leqq m-1,$$

(U is nonempty by (3.17)) and define the mapping $M: U \to \mathscr{D}^{m-1}[0, \infty)$ by

$$(Mu)(t) = \Psi_N^m f(\cdot, \bar{u})(t), \qquad t \geqq 0.$$

It is verified routinely that (a) M maps U into U, (b) M is continuous on U, and (c) $M(U)$ is relatively compact, so that M has a fixed point $u \in U$ by the Schauder-Tychonoff fixed point theorem. This fixed point u generates an entire solution $u(|x|)$ of (0.1) having the properties: $u(|x|) > 0$ for $x \in \boldsymbol{R}^N$ and $u(|x|) \to 0$ as $|x| \to \infty$. It is also clear that $u \in \mathscr{K}_0[\min]$ if $v \in \mathscr{K}_0[\min]$. This completes the proof of Theorem 3.2.

4. Existence of radial entire solutions I

We are now ready to develop the main results of this paper giving criteria for the existence of radial entire solutions of equation (0.1) (and its particular cases) belonging to the Kiguradze classes $\mathscr{K}_j$ and their subclasses. The results are presented in this and the next sections. Our purpose here is to characterize, under appropriate assumptions on the nonlinearity of (0.1), the classes $\mathscr{K}_j$, $\mathscr{K}_j[\max]$, $\mathscr{K}_j[\mathrm{int}]$ and $\mathscr{K}_j[\min]$ for j, $1 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$, and the classes $\mathscr{K}_m[\min]$ and $\mathscr{K}_0[\max]$.

DEFINITION 4.1. Let h be a continuous function on $[0, \infty) \times \boldsymbol{R}^m$ and let d be an integer such that $0 \leqq d \leqq m-1$. We say that the function h satisfies the condition (N_d) if

$$h(t, \bar{p}_d(t)) \not\equiv 0 \qquad \text{on } [0, \infty)$$

for every polynomial p_d of the form $p_d(t) = \sum_{k=0}^{d} c_k t^{2k}$ with $c_0 \neq 0$, where

$$\bar{p}_d(t) = (p_d(t), \Delta p_d(t), \ldots, \Delta^{m-1} p_d(t)).$$

For example, the function $h(t, u_0, u_1, \ldots, u_{m-1}) = p(t)|u_0|^\gamma \operatorname{sgn} u_0$, where $\gamma > 0$ and $p \in C[0, \infty)$, $p(t) \geqq 0$, $\not\equiv 0$ on $[0, \infty)$, satisfies conditions (N_d) for all d, $0 \leqq d \leqq m-1$.

If $0 \leqq j \leqq m-1$ and $u \in \mathscr{K}_j$ is a radial entire solution of (0.1), then, by Theorem 2.2, u satisfies (2.5) or (2.6) according as $1 \leqq j \leqq m-1$ or $j = 0$. Assume that $\Delta^m u(t) \equiv 0$ for $t \geqq 0$. Then, $u(t)$ is of the form $u(t) = p_j(t) = \sum_{k=0}^{j} c_k t^{2k}$ for some c_k $(0 \leqq k \leqq j)$, $c_0 \neq 0$. From equation (0.1) it follows that $f(t, \bar{p}_j(t)) \equiv 0$ for $t \geqq 0$. Therefore we can conclude that, if f satisfies condition (N_j), then $\Delta^m u(t) \not\equiv 0$ on $[0, \infty)$ for any radial entire solution u of (0.1) in the class $\mathscr{K}_j$ $(0 \leqq j \leqq m-1)$. Likewise we can prove by Theorem 2.4 that, if f satisfies condition (N_{j-1}), then $\Delta^m u(t) \not\equiv 0$ on $[0, \infty)$ for any radial entire solution u of (0.1) in $\mathscr{K}_j[\min]$ $(1 \leqq j \leqq m)$.

THEOREM 4.1. (i) *Suppose that j is an integer such that $0 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$ and f in* (0.1) *satisfies condition* (N_j). *If equation* (0.1) *has a radial entire solution of class $\mathscr{K}_j$, then $N \geqq 2(m-j)+1$.*

(ii) *Suppose that j is an integer such that $1 \leqq j \leqq m$, $(-1)^{j+1}\sigma = 1$ and f in* (0.1) *satisfies condition* (N_{j-1}). *If equation* (0.1) *has a radial entire solution of class $\mathscr{K}_j[\min]$, then $N \geqq 2(m-j)+3$.*

PROOF. (i) Let $u \in \mathscr{K}_j$ be a radial entire solution of (0.1). As mentioned above, $\Delta^m u(t) \not\equiv 0$ on $[0, \infty)$. Furthermore, from Theorem 2.2, $\Psi_N^{m-j}(\Delta^m u)(t)$ is well defined on $[0, \infty)$. Then Lemma 1.8 shows in particular that $N \geqq 2(m-j)+1$. This proves part (i). Part (ii) can be proved similarly with the aid of Theorem 2.4.

DEFINITION 4.2. Let d be an integer with $0 \leqq d \leqq m-1$ and let h be a continuous function defined on $[0, \infty) \times \boldsymbol{R}^{d+1}$ such that

$$u_0 h(t, u_0, u_1, \ldots, u_d) \geqq 0, \not\equiv 0 \quad \text{on } [0, \infty) \times \boldsymbol{R}^{d+1}. \tag{4.1}$$

We say in this paper that such a function h is *restrictively nondecreasing* on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if

$$h(t, u_0, u_1, \ldots, u_d) \operatorname{sgn} u_0 \leqq h(t, v_0, v_1, \ldots, v_d) \operatorname{sgn} v_0$$

for all $(t, u_0, u_1, \ldots, u_d)$ and $(t, v_0, v_1, \ldots, v_d)$ satisfying

$$t \geqq 0, \quad u_0 v_0 > 0, \quad 0 < u_i \operatorname{sgn} u_0 \leqq v_i \operatorname{sgn} v_0 \quad (0 \leqq i \leqq d). \tag{4.2}$$

A continuous function h on $[0, \infty) \times \boldsymbol{R}^{d+1}$ satisfying (4.1) is said to be *superlinear* [resp. *sublinear*] on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if it is restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}^{d+1}$ and the function

$$\frac{1}{\lambda} h(t, \lambda u_0, \lambda u_1, \ldots, \lambda u_d) \operatorname{sgn} u_0$$

is nondecreasing [resp. nonincreasing] in $\lambda \in (0, \infty)$ for each fixed $(t, u_0, u_1, \ldots, u_d)$ such that $t \geqq 0$ and $u_i \operatorname{sgn} u_0 > 0$ $(0 \leqq i \leqq d)$. A continuous function h on $[0, \infty) \times \boldsymbol{R}^{d+1}$ satisfying (4.1) is said to be *strongly superlinear* [resp. *strongly sublinear*] on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if it is superlinear [resp. sublinear] on $[0, \infty) \times \boldsymbol{R}^{d+1}$ and

$$\frac{1}{\lambda} h(t, \lambda u_0, \lambda u_1, \ldots, \lambda u_d) \to 0 \quad \text{as} \quad \lambda \to +0 \text{ [resp. } \lambda \to +\infty]$$

for each fixed $(t, u_0, u_1, \ldots, u_d)$, $t \geqq 0$, $u_i \operatorname{sgn} u_0 > 0$ $(0 \leqq i \leqq d)$. Furthermore, a continuous function on $[0, \infty) \times \boldsymbol{R}^{d+1}$ satisfying (4.1) is said to be *strictly* *superlinear* [resp. *strictly sublinear*] on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if it is restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}^{d+1}$ and there exists a number γ such that $\gamma > 1$ [resp. $0 < \gamma < 1$] and the function

$$\frac{1}{\lambda^{\gamma}} h(t, \lambda u_0, \lambda u_1, \dots, \lambda u_d) \operatorname{sgn} u_0$$

is nondecreasing [resp. nonincreasing] in $\lambda \in (0, \infty)$ for each fixed $(t, u_0, u_1, \dots, u_d)$, $t \geqq 0$, $u_i \operatorname{sgn} u_0 > 0$ $(0 \leqq i \leqq d)$.

As is easily verified, if h is strictly superlinear [resp. strictly sublinear] on $[0, \infty) \times \boldsymbol{R}^{d+1}$, then it is strongly superlinear [resp. strongly sublinear] on $[0, \infty) \times \boldsymbol{R}^{d+1}$. The function $h(t, u_0, u_1, \dots, u_d) = p(t)|u_0|^{\gamma} \operatorname{sgn} u_0$ with $\gamma > 0$ and $p \in C[0, \infty)$, $p(t) \geqq 0$, $\not\equiv 0$ on $[0, \infty)$, is superlinear or sublinear on $[0, \infty) \times \boldsymbol{R}^{d+1}$ according as $\gamma \geqq 1$ or $0 < \gamma \leqq 1$. It is strictly (and hence strongly) superlinear on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if $\gamma > 1$; and is strictly (and hence strongly) sublinear on $[0, \infty) \times \boldsymbol{R}^{d+1}$ if $0 < \gamma < 1$.

For an integer j, $0 \leqq j \leqq m$, we denote by D_j the set of all points $(t, u_0, u_1, \dots, u_{m-1}) \in [0, \infty) \times \boldsymbol{R}^m$ such that

$$\begin{cases} t \geqq 0, \quad u_0 \neq 0, \quad u_i \operatorname{sgn} u_0 > 0 & (0 \leqq i \leqq j-1), \\ (-1)^{i-j} u_i \operatorname{sgn} u_0 \geqq 0 & (j \leqq i \leqq m-1). \end{cases} \tag{4.3}$$

THEOREM 4.2. *Let j be an integer with $0 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$. Suppose that h_j is continuous and restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}^{j+1}$ and satisfies*

$$0 \leqq h_j(t, u_0, u_1, \dots, u_j) \operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \quad \text{on } D_j. \tag{4.4}$$

If equation (0.1) *has a radial entire solution of class $\mathscr{K}_j$[max], then*

$$\int_0^{\infty} t^{2(m-j)-1} |h_j(t, ct^{2j}, ct^{2(j-1)}, \dots, c)|\, dt < \infty \qquad \text{for some } c \neq 0. \tag{4.5}$$

PROOF. Let $u \in \mathscr{K}_j$[max] be a solution of (0.1) such that $u(t) > 0$ for $t \geqq 0$. By Theorem 2.2, $\Psi_N^{m-j}(\Delta^m u) = (-1)^{m-j} \Psi_N^{m-j} f(\cdot, \bar{u})$ is well defined on $[0, \infty)$, where $\bar{u}$ is defined by (3.5). Lemma 1.8 shows in particular that

$$\int_0^{\infty} t^{2(m-j)-1} f(t, \bar{u}(t))\, dt < \infty,$$

and hence (4.4) implies that

$$\int_0^{\infty} t^{2(m-j)-1} h_j(t, u(t), \Delta u(t), \dots, \Delta^j u(t))\, dt < \infty. \tag{4.6}$$

Since $u \in \mathscr{K}_j[\max]$ means that $\lim_{t\to\infty} \Delta^i u(t)/t^{2(j-i)}$, $0 \leqq i \leqq j$, exist and are positive, there exist $c_* > 0$, $c^* > 0$ and $T \geqq 1$ such that

$$c_* t^{2(j-i)} \leqq \Delta^i u(t) \leqq c^* t^{2(j-i)}, \qquad t \geqq T,\ 0 \leqq i \leqq j.$$

In view of the restrictively nondecreasing property of h_j we see that

$$\text{(4.7)} \qquad h_j(t, u(t), \Delta u(t), \dots, \Delta^j u(t)) \geqq h_j(t, c_* t^{2j}, c_* t^{2(j-1)}, \dots, c_*)$$

for $t \geqq T$. Then, (4.6) and (4.7) together imply that (4.5) holds for $c = c_*$. The case where $u \in \mathscr{K}_j[\max]$ is negative in $\boldsymbol{R}^N$ is similarly proved. This completes the proof of Theorem 4.2.

THEOREM 4.3. *Let j be an integer with $0 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$. Suppose that g_j is a continuous function on $[0, \infty) \times \boldsymbol{R}^{j+1}$ which is either strongly superlinear or strongly sublinear on $[0, \infty) \times \boldsymbol{R}^{j+1}$ and satisfies*

$$\text{(4.8)} \qquad f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \leqq g_j(t, u_0, u_1, \dots, u_j) \operatorname{sgn} u_0 \quad \textit{on } D_j,$$

where D_j is the set of all points $(t, u_0, u_1, \dots, u_{m-1})$ satisfying (4.3). *Suppose moreover that $N \geqq 2(m-j)+1$ and*

$$\text{(4.9)} \qquad \int_0^\infty t^{2(m-j)-1} |g_j(t, ct^{2j}, ct^{2(j-1)}, \dots, c)|\, dt < \infty \qquad \textit{for some } c \neq 0.$$

Then equation (0.1) *has a radial entire solution u_1 of class $\mathscr{K}_j^*[\max]$. If $1 \leqq j \leqq m-1$, then, in addition to this $u_1 \in \mathscr{K}_j^*[\max]$, equation* (0.1) *has another radial entire solution u_2 which belongs to $\mathscr{K}_j^*[\mathrm{int}] \cup \mathscr{K}_j^*[\min]$.*

PROOF. Without loss of generality we may assume that $c > 0$ in (4.9). Let

$$\text{(4.10)} \qquad \tau(t) = \max\{1, t\}, \qquad t \geqq 0.$$

The strong superlinearity [resp. strong sublinearity] of g_j on $[0, \infty) \times \boldsymbol{R}^{j+1}$ implies

$$0 \leqq \frac{1}{\lambda} g_j(t, \lambda[\tau(t)]^{2j}, \lambda[\tau(t)]^{2(j-1)}, \dots, \lambda)$$
$$\leqq \frac{1}{c} g_j(t, c[\tau(t)]^{2j}, c[\tau(t)]^{2(j-1)}, \dots, c)$$

for all $t \geqq 0$ and all λ, $0 < \lambda \leqq c$ [resp. $\lambda \geqq c$], and

$$\frac{1}{\lambda} g_j(t, \lambda[\tau(t)]^{2j}, \lambda[\tau(t)]^{2(j-1)}, \dots, \lambda) \to 0$$

as $\lambda \to +0$ [resp. $\lambda \to +\infty$]. Therefore we have

$$\frac{1}{\lambda}\int_0^\infty t^{2(m-j)-1} g_j(t, \lambda[\tau(t)]^{2j}, \lambda[\tau(t)]^{2(j-1)},\dots,\lambda)\,dt \to 0$$

as $\lambda \to +0$ [resp. $\lambda \to +\infty$]. Consequently there exists a positive number $\lambda_0 > 0$ such that

$$\int_0^\infty t^{2(m-j)-1} g_j(t, \lambda[\tau(t)]^{2j}, \lambda[\tau(t)]^{2(j-1)},\dots,\lambda)\,dt < \frac{\lambda}{2A_N(j-i, m-j)} \tag{4.11}$$

for all i, $0 \leqq i \leqq j$, and all $\lambda \in (0, \lambda_0]$ [resp. $\lambda \in [\lambda_0, \infty)$], where $A_N(j-i, m-j)$ is a positive constant appearing in (1.21) with (i, k) replaced by $(j-i, m-j)$.

Let $0 \leqq j \leqq m-1$. For any $\lambda(>0)$ satisfying (4.11), define the function $v_1(t)$ by

$$v_1(t) = \frac{\lambda}{2(j+1)}\sum_{k=0}^{j} \rho_N(k)t^{2k} + \Phi_N^j \Psi_N^{m-j} g_j(\cdot, \lambda\tau^{2j}, \lambda\tau^{2(j-1)},\dots,\lambda)(t)$$

for $t \geqq 0$. In view of Lemma 1.8 we see that $v_1(t)$ is well defined for $t \geqq 0$. Differentiation of v_1 gives

$$\Delta^i v_1(t) = \frac{\lambda}{2(j+1)}\sum_{k=i}^{j} \rho_N(k-i)t^{2(k-i)} + \Phi_N^{j-i} \Psi_N^{m-j} g_j(\cdot, \lambda\tau^{2j}, \lambda\tau^{2(j-1)},\dots,\lambda)(t),$$
$$t \geqq 0,\ 0 \leqq i \leqq j-1;$$

$$\Delta^j v_1(t) = \frac{\lambda}{2(j+1)} + \Psi_N^{m-j} g_j(\cdot, \lambda\tau^{2j}, \lambda\tau^{2(j-1)},\dots,\lambda)(t), \qquad t \geqq 0;$$

and

$$(-1)^{i-j}\Delta^i v_1(t) = \Psi_N^{m-i} g_j(\cdot, \lambda\tau^{2j}, \lambda\tau^{2(j-1)},\dots,\lambda)(t), \qquad t \geqq 0,\ j+1 \leqq i \leqq m.$$

It is clear that v_1 is of class $\mathscr{K}_j^*[\max]$, because v_1 satisfies $\Delta^i v_1(t) > 0$ $(0 \leqq i \leqq j-1)$ and $(-1)^{i-j}\Delta^i v_1(t) \geqq 0$ $(j \leqq i \leqq m)$ for $t \geqq 0$ and $\Delta^j v_1(t)$ has the positive finite limit $\lambda/[2(j+1)]$ as $t \to \infty$. Note that

$$\sigma(-1)^m \Delta^m v_1(t) + g_j(t, \lambda[\tau(t)]^{2j}, \lambda[\tau(t)]^{2(j-1)},\dots,\lambda) = 0 \tag{4.12}$$

for $t \geqq 0$. If $0 \leqq i \leqq j$, then, by means of (1.21) with (i, k) replaced by $(j-i, m-j)$, we obtain

$$\Delta^i v_1(t)$$
$$\leqq \frac{\lambda}{2(j+1)}\sum_{k=i}^{j} \rho_N(k-i)t^{2(k-i)}$$

$$+ A_N(j-i, m-j)t^{2(j-i)} \int_0^\infty s^{2(m-j)-1} g_j(s, \lambda[\tau(s)]^{2j},\dots,\lambda)\,ds$$

$$\leqq \frac{\lambda}{2(j+1)}(j-i+1)[\tau(t)]^{2(j-i)}$$

$$+ A_N(j-i, m-j)[\tau(t)]^{2(j-i)} \int_0^\infty s^{2(m-j)-1} g_j(s, \lambda[\tau(s)]^{2j},\dots,\lambda)\,ds$$

$$\leqq \left\{\frac{\lambda}{2} + A_N(j-i, m-j)\int_0^\infty s^{2(m-j)-1} g_j(s, \lambda[\tau(s)]^{2j},\dots,\lambda)\,ds\right\}[\tau(t)]^{2(j-i)}$$

$$\leqq \lambda[\tau(t)]^{2(j-i)}$$

for $t \geqq 0$, where (4.11) has been used in the last step. These inequalities combined with (4.12) yield

$$\sigma(-1)^m \Delta^m v_1(t) + g_j(t, v_1(t), \Delta v_1(t),\dots,\Delta^j v_1(t)) \leqq 0 \qquad \text{in } \boldsymbol{R}^N,$$

where $t = |x|$. Then, applying Theorem 3.1 to the case of $g(t, v_0, v_1,\dots,v_{m-1}) = g_j(t, v_0, v_1,\dots,v_j)$ and $v(t) = v_1(t)$, we conclude that there exists a radial entire solution u_1 of (0.1) of class $\mathscr{K}_j^*[\max]$.

Let $1 \leqq j \leqq m-1$. Then, for any $\lambda(>0)$ satisfying (4.11), define the function $v_2(t)$ by

$$v_2(t) = \frac{\lambda}{2(j+1)}\sum_{k=0}^{j-1} \rho_N(k)t^{2k} + \Phi_N^j \Psi_N^{m-j} g_j(\cdot, \lambda\tau^{2j}, \lambda\tau^{2(j-1)},\dots,\lambda)(t)$$

for $t \geqq 0$. As in the above, it is shown that v_2 belongs to $\mathscr{K}_j^*[\text{int}] \cup \mathscr{K}_j^*[\min]$ and satisfies

$$\sigma(-1)^m \Delta^m v_2(t) + g_j(t, v_2(t), \Delta v_2(t),\dots,\Delta^j v_2(t)) \leqq 0 \qquad \text{in } \boldsymbol{R}^N,$$

where $t = |x|$. Part (ii) of Theorem 3.1 shows that there is a radial entire solution u_2 of (0.1) belonging to $\mathscr{K}_j^*[\text{int}] \cup \mathscr{K}_j^*[\min]$. The proof of Theorem 4.3 is complete.

The next theorem is concerned with equations of the form

$$(0.1)_j \qquad (-1)^m \Delta^m u + \sigma f_j(|x|, u, \Delta u,\dots,\Delta^j u) = 0, \qquad x \in \boldsymbol{R}^N,$$

where $m \geqq 2$, $\sigma = +1$ or $\sigma = -1$, $N \geqq 3$, $0 \leqq j \leqq m-1$ and f_j is continuous and satisfies $u_0 f_j(t, u_0, u_1,\dots,u_j) \geqq 0$, $\not\equiv 0$ on $[0,\infty) \times \boldsymbol{R}^{j+1}$. Condition (4.5) in Theorem 4.2 for the case of $f = h_j = f_j$ and condition (4.9) in Theorem 4.3 for the case of $f = g_j = f_j$ reduce to the same condition

$$(4.13)_j \qquad \int_0^\infty t^{2(m-j)-1} |f_j(t, ct^{2j}, ct^{2(j-1)},\dots,c)|\,dt < \infty \qquad \text{for some } c \neq 0.$$

THEOREM 4.4. *Let j be an integer with $0 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$ and consider equation $(0.1)_j$. Suppose that f_j satisfies condition (N_j) and is either strongly superlinear or strongly sublinear on $[0, \infty) \times \boldsymbol{R}^{j+1}$. Then the following three statements are equivalent:*

(i) *equation $(0.1)_j$ has a radial entire solution of class $\mathscr{K}_j^*[\max]$;*

(ii) *equation $(0.1)_j$ has a radial entire solution of class $\mathscr{K}_j[\max]$;*

(iii) *the dimensional condition $N \geqq 2(m-j)+1$ and the integral condition $(4.13)_j$ are satisfied.*

PROOF. It is trivial that (i) implies (ii). Theorem 4.1 and Theorem 4.2 applied to the case $f = h_j = f_j$ show that (ii) implies (iii), and Theorem 4.3 applied to the case $f = g_j = f_j$ shows that (iii) implies (i).

THEOREM 4.5. *Let j be an integer with $1 \leqq j \leqq m$, $(-1)^{j+1}\sigma = 1$. Suppose that h_{j-1} is continuous and restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}^j$ and satisfies*

(4.14)

$$0 \leqq h_{j-1}(t, u_0, u_1, \dots, u_{j-1}) \operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \qquad \text{on } D_j.$$

If (0.1) has a radial entire solution of class $\mathscr{K}_j[\min]$, then

$$\int_0^\infty t^{2(m-j)+1} |h_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \dots, c)|\, dt < \infty \qquad \text{for some } c \neq 0. \tag{4.15}$$

PROOF. Let $u \in \mathscr{K}_j[\min]$ be a solution of (0.1) such that $u(t) > 0$ for $t \geqq 0$. Then there exists $c_* > 0$, $c^* > 0$ and $T \geqq 1$ such that

$$c_* t^{2(j-i-1)} \leqq \Delta^i u(t) \leqq c^* t^{2(j-i-1)}, \qquad t \geqq T,\ 0 \leqq i \leqq j-1.$$

By Theorem 2.4 we see that $\Psi_N^{m-j+1} f(\cdot, \bar{u})$ is well defined on $[0, \infty)$, and hence Lemma 1.8 and (4.14) yield

$$\int_0^\infty t^{2(m-j)+1} h_{j-1}(t, u(t), \Delta u(t), \dots, \Delta^{j-1} u(t))\, dt < \infty.$$

Therefore (4.15) holds for $c = c_*$. The case of $u(t) < 0$ on $[0, \infty)$ can be similarly proved. This completes the proof of Theorem 4.5.

THEOREM 4.6. *Let j be an integer with $1 \leqq j \leqq m$, $(-1)^{j+1}\sigma = 1$. Suppose that g_{j-1} is a continuous function on $[0, \infty) \times \boldsymbol{R}^j$ which is either strongly superlinear or strongly sublinear on $[0, \infty) \times \boldsymbol{R}^j$ and satisfies*

$$f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \leqq g_{j-1}(t, u_0, u_1, \dots, u_{j-1}) \operatorname{sgn} u_0 \qquad \text{on } D_j. \tag{4.16}$$

If $N \geqq 2(m-j)+3$ and

$$(4.17)\qquad \int_0^\infty t^{2(m-j)+1}|g_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)},\dots,c)|\,dt < \infty \qquad \textit{for some } c \neq 0,$$

then equation (0.1) *has a radial entire solution in the class* $\mathscr{K}_j^*$ [min].

PROOF. We may suppose that $c > 0$ in (4.17). Arguing as in the proof of Theorem 4.3, we can show that

$$\frac{1}{\lambda}\int_0^\infty t^{2(m-j)+1} g_{j-1}(t, \lambda[\tau(t)]^{2(j-1)}, \lambda[\tau(t)]^{2(j-2)},\dots,\lambda)\,dt \to 0$$

as $\lambda \to +0$ or as $\lambda \to +\infty$ according as g_{j-1} is strongly superlinear or g_{j-1} is strongly sublinear, where $\tau(t)$ is the function defined by (4.10). There is a number $\lambda_0 > 0$ such that

$$(4.18)\qquad \int_0^\infty t^{2(m-j)+1} g_{j-1}(t, \lambda[\tau(t)]^{2(j-1)}, \lambda[\tau(t)]^{2(j-2)},\dots,\lambda)\,dt < \frac{\lambda\rho_N(j-i-1)}{jA_N(j-i-1, m-j+1)}$$

for all i, $0 \leqq i \leqq j-1$, and all $\lambda \in (0, \lambda_0]$ or all $\lambda \in [\lambda_0, \infty)$ according as g_{j-1} is strongly superlinear or g_{j-1} is strongly sublinear. For any such λ, define $v(t)$ as follows:

$$v(t) = \frac{\lambda}{j}\sum_{k=0}^{j-1}\rho_N(k)t^{2k} - \Phi_N^{j-1}\Psi_N^{m-j+1}g_{j-1}(\cdot, \lambda\tau^{2(j-1)}, \lambda\tau^{2(j-2)},\dots,\lambda)(t)$$

for $t \geqq 0$. An easy computation shows that

$$(4.19)\qquad \Delta^i v(t) = \frac{\lambda}{j}\sum_{k=i}^{j-1}\rho_N(k-i)t^{2(k-i)} - \Phi_N^{j-i-1}\Psi_N^{m-j+1}g_{j-1}(\cdot, \lambda\tau^{2(j-1)}, \lambda\tau^{2(j-2)},\dots,\lambda)(t),\quad t \geqq 0,\ 0 \leqq i \leqq j-1;\ \text{and}$$

$$(4.20)\qquad (-1)^{i-j}\Delta^i v(t) = \Psi_N^{m-i}g_{j-1}(\cdot, \lambda\tau^{2(j-1)}, \lambda\tau^{2(j-2)},\dots,\lambda)(t),\quad t \geqq 0,\ j \leqq i \leqq m.$$

It then follows from (1.21) with (i, k) replaced by $(j-i-1, m-j+1)$ that

$$\Delta^i v(t) \geqq \frac{\lambda}{j}\sum_{k=i}^{j-1}\rho_N(k-i)t^{2(k-i)} - A_N(j-i-1, m-j+1)\,t^{2(j-i-1)} \times \int_0^\infty s^{2(m-j)+1}g_{j-1}(s, \lambda[\tau(s)]^{2(j-1)},\dots,\lambda)\,ds$$

$$\geqq \frac{\lambda}{j} \sum_{k=i}^{j-2} \rho_N(k-i) t^{2(k-i)} + \left[\frac{\lambda \rho_N(j-i-1)}{j} - A_N(j-i-1, m-j+1) \right.$$
$$\left. \times \int_0^\infty s^{2(m-j)+1} g_{j-1}(s, \lambda[\tau(s)]^{2(j-1)}, \dots, \lambda)\, ds \right] t^{2(j-i-1)}$$
$$> 0 \quad \text{for } t \geqq 0 \text{ and } 0 \leqq i \leqq j-1.$$

Therefore, noting that $\Delta^{j-1}v(t) \to \lambda/j \in (0, \infty)$ as $t \to \infty$, we see that v is of class $\mathscr{K}_j^*[\min]$. Furthermore, the function v satisfies

$$\text{(4.21)} \qquad \sigma(-1)^m \Delta^m v(t) + g_{j-1}(t, \lambda[\tau(t)]^{2(j-1)}, \lambda[\tau(t)]^{2(j-2)}, \dots, \lambda) = 0$$

for $t \geqq 0$. In view of (4.19), we have

$$\Delta^i v(t) \leqq \frac{\lambda}{j} \sum_{k=i}^{j-1} \rho_N(k-i) t^{2(k-i)}$$
$$\leqq \frac{\lambda}{j} \sum_{k=i}^{j-1} [\tau(t)]^{2(j-i-1)}$$
$$\leqq \lambda[\tau(t)]^{2(j-i-1)}, \qquad t \geqq 0,\ 0 \leqq i \leqq j-1,$$

and hence (4.21) gives

$$\sigma(-1)^m \Delta^m v(t) + g_{j-1}(t, v(t), \Delta v(t), \dots, \Delta^{j-1} v(t)) \leqq 0 \qquad \text{in } \boldsymbol{R}^N,$$

where $t = |x|$. Applying Theorem 3.1 to the case $g = g_{j-1}(t, v_0, \dots, v_{j-1})$, we conclude that equation (0.1) has a radial entire solution of class $\mathscr{K}_j^*[\min]$. This completes the proof of Theorem 4.6.

Consider the equation

$$(0.1)_{j-1} \qquad (-1)^m \Delta^m u + \sigma f_{j-1}(|x|, u, \Delta u, \dots, \Delta^{j-1} u) = 0, \quad x \in \boldsymbol{R}^N,$$

where $m \geqq 2$, $\sigma = +1$ or $\sigma = -1$, $N \geqq 3$, $1 \leqq j \leqq m$ and f_{j-1} is continuous and satisfies $u_0 f_{j-1}(t, u_0, u_1, \dots, u_{j-1}) \geqq 0$, $\not\equiv 0$ on $[0, \infty) \times \boldsymbol{R}^j$. Then condition (4.15) in Theorem 4.5 applied to the case of $f = h_{j-1} = f_{j-1}$ and condition (4.17) in Theorem 4.6 applied to the case of $f = g_{j-1} = f_{j-1}$ become

$(4.22)_{j-1}$
$$\int_0^\infty t^{2(m-j)+1} |f_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \dots, c)|\, dt < \infty \qquad \text{for some } c \neq 0.$$

This observation combined with (ii) of Theorem 4.1 yields the next theorem.

THEOREM 4.7. *Let j be an integer with $1 \leqq j \leqq m$, $(-1)^{j+1}\sigma = 1$ and* *consider equation* $(0.1)_{j-1}$. *Suppose that* f_{j-1} *satisfies condition* (N_{j-1}) *and is either strongly superlinear or strongly sublinear on* $[0, \infty) \times \boldsymbol{R}^j$. *Then the following three statements are equivalent*:

(i) *equation* $(0.1)_{j-1}$ *has a radial entire solution of class* $\mathscr{K}_j^*[\min]$;

(ii) *equation* $(0.1)_{j-1}$ *has a radial entire solution of class* $\mathscr{K}_j[\min]$;

(iii) *the dimensional condition* $N \geqq 2(m-j)+3$ *and the integral condition* $(4.22)_{j-1}$ *are satisfied.*

Now consider the equation

$$(-1)^m \Delta^m u + \sigma F(|x|, u) = 0, \qquad x \in \boldsymbol{R}^N, \tag{4.23}$$

where m, σ and N are as above and F is assumed to satisfy the conditions:

(4.24) F is continuous on $[0, \infty) \times \boldsymbol{R}$;

(4.25) $uF(t, u) > 0$ for $t \geqq 0$, $u \in \boldsymbol{R} - \{0\}$; and

(4.26) $|F(t, u_1)| \leqq |F(t, u_2)|$ for $t \geqq 0$, $u_1 u_2 > 0$, $|u_1| \leqq |u_2|$.

The next result follows from Theorems 4.4 and 4.7.

THEOREM 4.8. *Consider equation* (4.23) *under the above conditions and suppose in addition that* F *satisfies either*

$$\begin{cases} |u_1|^{-1}|F(t, u_1)| \leqq |u_2|^{-1}|F(t, u_2)| & \text{for } t \geqq 0,\ u_1 u_2 > 0,\ |u_1| \leqq |u_2|, \\ \lim\limits_{u \to \pm 0} u^{-1} F(t, u) = 0 & \text{for each fixed } t \geqq 0 \end{cases}$$

or

$$\begin{cases} |u_1|^{-1}|F(t, u_1)| \geqq |u_2|^{-1}|F(t, u_2)| & \text{for } t \geqq 0,\ u_1 u_2 > 0,\ |u_1| \leqq |u_2|, \\ \lim\limits_{u \to \pm \infty} u^{-1} F(t, u) = 0 & \text{for each fixed } t \geqq 0. \end{cases}$$

Let $k \in \{0, 1, \dots, m-1\}$. *Then* (4.23) *has a radial entire solution* u *which has no zero in* $\boldsymbol{R}^N$ *and has the property that*

$$\lim_{|x| \to \infty} \frac{u(|x|)}{|x|^{2k}} \text{ exists and is a nonzero finite value}$$

if and only if

$$N \geqq 2(m-k)+1 \quad \text{and} \quad \int_0^\infty t^{2(m-k)-1} |F(t, ct^{2k})|\, dt < \infty \quad \text{for some } c \neq 0.$$

THEOREM 4.9. *Let* j *be an integer such that* $1 \leqq j \leqq m-1$ *and* $(-1)^{j+1}\sigma = 1$.

Suppose that h_{j-1} *is a continuous and strictly superlinear function on* $[0, \infty) \times \boldsymbol{R}^j$ *satisfying*

(4.14)
$$0 \leqq h_{j-1}(t, u_0, u_1, \dots, u_{j-1}) \operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \qquad \text{on } D_j.$$

If (0.1) *has a radial entire solution of class* $\mathscr{K}_j$, *then*

(4.15)
$$\int_0^\infty t^{2(m-j)+1} |h_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \dots, c)| \, dt < \infty \qquad \text{for some } c \neq 0.$$

PROOF. Let u be a radial entire solution of (0.1) in the class $\mathscr{K}_j$. We suppose that $u(t) > 0$ for $t \geqq 0$. It follows from (ii) of Theorem 2.2 that

$$\Delta^i u(t) = \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0) t^{2(k-i)} + \rho_N(j-i)(\Delta^j u)(\infty) t^{2(j-i)}$$
$$+ \Phi_N^{j-i} \Psi_N^{m-j} f(\cdot, \bar{u})(t), \qquad t \geqq 0, \ 0 \leqq i \leqq j-1;$$

hence, by (4.14) and the fact that $\Delta^j u(\infty) \geqq 0$,

(4.27)
$$\Delta^i u(t) \geqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0) t^{2(k-i)} + \Phi_N^{j-i} \Psi_N^{m-j} h_{j-1}(\cdot, u, \Delta u, \dots, \Delta^{j-1} u)(t),$$
$$t \geqq 0, \ 0 \leqq i \leqq j-1.$$

From (i) of Lemma 1.11 we easily see that

$$\Delta^i u(t) \geqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0) t^{2(k-i)} \tag{4.28}$$
$$+ B_N(j-i, m-j) t^{2(j-i-1)} \int_0^t s^{2(m-j)+1} h_{j-1}(s, u(s), \dots, \Delta^{j-1} u(s)) \, ds$$

for $t \geqq 0$ and $0 \leqq i \leqq j-1$, where $B_N(j-i, m-j)$ is a positive constant.

Assume that the conclusion (4.15) does not hold in the following sense:

$$\int_0^\infty s^{2(m-j)+1} h_{j-1}(s, cs^{2(j-1)}, cs^{2(j-2)}, \dots, c) \, ds = \infty \qquad \text{for every } c > 0. \tag{4.29}$$

Since u satisfies $\Delta^i u(t) \geqq c_* t^{2(j-i-1)}$, $t \geqq T$, $0 \leqq i \leqq j-1$, for some $c_* > 0$ and $T \geqq 1$ (see (2.22) in Theorem 2.3), (4.29) implies

$$\int_0^\infty s^{2(m-j)+1} h_{j-1}(s, u(s), \Delta u(s), \dots, \Delta^{j-1} u(s)) \, ds = \infty.$$

Then, using (4.28), we see that there exist constants T_1 and c_1 such that $T_1 \geqq T$, $0 < c_1 < B_N(j-i, m-j)$ for all $i \in \{0, 1, \ldots, j-1\}$, and

$$\Delta^i u(t) \geqq c_1 t^{2(j-i-1)} \int_0^t s^{2(m-j)+1} h_{j-1}(s, u(s), \ldots, \Delta^{j-1} u(s))\, ds \tag{4.30}$$

for $t \geqq T_1$, $0 \leqq i \leqq j-1$. Define $I(t)$ by

$$I(t) = \int_0^t s^{2(m-j)+1} h_{j-1}(s, u(s), \ldots, \Delta^{j-1} u(s))\, ds$$

and take a number $T_2 \geqq T_1$ so that $I(t) > 0$ for every $t \geqq T_2$. By (4.30) we have

$$\Delta^i u(t) \geqq c_1 t^{2(j-i-1)} I(t), \qquad t \geqq T_2,\ 0 \leqq i \leqq j-1,$$

and hence

$$\begin{aligned}\frac{d}{dt} I(t) &= t^{2(m-j)+1} h_{j-1}(t, u(t), \ldots, \Delta^{j-1} u(t)) \\ &\geqq t^{2(m-j)+1} h_{j-1}(t, c_1 t^{2(j-1)} I(t), \ldots, c_1 I(t))\end{aligned}$$

for $t \geqq T_2$. The strict superlinearity of h_{j-1} implies that there is a constant $\gamma > 1$ such that

$$\begin{aligned}&[I(t)]^{-\gamma} h_{j-1}(t, c_1 t^{2(j-1)} I(t), \ldots, c_1 I(t)) \\ &\geqq [I(T_2)]^{-\gamma} h_{j-1}(t, c_1 t^{2(j-1)} I(T_2), \ldots, c_1 I(T_2)), \qquad t \geqq T_2.\end{aligned}$$

Therefore we have

$$\frac{d}{dt} I(t) \geqq [I(T_2)]^{-\gamma} t^{2(m-j)+1} h_{j-1}(t, c_1 I(T_2) t^{2(j-1)}, \ldots, c_1 I(T_2)) [I(t)]^{\gamma}$$

for $t \geqq T_2$. Dividing this inequality by $[I(t)]^{\gamma}$ and integrating over $[T_2, t]$, we find that

$$\begin{aligned}&-\frac{1}{\gamma-1}[I(t)]^{1-\gamma} + \frac{1}{\gamma-1}[I(T_2)]^{1-\gamma} \\ &\geqq [I(T_2)]^{-\gamma} \int_{T_2}^t s^{2(m-j)+1} h_{j-1}(s, c_1 I(T_2) s^{2(j-1)}, \ldots, c_1 I(T_2))\, ds\end{aligned}$$

for $t \geqq T_2$, which in the limit as $t \to \infty$ yields

$$\int_{T_2}^{\infty} s^{2(m-j)+1} h_{j-1}(s, c_1 I(T_2) s^{2(j-1)}, \ldots, c_1 I(T_2))\, ds < \infty.$$

But this is a contradiction to (4.29). Thus the integral in (4.15) converges for some $c > 0$. It is similarly shown that if $u(t) < 0$ on $[0, \infty)$, then the integral in (4.15) converges for some $c < 0$. The proof of Theorem 4.9 is complete.

THEOREM 4.10. *Let j be an integer with $1 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$ and consider equation $(0.1)_{j-1}$. Suppose that $N \geqq 2(m-j)+3$ and that f_{j-1} is strictly superlinear on $[0, \infty) \times \boldsymbol{R}^j$. Then the following three statements are equivalent*:

(i) *there exists a radial entire solution of $(0.1)_{j-1}$ in the class $\mathscr{K}_j^*$*;

(ii) *there exists a radial entire solution of $(0.1)_{j-1}$ in the class $\mathscr{K}_j$*;

(iii) *the integral condition $(4.22)_{j-1}$ is satisfied.*

PROOF. It is trivial that (i) implies (ii). Theorem 4.9 for the case $f = h_{j-1} = f_{j-1}$ shows that (ii) implies (iii). Theorem 4.6 for the case $f = g_{j-1} = f_{j-1}$ shows, under condition $(4.22)_{j-1}$, that $(0.1)_{j-1}$ has a radial entire solution of class $\mathscr{K}_j^*[\min]$, which is obviously of class $\mathscr{K}_j^*$. This means that (iii) implies (i).

THEOREM 4.11. *Let j be an integer such that $1 \leqq j \leqq m-1$ and $(-1)^{j+1}\sigma = 1$. Suppose that h_{j-1} is a continuous and strictly sublinear function on $[0, \infty) \times \boldsymbol{R}^j$ satisfying*

(4.14)
$$0 \leqq h_{j-1}(t, u_0, u_1, \dots, u_{j-1}) \operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \qquad \text{on } D_j.$$

If (0.1) has a radial entire solutio of class $\mathscr{K}_j$, then

$$\int_0^\infty t^{2(m-j)-1} |h_{j-1}(t, ct^{2j}, ct^{2(j-1)}, \dots, ct^2)| \, dt < \infty \qquad \text{for some } c \neq 0. \tag{4.31}$$

PROOF. Let $u \in \mathscr{K}_j$ be a radial entire solution of (0.1) such that $u(t) > 0$ for $t \geqq 0$. As in the proof of Theorem 4.9, inequality (4.27) can be derived. By (i) of Lemma 1.11 we have

$$\begin{aligned} \Delta^i u(t) \geqq {} & \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0) t^{2(k-i)} \\ & + B_N(j-i, m-j) \left\{ t^{2(j-i)} \int_t^\infty s^{2(m-j)-1} h_{j-1}(s, u(s), \dots, \Delta^{j-1} u(s)) \, ds \right. \\ & \left. + t^{2(j-i-1)} \int_0^t s^{2(m-j)+1} h_{j-1}(s, u(s), \dots, \Delta^{j-1} u(s)) \, ds \right\} \end{aligned} \tag{4.32}$$

for $t \geqq 0$, $0 \leqq i \leqq j-1$. The two possibilities occur:

$$(4.33)\qquad \int_0^\infty t^{2(m-j)+1} h_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t))\, dt < \infty; \text{ or}$$

$$(4.34)\qquad \int_0^\infty t^{2(m-j)+1} h_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t))\, dt = \infty.$$

Suppose that (4.33) occurs. By Theorem 2.3 we have

$$c_* t^{2(j-i-1)} \leqq \Delta^i u(t) \leqq c^* t^{2(j-i)}, \qquad t \geqq T,\ 0 \leqq i \leqq j-1,$$

for some $c_* > 0$, $c^* > 0$ and $T \geqq 1$. From the sublinearity of h_{j-1} it follows that, for each fixed $t \geqq T$, if $\lambda \geqq 1$ then

$$h_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t)) \geqq \lambda^{-1} h_{j-1}(t, \lambda u(t), \ldots, \lambda \Delta^{j-1}u(t)).$$

Taking

$$\lambda = \max\{c^* t^{2(j-i)}/\Delta^i u(t) : 0 \leqq i \leqq j-1\}$$

and noticing that $1 \leqq \lambda \leqq c^* t^2/c_*$, we see that

$$h_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t)) \geqq (c_*/c^*) t^{-2} h_{j-1}(t, c^* t^{2j}, \ldots, c^* t^2).$$

Then assumption (4.33) implies that (4.31) holds for $c = c^*$.

Suppose that (4.34) occurs. Then from (4.32) it follows that there exists $T_1 \geqq 1$ such that

$$(4.35)\qquad \Delta^i u(t) \geqq c_1 t^{2(j-i)} \int_t^\infty s^{2(m-j)-1} h_{j-1}(s, u(s), \ldots, \Delta^{j-1}u(s))\, ds$$

for $t \geqq T_1$, $0 \leqq i \leqq j-1$, where c_1 is a positive constant such that $B_N(j-i, m-j) > c_1$ for all $i \in \{0, 1, \ldots, j-1\}$. Let $J(t)$ denote the integral on the right-hand side of (4.35); thus

$$\Delta^i u(t) \geqq c_1 t^{2(j-i)} J(t), \qquad t \geqq T_1,\ 0 \leqq i \leqq j-1,$$

and consequently

$$\begin{aligned} -\frac{d}{dt} J(t) &= t^{2(m-j)-1} h_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t)) \\ &\geqq t^{2(m-j)-1} h_{j-1}(t, c_1 t^{2j} J(t), \ldots, c_1 t^2 J(t)) \end{aligned}$$

for $t \geqq T_1$. Note that $0 < J(t) \leqq J(T_1)$ for $t \geqq T_1$. Then the strict sublinearity of h_{j-1} implies that there is a constant γ such that $0 < \gamma < 1$ and

$$\begin{aligned} &[J(t)]^{-\gamma} h_{j-1}(t, c_1 t^{2j} J(t), \ldots, c_1 t^2 J(t)) \\ &\geqq [J(T_1)]^{-\gamma} h_{j-1}(t, c_1 t^{2j} J(T_1), \ldots, c_1 t^2 J(T_1)) \end{aligned}$$

for $t \geqq T_1$. Therefore we have

$$-\frac{d}{dt}J(t) \geqq [J(T_1)]^{-\gamma} t^{2(m-j)-1} h_{j-1}(t, c_1 J(T_1) t^{2j}, \ldots, c_1 J(T_1) t^2)[J(t)]^{\gamma}$$

for $t \geqq T_1$. This gives

$$-\frac{1}{1-\gamma}[J(t)]^{1-\gamma} + \frac{1}{1-\gamma}[J(T_1)]^{1-\gamma}$$

$$\geqq [J(T_1]]^{-\gamma} \int_{T_1}^{t} s^{2(m-j)-1} h_{j-1}(s, c_1 J(T_1) s^{2j}, \ldots, c_1 J(T_1) s^2)\, ds$$

for $t \geqq T_1$, from which it follows that

$$\int_{T_1}^{\infty} s^{2(m-j)-1} h_{j-1}(s, c_1 J(T_1) s^{2j}, \ldots, c_1 J(T_1) s^2)\, ds < \infty.$$

Therefore (4.31) holds for $c = c_1 J(T_1)$. This completes the proof of Theorem 4.11.

The following theorem is an easy consequence of (i) of Theorem 4.1, and Theorems 4.3 and 4.11.

THEOREM 4.12. *Let j be an integer with $1 \leqq j \leqq m-1$, $(-1)^{j+1}\sigma = 1$ and consider equation $(0.1)_{j-1}$. Suppose that f_{j-1} satisfies condition (N_j) and is strictly sublinear on $[0, \infty) \times \boldsymbol{R}^j$. Then the following three statements are equivalent:*

(i) *there exists a radial entire solution of $(0.1)_{j-1}$ in the class $\mathscr{K}_j^*$;*

(ii) *there exists a radial entire solution of $(0.1)_{j-1}$ in the class $\mathscr{K}_j$;*

(iii) *the conditions $N \geqq 2(m-j)+1$ and*

$$\int_0^{\infty} t^{2(m-j)-1} |f_{j-1}(t, ct^{2j}, ct^{2(j-1)}, \ldots, ct^2)|\, dt < \infty \qquad \text{for some } c \neq 0 \tag{4.36}$$

are satisfied.

THEOREM 4.13. *Let j be an integer with $1 \leqq j \leqq m-1$ and $(-1)^{j+1}\sigma = 1$. Suppose that $N \geqq 2(m-j)+3$ and g_{j-1} is a continuous and sublinear function on $[0, \infty) \times \boldsymbol{R}^j$ satisfying*

$$f(t, u_0, u_1, \ldots, u_{m-1}) \operatorname{sgn} u_0 \leqq g_{j-1}(t, u_0, u_1, \ldots, u_{j-1}) \operatorname{sgn} u_0 \qquad \text{on } D_j. \tag{4.16}$$

If equation (0.1) has a radial entire solution of class $\mathscr{K}_j[\mathrm{int}]$, then either

(4.37$^+$)

$$\int_0^{\infty} t^{2(m-j)+1} |g_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \ldots, c)|\, dt = \infty \qquad \text{for every } c > 0$$

or

(4.37⁻)

$$\int_0^\infty t^{2(m-j)+1}|g_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \ldots, c)|\,dt = \infty \qquad \textit{for every } c < 0.$$

PROOF. Let $u \in \mathscr{K}_j[\mathrm{int}]$ be a radial entire solution of (0.1), and suppose that $u(t) > 0$ for $t \geqq 0$. We claim that (4.37^+) holds. Since $u \in \mathscr{K}_j[\mathrm{int}]$ satisfies $\Delta^j u(\infty) = 0$, it follows from (ii) of Theorem 2.2 that

$$\Delta^i u(t) = \sum_{k=i}^{j-1} \rho_N(k-i)\,(\Delta^k u)(0)\,t^{2(k-i)} + \Phi_N^{j-i}\Psi_N^{m-j} f(\cdot, \bar{u})(t), \tag{4.38}$$

$$t \geqq 0,\ 0 \leqq i \leqq j-1.$$

Then (ii) of Lemma 1.11 shows that

$$\begin{aligned} \Delta^i u(t) \leqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0)\,t^{2(k-i)} \\ + C_N(j-i, m-j)\left\{t^{2(j-i)} \int_t^\infty s^{2(m-j)-1} f(s, \bar{u}(s))\,ds \right. \\ \left. + t^{2(j-i-1)} \int_0^t s^{2(m-j)+1} f(s, \bar{u}(s))\,ds \right\} \end{aligned} \tag{4.39}$$

for $t \geqq 0$, $0 \leqq i \leqq j-1$, where $C_N(j-i, m-j)$ is a positive constant and $\bar{u}$ is defined by (3.5).

On the other hand, (i) of Lemma 1.11 shows that

$$\begin{aligned} \Phi_N \Psi_N^{m-j} f(\cdot, \bar{u})(t) \\ \geqq B_N(1, m-j)\left\{t^2 \int_t^\infty s^{2(m-j)-1} f(s, \bar{u}(s))\,ds + \int_0^t s^{2(m-j)+1} f(s, \bar{u}(s))\,ds \right\} \end{aligned} \tag{4.40}$$

for $t \geqq 0$, where $B_N(1, m-j)$ is a positive constant. Therefore (4.39) and (4.40) together imply that

$$\begin{aligned} \Delta^i u(t) \leqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0)\,t^{2(k-i)} \\ + \frac{C_N(j-i, m-j)}{B_N(1, m-j)}\,t^{2(j-i-1)}\,\Phi_N \Psi_N^{m-j} f(\cdot, \bar{u})(t), \end{aligned}$$

so that, in view of (4.38),

$$\Delta^i u(t) \leqq \sum_{k=i}^{j-1} \rho_N(k-i)(\Delta^k u)(0)t^{2(k-i)} + \frac{C_N(j-i, m-j)}{B_N(1, m-j)} t^{2(j-i-1)}[\Delta^{j-1}u(t) - \Delta^{j-1}u(0)]$$

for $t \geqq 0$, $0 \leqq i \leqq j-1$. Since $\Delta^{j-1}u(t)$ tends to $+\infty$ as $t \to \infty$, using the above inequality, we conclude that there exist constants $M \geqq 1$ and $T \geqq 1$ such that

$$\text{(4.41)} \qquad 0 < \Delta^i u(t) \leqq Mt^{2(j-i-1)}\Delta^{j-1}u(t), \qquad t \geqq T,\ 0 \leqq i \leqq j-1.$$

Assume now that (4.37^+) does not hold, i.e., there is a constant $c > 0$ such that

$$\text{(4.42)} \qquad \int_0^\infty t^{2(m-j)+1} g_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \ldots, c)\,dt < \infty.$$

We will derive a contradiction. Take a number $T_1 \geqq T$ such that $\Delta^{j-1}u(t) \geqq c/M$ for $t \geqq T_1$, where M and c are constants satisfying (4.41) and (4.42), respectively. This is possible because $\Delta^{j-1}u(t) \to +\infty$ as $t \to \infty$. On the other hand, by (ii) of Theorem 2.3, there is a constant $c^* > 0$ such that $\Delta^{j-1}u(t) \leqq c^* t^2$ for $t \geqq T_1$. Therefore, using the sublinearity of g_{j-1} and (4.41), we have

$$\text{(4.43)} \qquad \begin{aligned} & g_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t)) \\ & \leqq g_{j-1}(t, Mt^{2(j-1)}\Delta^{j-1}u(t), \ldots, M\Delta^{j-1}u(t)) \\ & \leqq (M/c)(\Delta^{j-1}u)(t) g_{j-1}(t, ct^{2(j-1)}, \ldots, c) \\ & \leqq (Mc^*/c)t^2 g_{j-1}(t, ct^{2(j-1)}, \ldots, c) \end{aligned}$$

for $t \geqq T_1$. This implies that

$$\int_0^\infty t^{2(m-j)-1} g_{j-1}(t, u(t), \ldots, \Delta^{j-1}u(t))\,dt < \infty$$

by (4.42); and it follows from (4.16) and (4.39) that

$$\text{(4.44)} \qquad \begin{aligned} \Delta^{j-1}u(t) \leqq{} & \Delta^{j-1}u(0) \\ & + C_N(1, m-j)\left\{t^2 \int_t^\infty s^{2(m-j)-1} g_{j-1}(s, u(s), \ldots, \Delta^{j-1}u(s))\,ds \right. \\ & \left. + \int_0^t s^{2(m-j)+1} g_{j-1}(s, u(s), \ldots, \Delta^{j-1}u(s))\,ds \right\} \end{aligned}$$

for $t \geqq 0$. Note that

(4.45) $$g_{j-1}(t, u(t),\dots,\Delta^{j-1}u(t)) \leqq (M/c)(\Delta^{j-1}u)(t)g_{j-1}(t, ct^{2(j-1)},\dots,c)$$

for $t \geqq T_1$ (see (4.43)), and recall that $\Delta^{j-1}u(t)$ tends monotonically to $+\infty$ as $t \to \infty$. Then, for $t \geqq \tau \geqq T_1$,

$$\begin{aligned}
&\frac{1}{\Delta^{j-1}u(t)}\int_0^t s^{2(m-j)+1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds \\
&\leqq \frac{1}{\Delta^{j-1}u(t)}\int_0^\tau s^{2(m-j)+1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds \\
&\quad + \frac{M}{c\Delta^{j-1}u(t)}\int_\tau^t \Delta^{j-1}u(s)\cdot s^{2(m-j)+1}g_{j-1}(s, cs^{2(j-1)},\dots,c)\,ds \\
&\leqq \frac{1}{\Delta^{j-1}u(t)}\int_0^\tau s^{2(m-j)+1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds \\
&\quad + \frac{M}{c}\int_\tau^t s^{2(m-j)+1}g_{j-1}(s, cs^{2(j-1)},\dots,c)\,ds,
\end{aligned}$$

which in the upper limit as $t \to \infty$ gives

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{\Delta^{j-1}u(t)}\int_0^t s^{2(m-j)+1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds \\
&\leqq \frac{M}{c}\int_\tau^\infty s^{2(m-j)+1}g_{j-1}(s, cs^{2(j-1)},\dots,c)\,ds.
\end{aligned}$$

Since $\tau \geqq T_1$ is arbitrary, the left-hand side in the above is zero; and consequently

(4.46) $$\lim_{t\to\infty}\frac{1}{\Delta^{j-1}u(t)}\int_0^t s^{2(m-j)+1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds = 0.$$

By (4.44) and (4.46) we can conclude without difficulty that there exist $L > 0$ and $T_2 \geqq T_1$ such that

(4.47) $$\Delta^{j-1}u(t) \leqq Lt^2\int_t^\infty s^{2(m-j)-1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds$$

for $t \geqq T_2$. Then it follows from (4.47) and (4.45) that

$$\begin{aligned}
&g_{j-1}(t, u(t),\dots,\Delta^{j-1}u(t)) \\
&\leqq (LM/c)t^2 g_{j-1}(t, ct^{2(j-1)},\dots,c)\int_t^\infty s^{2(m-j)-1}g_{j-1}(s, u(s),\dots,\Delta^{j-1}u(s))\,ds
\end{aligned}$$

for $t \geqq T_2$, which implies that

$$K(t) = \int_t^\infty s^{2(m-j)-1} g_{j-1}(s, u(s), \dots, \Delta^{j-1} u(s))\, ds$$

satisfies

$$K(t) \leqq (LM/c) \int_t^\infty s^{2(m-j)+1} g_{j-1}(s, cs^{2(j-1)}, \dots, c) K(s)\, ds$$

for $t \geqq T_2$. Noticing that $K(t)$ is positive and nonincreasing on $[T_2, \infty)$, we have

$$1 \leqq (LM/c) \int_t^\infty s^{2(m-j)+1} g_{j-1}(s, cs^{2(j-1)}, \dots, c)\, ds$$

for $t \geqq T_2$. However this is a contradiction since the right-hand side tends to 0 as $t \to \infty$. Thus we conclude that, if (0.1) has a positive radial entire solution $u \in \mathscr{K}_j[\mathrm{int}]$, then ($4.37^+$) holds.

Likewise we can show that (4.37^-) holds if (0.1) has a negative radial entire solution in $\mathscr{K}_j[\mathrm{int}]$.

THEOREM 4.14. *Let j be an integer with $1 \leqq j \leqq m-1$ and $(-1)^{j+1}\sigma = 1$, and consider equation $(0.1)_{j-1}$. Suppose that $N \geqq 2(m-j)+3$ and that f_{j-1} is strictly sublinear on $[0, \infty) \times \boldsymbol{R}^j$ and satisfies*

$$f_{j-1}(t, -u_0, -u_1, \dots, -u_{j-1}) = -f_{j-1}(t, u_0, u_1, \dots, u_{j-1})$$

for $t \geqq 0$, $u_i > 0$ $(0 \leqq i \leqq j-1)$. Then the following statements are equivalent:

(i) *there exists a radial entire solution of $(0.1)_{j-1}$ of class $\mathscr{K}_j^*[\mathrm{int}]$;*

(ii) *there exists a radial entire solution of $(0.1)_{j-1}$ of class $\mathscr{K}_j[\mathrm{int}]$;*

(iii) *the following two integral conditions are satisfied:*

$$(4.36) \qquad \int_0^\infty t^{2(m-j)-1} |f_{j-1}(t, ct^{2j}, ct^{2(j-1)}, \dots, ct^2)|\, dt < \infty \qquad \text{for some } c \neq 0$$

and

$$(4.48) \qquad \int_0^\infty t^{2(m-j)+1} |f_{j-1}(t, ct^{2(j-1)}, ct^{2(j-2)}, \dots, c)|\, dt = \infty \qquad \text{for every } c \neq 0.$$

PROOF. Suppose that $(0.1)_{j-1}$ has a radial entire solution in $\mathscr{K}_j[\mathrm{int}]$. Then, using Theorem 4.11 with $f = h_{j-1} = f_{j-1}$ and Theorem 4.13 with $f = g_{j-1} = f_{j-1}$, we have (4.36) and (4.48). This means that (ii) implies (iii). Suppose that (4.36) and (4.48) hold. Theorem 4.3 applied to the case $f = g_j = f_{j-1}$ ensures the existence of a radial entire solution u_2 of $(0.1)_{j-1}$ belonging to $\mathscr{K}_j^*[\mathrm{int}] \cup \mathscr{K}_j^*[\mathrm{min}]$. But it follows from Theorem 4.5 with $f = h_{j-1} = f_{j-1}$ that this u_2 does not belong to $\mathscr{K}_j^*[\mathrm{min}]$, and consequently

$u_2 \in \mathscr{K}_j^*[\mathrm{int}]$. This proves that (iii) implies (i). It is trivial that (i) implies (ii).

Theorems 0.5–0.7 mentioned in the Introduction are the special versions of Theorems 4.4, 4.7, 4.10, 4.12 and 4.14.

5. Existence of radial entire solutions II

In this section the existence of radial entire solutions of (0.1) belonging to $\mathscr{K}_0, \mathscr{K}_0[\mathrm{int}]$ and $\mathscr{K}_0[\min]$ is discussed. Note that these classes are nonempty only for the case $\sigma = -1$.

THEOREM 5.1. *Let $\sigma = -1$ in* (0.1). *If equation* (0.1) *has a non-constant radial entire solution u of class $\mathscr{K}_0$, then $N \geqq 2m+1$.*

PROOF. We assume that $u(t) > 0$, $\not\equiv$ const. on $[0, \infty)$. By (iii) of Theorem 2.2, $\Psi_N^m(\Delta^m u)$ is well defined on $[0, \infty)$ and

$$u(t) = u(\infty) + (-1)^m \Psi_N^m(\Delta^m u)(t), \qquad t \geqq 0.$$

From the assumption, we see that $(-1)^m \Delta^m u(t) \geqq 0$, $\not\equiv 0$ on $[0, \infty)$. Then it follows from Theorem 1.8 that $N \geqq 2m+1$. This completes the proof of Theorem 5.1.

Let D_0 be the set of all $(t, u_0, u_1, \dots, u_{m-1}) \in [0, \infty) \times \boldsymbol{R}^m$ satisfying

$$(5.1) \qquad t \geqq 0, \quad u_0 \neq 0, \quad (-1)^i u_i \operatorname{sgn} u_0 \geqq 0 \quad (1 \leqq i \leqq m-1).$$

THEOREM 5.2. *Let $\sigma = -1$ in* (0.1). *Suppose that $h_0(t, u_0)$ is continuous and restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}$ and satisfies*

$$(5.2) \qquad 0 \leqq h_0(t, u_0) \operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \qquad \text{on } D_0.$$

If equation (0.1) *has a radial entire solution of class $\mathscr{K}_0[\min]$, then*

$$(5.3) \qquad \int_0^\infty t^{N-1} |h_0(t, ct^{-N+2m})| \, dt < \infty \qquad \text{for some } c \neq 0.$$

PROOF. Suppose that u is a radial entire solution of (0.1) such that $u \in \mathscr{K}_0[\min]$ and $u(t) > 0$ for $t \geqq 0$. It follows from (iii) of Theorem 2.2 that

$$u(t) = (-1)^m \Psi_N^m(\Delta^m u)(t) = \Psi_N^m f(\cdot, \bar{u})(t), \qquad t \geqq 0,$$

where $\bar{u} = (u, \Delta u, \dots, \Delta^{m-1} u)$. By (1.22) in Lemma 1.9 we have

$$(5.4) \qquad \lim_{t \to \infty} t^{N-2m} u(t) = e_N(m) \int_0^\infty s^{N-1} f(s, \bar{u}(s)) \, ds \in (0, \infty].$$

By the definition of $\mathscr{K}_0[\min]$, $\lim_{t \to \infty} t^{N-2m} u(t)$ exists and is a positive finite value, and so there are $c_* > 0$, $c^* > 0$ and $T \geqq 1$ such that

$$c_* t^{-N+2m} \leqq u(t) \leqq c^* t^{-N+2m}, \qquad t \geqq T. \tag{5.5}$$

Combining (5.5) with the inequality

$$\int_0^\infty s^{N-1} f(s, \bar{u}(s))\, ds < \infty,$$

which follows from (5.4), we see that (5.3) holds as desired. This completes the proof of Theorem 5.2.

THEOREM 5.3. *Let $\sigma = -1$ and $N \geqq 2m+1$. Suppose that the functions $h_0(t, u_0)$ and $g_0(t, u_0)$ are continuous and restrictively nondecreasing on $[0, \infty) \times \boldsymbol{R}$ and satisfy*

$$\begin{aligned} 0 \leqq h_0(t, u_0) \operatorname{sgn} u_0 &\leqq f(t, u_0, u_1, \ldots, u_{m-1}) \operatorname{sgn} u_0 \\ &\leqq g_0(t, u_0) \operatorname{sgn} u_0 \qquad \text{on } D_0, \end{aligned} \tag{5.6}$$

where D_0 is the set of all points $(t, u_0, u_1, \ldots, u_{m-1})$ satisfying (5.1). *In addition, suppose that, for each fixed $t \geqq 0$, $h_0(t, u_0)/u_0$ is nonincreasing in $u_0 \in (0, \infty)$ and nondecreasing in $(-\infty, 0)$, and*

$$\lim_{u_0 \to \pm 0} \frac{h_0(t, u_0)}{u_0} = +\infty, \tag{5.7}$$

Suppose that g_0 is strongly sublinear on $[0, \infty) \times \boldsymbol{R}$ and that

$$g_0(t, \varphi(t)) \operatorname{sgn} \varphi(t) \geqq 0, \ \not\equiv 0 \qquad \text{on } [0, \infty)$$

for every bounded $\varphi \in C[0, \infty)$, $\varphi(t) \neq 0$ $(t \geqq 0)$.

(i) *If*

$$\int_1^\infty t^{2m-1} |g_0(t, c)|\, dt < \infty \qquad \text{for some } c \neq 0, \tag{5.8}$$

then equation (0.1) *has a radial entire solution u such that $u(|x|) \neq 0$ in $\boldsymbol{R}^N$ and $u(|x|) \to 0$ as $|x| \to \infty$.*

(ii) *If*

$$\int_1^\infty t^{N-1} |g_0(t, ct^{-N+2m})|\, dt < \infty \qquad \text{for some } c \neq 0, \tag{5.9}$$

then equation (0.1) *has a radial entire solution u such that $u(|x|) \neq 0$ in $\boldsymbol{R}^N$ and $\lim_{|x| \to \infty} |x|^{N-2m} u(|x|)$ exists and is a nonzero finite value.*

PROOF. (i) Without loss of generality we may assume that c in (5.8) is positive. The strong sublinearity of g_0, condition (5.8) and the Lebesgue dominated convergence theorem imply that

$$\lim_{\lambda\to+\infty} d_N(m)\int_0^\infty s^{2m-1}\frac{g_0(s,\lambda)}{\lambda}\,ds=0,$$

where $d_N(m)$ is a positive constant appearing in (1.20) with $i=m$. There exists $\lambda_0\geqq c$ such that

$$d_N(m)\int_0^\infty s^{2m-1}g_0(s,\lambda)\,ds\leqq\lambda \tag{5.10}$$

for all $\lambda\in[\lambda_0,\infty)$. For such a λ, define the function $v(t)$ by

$$v(t)=\Psi_N^m g_0(\cdot,\lambda)(t),\qquad t\geqq 0.$$

By the assumption on g_0 we see that $v(t)$ is well defined on $[0,\infty)$ (see Lemma 1.8) and that $v(t)>0$ on $[0,\infty)$ and $\lim_{t\to\infty}v(t)=0$. From inequality (1.20) with $i=m$, $h(t)=g_0(t,\lambda)$ it follows that

$$v(t)\leqq d_N(m)\int_0^\infty s^{2m-1}g_0(s,\lambda)\,ds,\qquad t\geqq 0,$$

which implies by (5.10) that $v(t)\leqq\lambda$ for $t\geqq 0$. Then it is clear that

$$\begin{aligned}0&=-(-1)^m\Delta^m v(t)+g_0(t,\lambda)\\&\geqq-(-1)^m\Delta^m v(t)+g_0(t,v(t)),\qquad t\geqq 0.\end{aligned}$$

Applying Theorem 3.2 to the case of $h=h_0$ and $g=g_0$, we conclude that equation (0.1) has a radial entire solution u such that $u(|x|)>0$ for $x\in\boldsymbol{R}^N$ and $u(|x|)\to 0$ as $|x|\to\infty$.

(ii) The proof of (ii) is similar to that of (i). Let $c>0$ in (5.9). It is shown that

$$\lim_{\lambda\to+\infty} d_N(m)\int_0^\infty \max\{s^{2m-1},s^{N-1}\}\frac{g_0(s,\lambda q_{N,m}(s))}{\lambda}\,ds=0,$$

where $q_{N,m}(t)=\min\{1,t^{-N+2m}\}$, $t\geqq 0$. Note that $\max\{t^{2m-1},t^{N-1}\}=t^{N-1}$ and $q_{N,m}(t)=t^{-N+2m}$ for $t\geqq 1$ because of $N\geqq 2m+1$. There is $\lambda_0\geqq c$ such that for $\lambda\in[\lambda_0,\infty)$

$$d_N(m)\int_0^\infty \max\{s^{2m-1},s^{N-1}\}\,g_0(s,\lambda q_{N,m}(s))\,ds\leqq\lambda.$$

Fix a constant λ satisfying the above inequality, and define $v(t)$ by

$$v(t)=\Psi_N^m g_0(\cdot,\lambda q_{N,m})(t),\qquad t\geqq 0.$$

It can be proved that $v(t) > 0$ for $t \geqq 0$; $v(t) \leqq \lambda q_{N,m}(t)$ for $t \geqq 0$ (see (ii) of Lemma 1.10); and

$$\lim_{t\to\infty} t^{N-2m} v(t) = e_N(m) \int_0^\infty s^{N-1} g_0(s, \lambda q_{N,m}(s))\, ds$$

(see Lemma 1.9). Since the right-hand side in the above is finite and positive, we find that $v \in \mathscr{K}_0[\min]$. It is easy to see that

$$-(-1)^m \Delta^m v(t) + g_0(t, v(t)) \leqq 0, \qquad t \geqq 0,$$

and so it follows from Theorem 3.2 that equation (0.1) has a radial entire solution u in the class $\mathscr{K}_0[\min]$. This completes the proof of Theorem 5.3.

REMARK 5.1. We know by Theorem 4.3 for the case $j = 0$ that (5.8) is sufficient for (0.1) to have a radial entire solution u of class $\mathscr{K}_0[\max]$. Thus, if (5.8) is satisfied, then (0.1) has at least two different kinds of solutions $u_1 \in \mathscr{K}_0[\max]$ and $u_2 \in \mathscr{K}_0[\mathrm{int}] \cup \mathscr{K}_0[\min]$.

Let $f_0(t, u)$ be continuous on $[0, \infty) \times \boldsymbol{R}$ and satisfy

(5.11) $$u f_0(t, u) \geqq 0, \ \not\equiv 0 \qquad \text{for } (t, u) \in [0, \infty) \times \boldsymbol{R}.$$

Consider the conditions on f_0:

(5.12) $f_0(t, \varphi(t)) \operatorname{sgn} \varphi(t) \geqq 0, \ \not\equiv 0$ on $[0, \infty)$
for every bounded $\varphi \in C[0, \infty)$, $\varphi(t) \neq 0$ $(t \geqq 0)$;

(5.13) $|f_0(t, u)| \leqq |f_0(t, v)|$ for $t \geqq 0$, $uv > 0$, $|u| \leqq |v|$;

(5.14) $u^{-1} f_0(t, u) \geqq v^{-1} f_0(t, v)$ for $t \geqq 0$, $uv > 0$, $|u| \leqq |v|$;

(5.15) $\lim_{u\to\pm\infty} u^{-1} f_0(t, u) = 0$ for each fixed $t \geqq 0$;

(5.16) $\lim_{u\to\pm 0} u^{-1} f_0(t, u) = +\infty$ for each fixed $t \geqq 0$;

(5.17) there exists a number γ, $0 < \gamma < 1$, such that
$|u|^{-\gamma} |f_0(t, u)| \geqq |v|^{-\gamma} |f_0(t, u)|$ for $t \geqq 0$, $uv > 0$, $|u| \leqq |v|$;

(5.18) $f_0(t, -u) = -f_0(t, u)$ for $t \geqq 0$, $u > 0$.

The function f_0 satisfying (5.11) is strongly sublinear [resp. strictly sublinear] on $[0, \infty) \times \boldsymbol{R}$ if and only if (5.13)–(5.15) [resp. (5.13) and (5.17)] hold.

The following theorem is concerned with equations of the form

(5.19) $$(-1)^m \Delta^m u = f_0(|x|, u), \qquad x \in \boldsymbol{R}^N,$$

where f_0 is continuous on $[0, \infty) \times \boldsymbol{R}$ and satisfies (5.11).

THEOREM 5.4. *Suppose that f_0 satisfies* (5.12)–(5.16). *Then equation* (5.19) *has a radial entire solution of class $\mathscr{K}_0[\min]$ if and only if*

$$(5.20)\qquad N \geqq 2m+1 \quad \text{and} \quad \int_1^\infty t^{N-1}|f_0(t, ct^{-N+2m})|\,dt < \infty \quad \text{for some } c \neq 0.$$

Theorem 5.4 follows from Theorems 5.1, 5.2 and (ii) of Theorem 5.3.

THEOREM 5.5. *Let $\sigma = -1$. Suppose that $h_0(t, u_0)$ is a continuous and strictly sublinear function on $[0, \infty) \times \boldsymbol{R}$ satisfying*

$$(5.2)\qquad 0 \leqq h_0(t, u_0)\operatorname{sgn} u_0 \leqq f(t, u_0, u_1, \dots, u_{m-1})\operatorname{sgn} u_0 \qquad \text{on } D_0.$$

If equation (0.1) *has a radial entire solution in the class $\mathscr{K}_0$, then*

$$(5.21)\qquad \int_1^\infty t^{2m-1}|h_0(t, c)|\,dt < \infty \qquad \text{for some } c \neq 0.$$

PROOF. Let $u \in \mathscr{K}_0$ be a radial entire solution of (0.1) such that $u(t) > 0$ for $t \geqq 0$. It is easily seen that

$$u(t) \geqq \Psi_N^m h_0(\cdot, u)(t), \qquad t \geqq 0.$$

By Lemma 1.8 we find that

$$(5.22)\qquad u(t) \geqq c_N(m)\int_t^\infty s^{2m-1}h_0(s, u(s))\,ds, \qquad t \geqq 0.$$

Assume that $h_0(t, u(t)) \equiv 0$ on $[t_1, \infty)$ for some $t_1 \geqq 0$. Since $0 < u(t) \leqq u(0)$ for $t \geqq 0$ and h_0 is strictly sublinear on $[0, \infty) \times \boldsymbol{R}$, we have $[u(t)]^{-\gamma}h_0(t, u(t)) \geqq [u(0)]^{-\gamma}h_0(t, u(0))$ for $t \geqq 0$, where γ is a constant such that $0 < \gamma < 1$. Then, $h_0(t, u(0)) \equiv 0$ for $t \geqq t_1$; and (5.21) is trivially satisfied.

Assume that $h_0(t, u(t)) \geqq 0$, $\not\equiv 0$ on $[t_1, \infty)$ for all $t_1 \geqq 0$. We denote the integral in (5.22) by $J(t)$. Observe that $J(t)$ is positive on $[0, \infty)$ and

$$-\frac{d}{dt}J(t) = t^{2m-1}h_0(t, u(t)) \geqq t^{2m-1}h_0(t, c_N(m)J(t)), \qquad t \geqq 0.$$

Since the strict sublinearity of h_0 implies

$$[J(t)]^{-\gamma}h_0(t, c_N(m)J(t)) \geqq [J(0)]^{-\gamma}h_0(t, c_N(m)J(0)), \qquad t \geqq 0,$$

we have

$$-[J(t)]^{-\gamma}\frac{d}{dt}J(t) \geqq [J(0)]^{-\gamma}t^{2m-1}h_0(t, c_N(m)J(0)), \qquad t \geqq 0.$$

An integration of the above inequality yields

$$- \frac{1}{1-\gamma}[J(t)]^{1-\gamma} + \frac{1}{1-\gamma}[J(0)]^{1-\gamma}$$

$$\geqq [J(0)]^{-\gamma} \int_0^t s^{2m-1} h_0(s, c_N(m)J(0))\, ds, \qquad t \geqq 0,$$

from which it follows that

$$\int_0^\infty s^{2m-1} h_0(s, c_N(m)J(0))\, ds < \infty.$$

Thus (5.21) also holds in this case. The proof of Theorem 5.5 is complete.

Combining Theorems 5.1 and 5.5 with Theorem 4.3 for the case $j = 0$, we have the following theorem.

THEOREM 5.6. *Consider equation* (5.19). *Suppose that* f_0 *satisfies* (5.12), (5.13) *and* (5.17). *Then equation* (5.19) *has a radial entire solution of class* $\mathscr{K}_0$ *if and only if*

$$N \geqq 2m+1 \quad \text{and} \quad \int_1^\infty t^{2m-1} |f_0(t, c)|\, dt < \infty \quad \text{for some } c \neq 0. \tag{5.23}$$

THEOREM 5.7. *Let* $\sigma = -1$. *Suppose that* g_0 *is continuous and sublinear on* $[0, \infty) \times \boldsymbol{R}$ *and satisfies*

$$f(t, u_0, u_1, \dots, u_{m-1}) \operatorname{sgn} u_0 \leqq g_0(t, u_0) \operatorname{sgn} u_0 \qquad \text{on } D_0. \tag{5.24}$$

If equation (0.1) *has a radial entire solution of class* $\mathscr{K}_0[\text{int}]$, *then either*

$$\int_1^\infty t^{N-1} |g_0(t, ct^{-N+2m})|\, dt = \infty \qquad \text{for every } c > 0 \tag{5.25$^+$}$$

or

$$\int_1^\infty t^{N-1} |g_0(t, ct^{-N+2m})|\, dt = \infty \qquad \text{for every } c < 0. \tag{5.25$^-$}$$

PROOF. Suppose that $u \in \mathscr{K}_0[\text{int}]$ is a radial entire solution of (0.1) and that $u(t) > 0$ for $t \geqq 0$. By Theorem 5.1 we have $N \geqq 2m+1$.

Assume now that there is a $c > 0$ such that

$$\int_1^\infty t^{N-1} g_0(t, ct^{-N+2m})\, dt < \infty. \tag{5.26}$$

Take a $T_1 \geqq 1$ so that $u(t) \geqq ct^{-N+2m}$ for $t \geqq T_1$, which is possible because of $u \in \mathscr{K}_0[\text{int}]$. From the sublinearity of g_0 it follows that

$$(5.27) \qquad g_0(t, u(t)) \leqq \frac{1}{c} u(t) t^{N-2m} g_0(t, ct^{-N+2m}), \qquad t \geqq T_1,$$

and hence

$$(5.28) \qquad g_0(t, u(t)) \leqq \frac{u(0)}{c} t^{N-2m} g_0(t, ct^{-N+2m}), \qquad t \geqq T_1.$$

In view of (5.26) and (5.28), we have

$$\int_1^\infty t^{2m-1} g_0(t, u(t))\, dt < \infty,$$

which combined with $N \geqq 2m + 1$ implies that $\Psi_N^m g_0(\cdot, u)$ is well defined on $[0, \infty)$. Then it is easily seen that

$$u(t) = (-1)^m \Psi_N^m(\Delta^m u)(t) \leqq \Psi_N^m g_0(\cdot, u)(t), \qquad t \geqq 0.$$

By Lemma 1.8 we have

(5.29)

$$u(t) \leqq d_N(m) \left\{ t^{-N+2m} \int_0^t s^{N-1} g_0(s, u(s))\, ds + \int_t^\infty s^{2m-1} g_0(s, u(s))\, ds \right\}, \qquad t \geqq 0.$$

Lemma 1.9 shows that $t^{N-2m} \Psi_N^m((-1)^m \Delta^m u)(t) = t^{N-2m} u(t)$ is nondecreasing in $[0, \infty)$. If $t \geqq \tau \geqq T_1$, then we can estimate as follows:

$$\begin{aligned} &\frac{t^{-N+2m}}{u(t)} \int_0^t s^{N-1} g_0(s, u(s))\, ds \\ &\leqq \frac{t^{-N+2m}}{u(t)} \int_0^\tau s^{N-1} g_0(s, u(s))\, ds + \frac{t^{-N+2m}}{cu(t)} \int_\tau^t s^{N-1} u(s) s^{N-2m} g_0(s, cs^{-N+2m})\, ds \\ &\leqq \frac{t^{-N+2m}}{u(t)} \int_0^\tau s^{N-1} g_0(s, u(s))\, ds + \frac{1}{c} \int_\tau^t s^{N-1} g_0(s, cs^{-N+2m})\, ds. \end{aligned}$$

Noting that $t^{-N+2m}/u(t) \to 0$ as $t \to \infty$, and taking the upper limit as $t \to \infty$, we have

$$\limsup_{t \to \infty} \frac{t^{-N+2m}}{u(t)} \int_0^t s^{N-1} g_0(s, u(s))\, ds \leqq \frac{1}{c} \int_\tau^\infty s^{N-1} g_0(s, cs^{-N+2m})\, ds.$$

Since the right-hand side of the above inequality tends to 0 as $\tau \to \infty$, the left-hand side is equal to 0, and hence

$$(5.30) \qquad \lim_{t \to \infty} \frac{t^{-N+2m}}{u(t)} \int_0^t s^{N-1} g_0(s, u(s))\, ds = 0.$$

In view of (5.29) and (5.30) there are $L>0$ and $T_2 \geqq T_1$ such that

$$u(t) \leqq L\int_t^\infty s^{2m-1} g_0(s, u(s))\,ds, \qquad t \geqq T_2. \tag{5.31}$$

Then it follows from (5.27) and (5.31) that

$$g_0(t, u(t)) \leqq \frac{L}{c} t^{N-2m} g_0(t, ct^{-N+2m}) \int_t^\infty s^{2m-1} g_0(s, u(s))\,ds$$

for $t \geqq T_2$, which implies that

$$K(t) = \int_t^\infty s^{2m-1} g_0(s, u(s))\,ds$$

satisfies

$$K(t) \leqq \frac{L}{c}\int_t^\infty s^{N-1} g_0(s, cs^{-N+2m}) K(s)\,ds$$

for $t \geqq T_2$. Noticing that $K(t)$ is positive (see (5.29)) and nonincreasing on $[T_2, \infty)$, we obtain

$$1 \leqq \frac{L}{c}\int_t^\infty s^{N-1} g_0(s, cs^{-N+2m})\,ds$$

for $t \geqq T_2$, which contradicts the fact that the right-hand side approaches 0 as $t \to \infty$. Thus we conclude that if (0.1) has a positive radial entire solution of class $\mathscr{K}_0$[int], then (5.25$^+$) holds.

Similarly we can prove that if (0.1) has a negative radial entire solution of class $\mathscr{K}_0$[int], then (5.25$^-$) holds. The proof of Theorem 5.7 is complete.

THEOREM 5.8. *Consider equation* (5.19). *Suppose that* f_0 *satisfies* (5.12), (5.13), (5.16)–(5.18). *Then equation* (5.19) *has a radial entire solution of class* $\mathscr{K}_0$[int] *if and only if*

$$\begin{cases} N \geqq 2m+1, \\ \displaystyle\int_1^\infty t^{2m-1}|f_0(t, c)|\,dt < \infty & \text{for some } c \neq 0, \\ \displaystyle\int_1^\infty t^{N-1}|f_0(t, ct^{-N+2m})|\,dt = \infty & \text{for all } c \neq 0. \end{cases} \tag{5.32}$$

PROOF. If there is a radial entire solution of (5.19) in $\mathscr{K}_0$[int], then, by Theorems 5.1, 5.5 (or 5.6) and 5.7, the desired conclusion (5.32) holds. Conversely, if (5.32) is satisfied, then (i) of Theorem 5.3 ensures the existence of a radial entire solution u of (5.19) such that $u(t) \neq 0$ for $t \geqq 0$ and $u(t) \to 0$ as $t \to \infty$. By Theorem 5.2 (or 5.4), this solution u does not belong to $\mathscr{K}_0[\min]$ and therefore belongs to $\mathscr{K}_0[\mathrm{int}]$. This completes the proof of Theorem 5.8.

Finally let us summarize some results on bounded radial entire solutions with no zero on $\boldsymbol{R}^N$ of the generalized Emden-Fowler equation

$$(-1)^m \Delta^m u = p(|x|)|u|^\gamma \operatorname{sgn} u, \qquad x \in \boldsymbol{R}^N, \tag{5.33}$$

where $\gamma > 0$ and $p \in C[0, \infty)$, $p(t) \geqq 0$, $\not\equiv 0$ for $t \geqq 0$. They follow from Theorem 4.4 with $j = 0$, and Theorems 5.4, 5.6, 5.8 specialized to (5.33).

Corollary 5.9. (i) *Let* $\gamma > 0$, $\neq 1$. *Equation* (5.33) *has a radial entire solution u such that* $u(t) \neq 0$ *for* $t \geqq 0$ *and*

$$\lim_{t\to\infty} u(t) \text{ exists and is a nonzero finite value} \tag{5.34}$$

if and only if

$$N \geqq 2m+1 \quad \text{and} \quad \int_1^\infty t^{2m-1} p(t)\,dt < \infty. \tag{5.35}$$

(ii) *Let* $0 < \gamma < 1$. *Equation* (5.33) *has a bounded radial entire solution u such that* $u(t) \neq 0$ *for* $t \geqq 0$ *and*

$$\lim_{t\to\infty} t^{N-2m} u(t) \text{ exists and is a nonzero finite value} \tag{5.36}$$

if and only if

$$N \geqq 2m+1 \quad \text{and} \quad \int_1^\infty t^{N-1-\gamma(N-2m)} p(t)\,dt < \infty. \tag{5.37}$$

(iii) *Let* $0 < \gamma < 1$. *Equation* (5.33) *has a bounded radial entire solution u such that* $u(t) \neq 0$ *for* $t \geqq 0$ *and*

$$\lim_{t\to\infty} u(t) = 0, \qquad \lim_{t\to\infty} t^{N-2m} u(t) = \pm\infty \tag{5.38}$$

if and only if

$$N \geqq 2m+1, \int_1^\infty t^{2m-1} p(t)\,dt < \infty \text{ and } \int_1^\infty t^{N-1-\gamma(N-2m)} p(t)\,dt = \infty. \tag{5.37}$$

Corollary 5.10. *Let* $0 < \gamma < 1$. *All of the following statements are equivalent*:

(i) *equation* (5.33) *has a bounded radial entire solution which has no zero in* $\boldsymbol{R}^N$;

(ii) *equation* (5.33) *has a radial entire solution which has no zero in* $\boldsymbol{R}^N$ *and satisfies* (5.34);

(iii) *equation* (5.33) *has a radial entire solution which has no zero in* $\boldsymbol{R}^N$ *and satisfies* $\lim_{|x|\to\infty} u(|x|) = 0$;

(iv) *condition* (5.35) *is satisfied.*

6. Supplementary results

In this section we consider the equation

$$(-1)^m \Delta^m u + \sigma f_0(|x|, u) = 0, \qquad x \in \boldsymbol{R}^N, \tag{6.1}$$

where $m \geqq 2$, $N \geqq 3$, $\sigma = +1$ or $\sigma = -1$ and f_0 is continuous on $[0, \infty)$, and state some results on the existence and asymptotic behavior of its radial entire solutions which can be obtained by suitably combining the theorems given in Sections 4 and 5. For simplicity it is assumed that

$$u f_0(t, u) > 0 \qquad \text{for } t \geqq 0,\ u \neq 0. \tag{6.2}$$

For equation (6.1), the integral condition of the type

$$\int_0^\infty t^{2(m-j)-1} |f_0(t, ct^{2j})| \, dt < \infty \qquad \text{for some } c \neq 0 \tag{6.3$_j$}$$

plays an important rule. Here j is an integer with $0 \leqq j \leqq m-1$.

Let f_0 be strictly superlinear on $[0, \infty) \times \boldsymbol{R}$, that is, let f_0 satisfy the condition

(6.4) there exists a number $\gamma > 1$ such that

$$|u|^{-\gamma} |f_0(t, u)| \leqq |v|^{-\gamma} |f_0(t, v)| \qquad \text{for } t \geqq 0,\ uv > 0,\ |u| \leqq |v|.$$

Then, noting that

$$t^{2(m-j)+1} |f_0(t, ct^{2(j-1)})| \leqq t^{2(m-j)-1} |f_0(t, ct^{2j})|$$

for $t \geqq 1$ and $c \neq 0$, we find that $(6.3)_j$ implies $(6.3)_{j-1}$; and in particular $(6.3)_j$ implies $(6.3)_{j-2}$, $(6.3)_{j-4}, \ldots$, and so on. We observe that, under the dimensional condition $N \geqq 2m+1$, the following statements hold:

(i) (6.1) has a radial entire solution of class $\mathscr{K}_j$ $(1 \leqq j \leqq m-1)$ if and only if $(-1)^{j+1}\sigma = 1$ and $(6.3)_{j-1}$ holds (see Theorem 4.10);

(ii) (6.1) has a radial entire solution of class $\mathscr{K}_0$ such that $\lim_{t\to\infty} u(t) \in \boldsymbol{R} - \{0\}$ if and only if $\sigma = -1$ and $(6.3)_0$ holds (see Theorem 4.4 with $j = 0$);

(iii) (6.1) has a radial entire solution of class $\mathscr{K}_m$ such that $\lim_{t\to\infty} u(t)/$

$t^{2(m-1)} \in \boldsymbol{R} - \{0\}$ if and only if $(-1)^{m+1}\sigma = 1$ and $(6.3)_{m-1}$ holds (see Theorem 4.7 with $j = m$).

Taking these facts into account and recalling the decomposition of $\mathscr{K}$ mentioned just before Theorem 2.2, we have the following theorems.

THEOREM 6.1. *Consider equation* (6.1). *Suppose that* m *is even*, $\sigma = +1$, $N \geqq 2m+1$ *and that* f_0 *satisfies* (6.2) *and* (6.4). *Then equation* (6.1) *has a radial entire solution with no zero in* $\boldsymbol{R}^N$ *if and only if*

$$\int_0^\infty t^{2m-1} |f_0(t, c)| \, dt < \infty \qquad \text{for some } c \neq 0.$$

THEOREM 6.2. *Consider equation* (6.1). *Suppose that* $N \geqq 2m+1$ *and that* f_0 *satisfies* (6.2) *and* (6.4). *Then the condition*

$$\int_0^\infty t^{2m-1} |f_0(t, c)| \, dt = \infty \qquad \text{for all } c \neq 0$$

is necessary and sufficient in order that the following situation occurs:

(i) *for* m *even and* $\sigma = +1$, *each radial entire solution of* (6.1) *has at least one zero*;

(ii) *for* m *odd and* $\sigma = -1$, *each radial entire solution* u *of* (6.1) *with no zero satisfies* $\lim_{t\to\infty} u(t) = 0$;

(iii) *for* m *even and* $\sigma = -1$, *each radial entire solution* u *of* (6.1) *with no zero satisfies either* $\lim_{t\to\infty} u(t) = 0$ *or* $\lim_{t\to\infty} |u(t)|/t^{2(m-1)} = +\infty$;

(iv) *for* m *odd and* $\sigma = +1$, *each radial entire solution* u *of* (6.1) *with no zero satisfies* $\lim_{t\to\infty} |u(t)|/t^{2(m-1)} = +\infty$.

Next consider the case where f_0 satisfies the conditions

(6.5) $$\lim_{u\to\pm 0} u^{-1} f_0(t, u) = +\infty \qquad \text{for fixed } t \geqq 0;$$

(6.6) $$|f_0(t, u)| \leqq |f_0(t, v)| \qquad \text{for } t \geqq 0,\ uv > 0,\ |u| \leqq |v|; \text{ and}$$

(6.7) there exists a number γ, $0 < \gamma < 1$, such that

$$|u|^{-\gamma} |f_0(t, u)| \geqq |v|^{-\gamma} |f_0(t, v)| \qquad \text{for } t \geqq 0,\ uv > 0,\ |u| \leqq |v|.$$

In this case, the condition

$(6.8)_j$ $$\begin{cases} N \geqq 2(m-j)+1 \text{ and} \\ \displaystyle\int_0^\infty t^{2(m-j)-1} |f_0(t, ct^{2j})| \, dt < \infty \qquad \text{for some } c \neq 0 \end{cases}$$

is crucial. It is easy to see that $(6.8)_j$ implies $(6.8)_{j+1}$; and hence $(6.8)_j$ implies

$(6.8)_{j+2}$, $(6.8)_{j+4}, \ldots$, and so on. Furthermore,

(i) equation (6.1) has a radial entire solution of class $\mathscr{K}_j$ $(0 \leqq j \leqq m-1)$ if and only if $(-1)^{j+1}\sigma = 1$ and $(6.8)_j$ holds (see Theorem 4.12 for $1 \leqq j \leqq m-1$ and Theorem 5.6 for $j = 0$);

(ii) equation (6.1) has a radial entire solution of class $\mathscr{K}_m$ such that $\lim_{t\to\infty} u(t)/t^{2(m-1)} \in \boldsymbol{R} - \{0\}$ if and only if $(-1)^{m+1}\sigma = 1$ and $(6.8)_{m-1}$ holds (see Theorem 4.7 with $j = m$).

Then the following theorems can be shown without difficulty.

THEOREM 6.3. *Consider equation* (6.1). *Suppose that* $(-1)^m\sigma = 1$ *and that* f_0 *satisfies* (6.2), (6.5)–(6.7). *Then equation* (6.1) *has a radial entire solution with no zero in* $\boldsymbol{R}^N$ *if and only if*

$$\int_0^\infty t|f_0(t, ct^{2(m-1)})|\,dt < \infty \qquad \textit{for some } c \neq 0.$$

THEOREM 6.4. *Consider equation* (6.1). *Suppose that* f_0 *satisfies* (6.2), (6.5)–(6.7). *Then the condition*

$$\int_0^\infty t|f_0(t, ct^{2(m-1)})|\,dt = \infty \qquad \textit{for all } c \neq 0$$

is necessary and sufficient for the following situation to occur:

(i) *for the case of* $(-1)^m\sigma = 1$, *each radial entire solution of* (6.1) *has at least one zero*;

(ii) *for the case of* $(-1)^{m+1}\sigma = 1$, *each radial entire solution* u *of* (6.1) *with no zero satisfies* $\lim_{t\to\infty}|u(t)|/t^{2(m-1)} = +\infty$.

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