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Asymptotic behavior of solutions to certain nonlinear parabolic evolution equations II

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1. Introduction

The purpose of this paper is to show that given a pair of solutions u_1, u_2 in $W^{1,1}_{\mathrm{loc}}((0, \infty); H)$ of the time-dependent evolution equation

$$(1.1) \qquad (d/dt)u(t) + \partial\psi^t(u(t)) \ni 0, \qquad a.e.\ t \geq 0,$$

the strong convergence

$$(1.2) \qquad \mathrm{s} - \lim_{t\to\infty}\{u_1(t) - u_2(t)\} = \mathrm{const.} \in H$$

is valid, where H is a real Hilbert space, and for each $t \in [0, \infty)$, ψ^t is a proper lower semi-continuous (l.s.c.) convex functional defined in H and $\partial\psi^t$ denotes the subdifferential of ψ^t.

A typical example of (1.1) is the following parabolic equation:

$$(1.3) \qquad \begin{cases} (\partial/\partial t)u(t, x) - \sum_{j=1}^{n}(\partial/\partial x_j)f_j(t, x, \nabla u) + g(t, x, u) = 0, \\ \qquad\qquad\qquad\qquad\qquad (t, x) \in \bigcup_{t\geq 0}\{t\} \times Q(t), \\ u(t, x) = 0, \qquad\qquad\qquad (t, x) \in \bigcup_{t\geq 0}\{t\} \times \Gamma(t). \end{cases}$$

Here, for each fixed (t, x), the family $\{f_j(t, x, y)\}$ is supposed to be completely integrable with respect to $y \in \mathbf{R}^n$ and an ellipticity condition

$$(1.4) \qquad \sum_{j,k=1}^{n}(\partial/\partial y_j)f_k(t, x, y)\ \xi_j\xi_k \geq r(t)a(x)|\xi|^2, \qquad \xi \in \mathbf{R}^n$$

holds for some positive smooth functions r on $[0, \infty)$ and a on $\mathbf{R}^n$. For each $t \in [0, \infty)$, the set $Q(t)$ denotes a domain in $\mathbf{R}^n$ with smooth compact boundary $\Gamma(t)$. In most of our results, we do not assume the boundedness of $Q(t)$. By means of the zero-extension, we formulate equation (1.3) in the real Hilbert space $L^2(\mathbf{R}^n)$.

The convergence (1.2) is interesting, for example, if $\partial\psi^{t+T} = \partial\psi^t$ with some $T > 0$ and $u_2(.)$ is a T-periodic solution of (1.2). The existence of periodic solutions of our example (1.3) is obtained in [5] (see also [7]).

As mentioned in the Introduction of our previous paper [4], to get the strong convergence (1.2) we need the following lemma.

LEMMA 1.1. *Let* u_2 *be a solution of* (1.1). *For any* $t \geq 0$ *such that* $u_2(t)$ *satisfies the relation* (1.1), *put*

$$\varphi^t(w) = \psi^t(w + u_2(t)) + ((d/dt)u_2(t), w) - \psi^t(u_2(t)), \qquad w \in H, \tag{1.5}$$

where $(.,.)$ *stands for the inner product of* H. *Then the family* $\{\varphi^t;\ \text{a.e. } t \geq 0\}$ *has the following properties* (i) *and* (ii).

(i) φ^t *are proper l.s.c. convex functionals on* H *satisfying*

$$\varphi^t(0) = \min_H \varphi^t = 0. \tag{1.6}$$

(ii) $u_1(.)$ *is a solution of* (1.1) *if and only if* $u(.) \equiv u_1(.) - u_2(.)$ *is a solution of the equation*

$$(d/dt)u(t) + \partial\varphi^t(u(t)) \ni 0, \qquad \text{a.e. } t \geq 0.$$

By Lemma 1.1, the problem of finding the strong convergence (1.2) for solutions u_1 and u_2 of (1.1) turns into the problem to show the strong convergence

$$\mathrm{s} - \lim_{t\to\infty} u(t) = \mathrm{const.} \in H \tag{1.7}$$

for a solution u of the equation

$$(d/dt)u(t) + \partial\varphi^t(u(t)) \ni 0 \tag{1.8}$$

under the condition (1.6) on $\{\varphi^t\}$.

In the case where $\varphi^{(\cdot)}$ is independent of t (i.e., $\varphi^t \equiv \varphi$), the strong convergence stated in (1.7) and similar problems on the weak convergence were studied by many authors (e.g. the references listed in [3]). Typical conditions as stated in (I) and (II) below are known to be sufficient for the strong convergence (1.7).

(I) (*Compactness condition*)
  (a) The minimum set of φ is nonempty.
  (b) For each $\varepsilon, R > 0$, the set $\{w;\ \varphi(w) \leq \min_H \varphi + \varepsilon,\ \|w\| \leq R\}$ is relatively compact.

(II) (*Evenness condition in a generalized sense*, [3])
  There is a constant $c > 0$ such that

$$\varphi(-cw) \leq \varphi(w), \qquad w \in \mathfrak{D}(\varphi).$$

As seen from our example (1.3) subject to the ellipticity condition (1.4), the compactness condition (I) is usually useful for the case in which the space domain of $\mathbf{R}^n_x$ is bounded, but condition (I) need not be fulfilled if the domain is unbounded. On the other hand, the evenness condition (II) can be formulated without imposing any restrictions on the domain. Now both of the conditions (I) and (II) are special cases of the following condition (III).

(III) ([3; Theorem 1])

(a) The minimum set of φ is nonempty.

(b)′ There is a Fréchet differentiable operator A such that for each $\varepsilon, R > 0$ the set

$$\{Aw;\ \varphi(w) \le \min_H \varphi + \varepsilon,\ \|w\| \le R\}$$

is relatively compact.

(c)′ There exists a constant $c > 0$ such that

$$\varphi(-cw + (1 + c)Aw) \le \varphi(w), \qquad w \in \mathfrak{D}(\varphi).$$

In the previous paper [4], we established the convergence (1.7) in the case where φ^t depends upon t and generalized each of (I), (II) and (III) to the time-dependent case (see [4; Theorem 2.1], [4; Theorem 2.2] and [4; Theorem 6.1], respectively).

In this paper, we show that the strong convergence (1.2) holds for the constant 0 for any pair of solutions $u_1, u_2 \in W^{1,1}_{\mathrm{loc}}((0, +\infty); L^2(\mathbf{R}^n))$ of equation (1.3), and that the results obtained in the previous paper [4] can be supplied to get the strong convergence (1.2) or a similar type of convergence result for any pair of solutions of (1.3). For this purpose, we first review [4; Theorem 6.1]. It is expected that our results can be applied to many other problems, although the results are stated in fairly general forms. Next, in Section 4, we show that equation (1.3) subject to some natural conditions is written in the form (1.1) defined in the real Hilbert space $L^2(\mathbf{R}^n)$. Finally, in Section 5, we apply our abstract result, Theorem 2.1, to equation (1.3) to get the convergence (1.2) with the constant 0. See Theorem 5.1. We also verify that all other results of [4] are applied to get the convergence (1.2) or a weak convergence for any pair of solutions of (1.3). See Remarks 5.1, 5.2 and 5.3.

2. Abstract results

Let H be a real Hilbert space with inner product $(.,.)$ and norm $\|.\|$. Let $\{\varphi^t\}$ be a family of proper l.s.c. convex functionals on H and put

$$F(\varphi^t) = \{w \in H;\ \varphi^t(w) = \min_H \varphi^t\}.$$

We show the strong convergence

$$\mathrm{s} - \lim_{t \to +\infty} u(t) \in H$$

for each solution of

(E) $$(d/dt)u(t) + \partial\varphi^t(u(t)) \ni 0, \qquad t \geq 0,$$

where we say that $u(.)$ is a solution of (E) if (i) $u(.) \in W^{1,1}_{\mathrm{loc}}(0, \infty; H)$; and (ii) the relations $u(t) \in \mathfrak{D}(\partial\varphi^t)$ and $-(d/dt)u(t) \in \partial\varphi^t(u(t))$ hold for a.e. $t \geq 0$.

In this section we extend the basic reselt in the previous paper [4] and apply the extended version to the equation (1.3) without assuming the boundedness of $\bigcup_{t\geq 0} Q(t)$.

The result is stated as follows:

THEOREM 2.1. *Suppose that there are proper l.s.c. convex functionals* ψ_i, $i = 1, 2$, *on* H, *an operator* A *in* H *and constants* $a \in (0, \infty)$, $b \in \mathbf{R}$, *and* $c \in (0, \infty)$ *satisfying the five conditions below*;

(D1) $F(\varphi^t) = F(\psi_i)\ (\equiv F) \ni 0,\ \varphi^t(0) = \psi_i(0) = 0,\ t \geq 0,\ i = 1, 2.$

(D2) $\{-au + bAu;\ u \in \mathfrak{D}(\varphi^t)\} \subset \mathfrak{D}(\varphi^s)$ if $s \leq t$.

(D3) *There is a positive measurable function* $r(.)$ *defined on all of* $[0, \infty)$ *such that*

(2.1) $$\int_0^\infty r(t)\, dt = \infty,$$

(2.2) $$\varphi^t(-au + bAu) \leq r(t)\psi_1(-au + bAu) \leq \varphi^t(u), \qquad t \geq 0,\ u \in \mathfrak{D}(\varphi^t),$$

(2.3) $$r(t)\psi_1(Au) \leq c\varphi^t(u), \qquad t \geq 0,\ u \in \mathfrak{D}(\varphi^t),$$

(2.4) $$r(t)\psi_2(u) \leq \varphi^t(u), \qquad t \geq 0,\ u \in \mathfrak{D}(\varphi^t).$$

(D4) *For each* $\varepsilon > 0$ *there is a positive constant* $\delta = \delta(\varepsilon)$ *such that*

$$\mathrm{dist}(Au, F) < \varepsilon \qquad \text{for } u \text{ with } \psi_1(Au) \leq \delta.$$

(D5) *The inclusion* $-F + f_0 \subset d(F - f_0)$ *holds for some* $f_0 \in F$ *and* $d \geq 1$.

Then every solution $u(.)$ *of* (E) *converges strongly to some point of* F *as* $t \to \infty$, *namely*

(2.5) $$\mathrm{s} - \lim_{t\to\infty} u(t) \in F.$$

By Lemma 1.1, the family $\{\varphi^t\}$ defined by (1.5) has the property

$$F(\varphi^t) \ni 0, \quad \varphi^t(0) = 0, \quad t \geq 0,$$

which is a pair of assumption (D1).

For the moment we consider an example of the operator A as mentioned in Theorem 2.1 to illustrate the above-mentioned condition. Let

$$H = L^2(\Omega),\ \Omega \text{ a domain of } \mathbf{R}^n, \text{ and}$$

$$\varphi^t(u) = \int_\Omega F(t, x, u(x), \nabla u(x))\,dx \qquad \text{for } u \in \mathfrak{D}(\varphi),$$

where, for each $(t, x) \in [0, \infty) \times \Omega$, the function $F(t, x, ., .)$ is supposed to be convex on $\mathbf{R} \times \mathbf{R}^n$ and

$$F(t, x, 0, 0) = \min_{\mathbf{R} \times \mathbf{R}^n} F(t, x, ., .) = 0.$$

Suppose that there is a subdomain Ω_1 of Ω with the following conditions.

(2.6) (*generalized evenness condition on* $\Omega \backslash \Omega_1$) There is $\varepsilon > 0$ such that

$$F(t, x, -\varepsilon u(x), -\varepsilon \nabla u(x)) \leq F(t, x, u(x), \nabla u(x))$$
$$\text{for } x \in \Omega \backslash \Omega_1 \text{ and } u \in \mathfrak{D}(\varphi^t).$$

(2.7) (*generalized compactness condition on* Ω_1)

$$\int_{\Omega_1} F(t, x, u(x), \nabla u(x))\,dx \geq g(\|u\|_{L^2(\Omega_1)}) \qquad \text{for } u \in \mathfrak{D}(\varphi^t)$$

hold for some continuous function g satisfying $g(0) = 0$ and $g(r) > 0$ for $r > 0$.

If we put

$$(A_1 u)(x) = \begin{cases} u(x), & x \in \Omega_1, \\ 0, & x \in \Omega \backslash \Omega_1, \end{cases} \tag{2.8}$$

then it follows from (2.7) that condition (D4) holds for any ψ_1 satisfying $\psi_1(A_1 u) \geq g(\|A_1 u\|_{L^2})$. On the other hand, by (2.6), one has

$$\varphi^t(-\varepsilon u + (1 + \varepsilon) A_1 u) \leq \varphi^t(u) \text{ provided that } -\varepsilon u + (1 + \varepsilon) A_1 u,\ u \in \mathfrak{D}(\varphi^t).$$

This estimate is condition (2.2) with $a = \varepsilon$ and $b = 1 + \varepsilon$. But the inclution $\{-\varepsilon u + (1 + \varepsilon) A_1 u;\ u \in \mathfrak{D}(\varphi^t)\} \subset \mathfrak{D}(\varphi^t)$ stated in condition (D2) does not hold since the relation $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ does not yield $-\varepsilon u + (1 + \varepsilon) A_1 u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ in general. To overcome this difficulty we put

$$(A_2 u)(x) = \alpha(x) u(x) \qquad \text{for } x \in \Omega, \tag{2.9}$$

where $\alpha(.) \in C_0^\infty(\Omega)$ and $\alpha(x) \equiv 1$ on Ω_1.

In Section 5, we employ this operator A_2 to apply Theorem 2.1 to the nonlinear parabolic equation (1.3). Clearly, the operator A_2 is not a projection operator. In our previous result [4; Theorem 6.1] we assumed that A was an orthogonal projection onto a closed linear subspace of H. This is the first reason why we necessiate to relax the assumptions of [4; Theorem 6.1]. The second reason for extendig the result is the following; For the family

$\{\varphi^t\}$ associated with (1.3), the inclusion relation $Q(t) \subset Q(s)$ is required to impose the condition $\mathfrak{D}(\varphi^t) \subset \mathfrak{D}(\varphi^s)$, but it does not necessary for the condition $\{-au + bAu; u \in \mathfrak{D}(\varphi^t)\} \subset \mathfrak{D}(\varphi^s)$ (see Section 5). Therefore in Theorem 2.1 we assume the condition as (D2). It should be noted that in [4; Theorem 6.1] we assumed $\mathfrak{D}(\varphi^t) \subset \mathfrak{D}(\varphi^s)$ for $s \leq t$.

Remark 2.1. The typical example (2.9) of the oparator A appearing in Theorem 2.1 is linear and smooth. But none of the linearity and the smoothness of A and the compactness of the level sets of φ^t are assumed in Theorem 2.1.

Theorem 2.2. *Assume the following condition* (D6) *in Theorem* 2.1 *instead of* (D5):

(D6) *For each* $R > 0$ *the set* $\{Au: u \in \mathfrak{D}(\partial\psi_1),\ \psi_1(Au) \leq \delta$ and $\|u\| \leq R\}$ *is relatively compact in* H *for some* $\delta \equiv \delta_R > 0$.

Then the convergence (2.5) *holds for solution* $u(.)$ *of* (E).

3. Proofs of Theorem 2.1 and Theorem 2.2.

Let $u(.)$ be a solution of (E). Put

$$I = \{t \geq 0: -(d/dt)u(t) \in \partial\varphi^t(u(t)) \text{ holds}\}.$$

In what follws, $u'(t)$ stands for the derivative $(d/dt)u(t)$. We first prepare three lemmas.

Lemma 3.1 ([4; Lemma 3.1]). *Suppose* (D1). *Then*

(i) *For each* $f \in F$, $\|u(t) - f\|$ *is nonincreasing in* t *and converges as* $t \to \infty$.

(ii) *There is a nonnegative function* $\beta \in L^1(0, +\infty)$ *such that*

$$0 \leq \varphi^t(u(t)) \leq \beta(t), \qquad \text{a.e. } t \geq 0.$$

Lemma 3.2. *Suppose* (D1) *and* (D5). *Then*

$$\lim_{t,T\to\infty} \sup_{f\in F} |\, \|u(t) - f\|^2 - \|u(T) - f\|^2| = 0.$$

Proof. Let $f \in F$. Then one has

$$(-u'(s), f - u(s)) \leq \varphi^s(f) - \varphi^s(u(s)) \leq 0 \qquad \text{for } s \in I.$$

Let f_0 be an element of F given by condition (D5). Then

$$(3.1)\qquad \begin{aligned}(u(t)-u(T),\ f-f_0) &= \int_t^T (-u'(s),\ f-f_0)\,ds \\ &= \int_t^T (-u'(s),\ f-u(s))\,ds + \int_t^T (-u'(s),\ u(s)-f_0)\,ds \\ &\le \int_t^T (-u'(s),\ u(s)-f_0)\,ds \\ &= 2^{-1}\{\|u(t)-f_0\|^2 - \|u(T)-f_0\|^2\}, \qquad 0\le t\le T.\end{aligned}$$

On the other hand, by (D5), there is $g\in F$ such that $f-f_0 = d(-g+f_0)$. In the same way as in (3.1) one has

$$(3.2)\qquad \begin{aligned}-(u(t)-u(T),\ f-f_0) &= d\,(u(t)-u(T),\ g-f_0) \\ &\le 2^{-1}d\{\|u(t)-f_0\|^2 - \|u(T)-f_0\|^2\}, \qquad 0\le t\le T.\end{aligned}$$

Estimates (3.1) and (3.2) together imply that

$$(3.3)\qquad |(u(t)-u(T),\ f-f_0)| \le 2^{-1}d\{\|u(t)-f_0\|^2 - \|u(T)-f_0\|^2\}.$$

Hence one has

$$(3.4)\qquad \begin{aligned}&|\,\|u(t)-f\|^2 - \|u(T)-f\|^2| \\ &= |\,\|u(t)-f_0\|^2 - \|u(T)-f_0\|^2 - 2(u(t)-u(T),\ f-f_0)| \\ &\le (1+d)|\,\|u(t)-f_0\|^2 - \|u(T)-f_0\|^2|\end{aligned}$$

for each $f\in F$. Lemma 3.1 (i) then implies that the right side of (3.4) converges to 0 as $t,\ T\to\infty$. This completes the proof.

LEMMA 3.3. *Suppose that* (D1)–(D3) *are satisfied, and that for each $n\ge 1$ there are sequences $\{T(n)\}$ and $\{t_n\}$ in I satisfying*

$$(3.5)\qquad T(n)\ge t_n, \qquad n\in\mathbf{N},$$

$$(3.6)\qquad \varphi^{T(n)}(u(T(n))) \le n^{-1}r(T(n)), \qquad n\in\mathbf{N},$$

$$(3.7)\qquad \varphi^{s}(u(s)) > n^{-1}r(s) \qquad \text{for a.e. } s\in[0,\ t_n),\ n\in\mathbf{N},$$

where $r(.)$ is a positive measurable function on $[0,\infty)$ *provided by condition* (D3). *Then there is a constant $C>0$ such that*

$$(3.8)\qquad \begin{aligned}&\|u(t)-u(T(n))\|^2 \\ &\le C\{|\,\|u(t)-y\|^2 - \|u(t_n)-y\|^2| + |\,\|u(t_n)\|^2 - \|u(t)\|^2| \\ &\quad + |(u(t_n)-u(t),\ Au(T(n))-y)|\} + \|u(t_n)-u(T(n))\|^2\end{aligned}$$

holds for each $n\ge 1$, $t\in[0,\ t_n]$ and $y\in H$.

PROOF. Let $n \geq 1$ and $y \in H$. By (D2), (2.2), (3.6) and (3.7) one has

$$\begin{aligned}\varphi^s(-au(T(n))+bAu(T(n))) &\leq r(s)\psi_1(-au(T(n))+bAu(T(n)))\\ &\leq r(s)\varphi^{T(n)}(u(T(n)))/r(T(n)) \leq n^{-1}r(s) \leq \varphi^s(u(s))\end{aligned}$$

for $s \in [0, t_n)$. Hence we see from the definition of subdifferential $\partial\varphi^s$

$$\begin{aligned}&(-u'(s),\ -au(T(n))+bAu(T(n))-u(s))\\ &\quad\leq \varphi^s(-au(T(n))+bAu(T(n)))-\varphi^s(u(s)) \leq 0,\end{aligned}$$

and so

$$(-u'(s),\ -u(T(n))) \leq a^{-1}\{(-u'(s),\ u(s))+b(u'(s),\ Au(T(n)))\}$$

for a.e. $s \in [0, t_n)$. Therefore we have

$$\begin{aligned}&\|u(t)-u(T(n))\|^2\\ &= \int_t^{t_n} -(d/ds)\|u(s)-u(T(n))\|^2\,ds + \|u(t_n)-u(T(n))\|^2\\ &= 2\int_t^{t_n}(-u'(s),\ u(s)-u(T(n)))\,ds + \|u(t_n)-u(T(n))\|^2\\ &\leq 2\int_t^{t_n}\{(-u'(s),\ u(s))+a^{-1}(-u'(s),\ u(s))\\ &\qquad + a^{-1}b(u'(s),\ Au(T(n)))\}\,ds + \|u(t_n)-u(T(n))\|^2\\ &= 2\int_t^{t_n}\{a^{-1}b(-u'(s),\ u(s)-y)+a^{-1}b(-u'(s),\ y-Au(T(n))\\ &\qquad +(1+a^{-1}-a^{-1}b)(-u'(s),\ u(s))\}\,ds\\ &\qquad + \|u(t_n)-u(T(n))\|^2\\ &\leq C\{|\,\|u(t)-y\|^2-\|u(t_n)-y\|^2| + |(u(t_n)-u(t),\ Au(T(n))-y)|\\ &\qquad + |\,\|u(t)\|^2-\|u(t_n)\|^2|\} + \|u(t_n)-u(T(n))\|^2\end{aligned}$$

for $t \in [0, t_n)$. From this the aimed estimate (3.8) is obtained.

PROOF OF THEOREM 2.1. By (D3), there is a positive function $r(.)$ with $\int_0^\infty r(t)\,dt = \infty$. In view of Lemma 3.1 (ii), put

$$t_n = \text{ess inf}\,\{t \in I;\ \varphi^t(u(t)) \leq n^{-1}r(t)\}, \qquad n = 1, 2, \ldots. \tag{3.9}$$

Then we have

$$0 \leq t_n < \infty \text{ and } t_n \leq t_{n+1} \qquad \text{for } n = 1, 2, \ldots.$$

With regard to the sequence $\{t_n\}$ so obtained there are two possible cases below:

(a) $\{t_n\}$ is bounded, i.e., $t_n \uparrow T$ for some $T > 0$ as $n \to \infty$;
(b) $\{t_n\}$ is unbounded, i.e., $t_n \uparrow \infty$ as $n \to \infty$.

In case that (a) holds, (2.4) yields

$$\begin{aligned}\psi_2(u(T)) &\leq \lim\ \inf_{n\to\infty} \psi_2(u(t_n)) \leq \lim\ \inf_{n\to\infty} r(t_n)^{-1} \varphi^{t_n}(u(t_n)) \\ &\leq \lim\ \inf_{n\to\infty} n^{-1} = 0.\end{aligned}$$

This shows that $u(T) \in F$, since $\min \psi_2 = 0$ by (D1). By Lemma 3.1 (i) we see $u(t) = u(T) \in F$ for $t \geq T$, so that $\mathrm{s}-\lim_{t\to\infty} u(t) = u(T) \in F$.

We next consider the second case (b). In view of (3.9) one can choose a sequence $\{T(n)\}$ such that

(3.10) $$t_n \leq T(n),\ u(T(n)) \in \mathfrak{D}(\varphi^{T(n)}),\ \varphi^{T(n)}(u(T(n))) \leq n^{-1} r(T(n)),$$

(3.11) $$\|u(t_n) - u(T(n))\| \leq n^{-1} \qquad \text{for } n = 1, 2, \ldots$$

Then by (3.10) and (3.11) the pair of the sequences $\{t_n\}$ and $\{T(n)\}$ satisfies the assumptions of Lemma 3.3. Hence by (3.8) and (3.11)

(3.12) $$\begin{aligned}&\|u(t) - u(T(n))\|^2 \\ &\leq \inf_{f\in F}[C\{|\,\|u(t) - f\|^2 - \|u(t_n) - f\|^2| + |\,\|u(t_n)\|^2 - \|u(t)\|^2| \\ &\quad + \|u(t_n) - u(t)\|\,\|Au(T(n)) - f\|\} + n^{-2}] \\ &\leq 2C \sup_{f\in F} |\,\|u(t) - f\|^2 - \|u(t_n) - f\|^2| \\ &\quad + C\|u(t_n) - u(t)\| \inf_{f\in F} \|Au(T(n)) - f\| + n^{-2}\end{aligned}$$

holds for $n \geq 1$ and $t \in [0, t_n)$, where we have used the condition (D1), namely, $0 \in F$.

By Lemma 3.2 we see that the first term of the right side of (3.12) converges 0 as $t, t_n \to \infty$. On the other hand by (2.3) and (3.10) one obtains

(3.13) $$\psi_1(Au(T(n))) \leq cr(T(n))^{-1} \varphi^{T(n)}(u(T(n)) \leq n^{-1}$$

for $n \geq 1$. Hence it follows from (D4) that

$$\lim_{n\to\infty} \inf_{f\in F} \|\,Au(T(n)) - f\,\| = 0.$$

Noting that $\|u(t_n) - u(t)\|$ is bounded by Lemma 3.1, we see that the second term on the right side of (3.12) converges 0 as $t_n \to \infty$. Consequently, it follows that

(3.14) $$\mathrm{s}-\lim_{t\to\infty} u(t)\ (= z) \text{ exists.}$$

Finally, we see that $z \in F$. In fact, by (2.4) and (3.6) (which holds for the sequence $\{T(n)\}$), one has

$$\psi_2(z) \leq \lim\inf_{t\to\infty} \psi_2(u(t)) \leq \lim\inf_{n\to\infty} \psi_2(u(T(n)))$$
$$\leq \lim\inf_{n\to\infty} r(T(n))^{-1} \varphi^{T(n)}(u(T(n)) \leq \lim\inf_{n\to\infty} n^{-1} = 0.$$

Since $\min \psi_2 = 0$, this shows that $z \in F$. The proof is now complete.

PROOF OF THEOREM 2.2. Since $\{u(t)\}$ is bounded, we see from (D6) and (3.13) that there exists a subsequence $\{n'\}$ of $\{n\}$ and an element $f_1 \in F$ satisfying

$$\text{s} - \lim_{n'\to\infty} Au(T(n')) = f_1 \in F. \tag{3.15}$$

By the first inequality of (3.12) we have

$$\|u(t) - u(T(n))\|^2 \leq C\{| \|u(t) - f_1\|^2 - \|u(t_n) - f_1\|^2| + | \|u(t_n)\|^2 - \|y(t)\|^2|$$
$$+ \|u(t_n) - u(t)\| \, \|Au(T(n)) - f_1\|\} + n^{-2}$$

for $0 \leq t \leq t_n$ and $n \geq 1$. Lemma 3.1(i) for f_1, $0 \in F$ and (3.15) together imply the convergence (2.5). Hence we obtain the same conclution as in Theorem 2.1.

4. An example of a subdifferential operator

The purpose of this section is to show that equation (1.3) is written in the form (1.1) defined in the real Hilbert space $L^2(\mathbf{R}^n)$. To this end, we define the following operator $\mathscr{A}$.

$$\mathfrak{D}(\mathscr{A}) = \{u \in L^2(\mathbf{R}^n) \colon u \in W^{1,1}_{\text{loc}}(\mathbf{R}^n),\ u(x) = 0, \text{ a.e. } x \in \mathbf{R}^n \backslash \Omega, \text{ and}$$
$$- \textstyle\sum_{j=1}^n (\partial/\partial x_j) f_j(.,\ \nabla u(.)) + g(.,\ u(.))|_\Omega \in L^2(\Omega)\} \tag{4.1}$$

$$\mathscr{A}u = \{h \in L^2(\mathbf{R}^n) \colon h(x) = - \textstyle\sum_{j=1}^n (\partial/\partial x_j) f_j(x,\ \nabla u(x)) + g(x,\ u(x)),$$
$$\text{a.e. } x \in \Omega\} \tag{4.2}$$

Here Ω is either $\mathbf{R}^n$ or a domain of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. We assume the following conditions on f and g.

(f1) $f_j(x, .) \in C^1(\mathbf{R}^n)$, $f_j(., 0) \in H^1(\Omega)$ and $f_j(., z)$ is measurable on Ω for each $x \in \Omega$, $z \in \mathbf{R}^n$ and $j = 1, 2, \ldots, n$.

(f2) For each $x \in \Omega$, the family $\{f_j(x, .)\}$ is completely integrable in the sense that the equation

$$(\partial/\partial z_k) f_j(x, z) = (\partial/\partial z_j) f_k(x, z) \ (\equiv a_{jk}(x, z))$$

holds for $z \in \mathbf{R}^n$ and $j, k = 1, 2, \ldots, n$.

($f3$) There is a function $a(.)\in C(\Omega)$ with $a(x)>0$ for $x\in\Omega$ such that

$$a(x)|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x, z)\,\xi_j\xi_k, \qquad (x, z)\in \mathbf{R}^n,\ (\xi_j)\in \mathbf{R}^n. \tag{4.3}$$

($g1$) Both $g(x, .)\in C^1(\mathbf{R})$ and $(\partial/\partial s)g(x, s)\geq 0$ hold for each fixed $x\in\Omega$. $g(., r)$ is measurable on Ω for each fixed $r\in\mathbf{R}$ and $g(., 0)\in L^2(\Omega)$.

Our result here is stated as follows:

PROPOSITION 4.1. *There is a proper l.s.c. convex functional Φ on the real Hilbert space $L^2(\mathbf{R}^n)$ such that*

$$\mathscr{A} = \partial\Phi$$

To prove Proposition 4.1, we first define the functional Φ in the following way: Fix any $x\in\Omega$. By ($f2$) there is a potential function $F(x, .)$ on $\mathbf{R}^n$ such that

$$\begin{cases} (\partial/\partial z_j)F(x, z) = f_j(x, z), & z\in\mathbf{R}^n, \\ F(x, 0) = 0. \end{cases}$$

By ($f3$), the functions $F(x, .)$, $x\in\Omega$, are convex on $\mathbf{R}^n$. By ($g1$) we define a convex function $G(x, .)$ on $\mathbf{R}$ by

$$G(x, r) = \int_0^r g(x, s)\,ds, \qquad r\in\mathbf{R}.$$

Now we define the convex functional Φ on the real Hilbert space $L^2(\mathbf{R}^n)$ by

$$\mathfrak{D}(\Phi) = \{w\in L^2(\mathbf{R}^n)\colon w\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n),\ w(x)=0, \text{ a.e. } x\in\mathbf{R}^n\backslash\Omega, \tag{4.4}$$
$$\text{and} \int_\Omega \{F(x, \nabla w(x)) + G(x, w(x))\}\,dx < +\infty\},$$

$$\Phi(w) = \begin{cases} \displaystyle\int_\Omega \{F(x, \nabla w(x)) + G(x, w(x))\}\,dx, & w(x)\in\mathfrak{D}(\Phi), \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.5}$$

Since $F(x, .)$, $G(x, .)$, $x\in\Omega$, are convex, the functional Φ is convex.

Next we see that Φ is l.s.c. in $L^2(\mathbf{R}^n)$ in the following three lemmas.

LEMMA 4.1. *Put*

$$\Phi_1(w) = \int_\Omega F(x, \nabla w(x))\,dx, \qquad \Phi_2(w) = \int_\Omega G(x, w(x))\,dx$$

for $w\in\mathfrak{D}(\Phi)$. *Then*

$$\Phi_1(w+v) \geq \int_\Omega \{F(x, \nabla w) + \textstyle\sum_{j=1}^n f_j(x, \nabla w)\dfrac{\partial v}{\partial x_j} + a(x)|\nabla v|^2\}\, dx \tag{4.6}$$

$$\Phi_2(w+v) \geq \int_\Omega \{G(x, w) + g(x, w)v(x)\}\, dx \tag{4.7}$$

for w, v satisfying w, $w+v \in \mathfrak{D}(\Phi)$.

Proof. Let w, $w+v \in \mathfrak{D}(\Phi)$. By ($f3$) one has

$$\begin{aligned}
F(x, \nabla(w+v)(x)) &= F(x, \nabla w(x)) + \textstyle\sum_{j=1}^n \dfrac{\partial F}{\partial z_j}(x, \nabla w(x))\dfrac{\partial v}{\partial x_j}(x) \\
&\quad + \textstyle\sum_{j,k=1}^n \dfrac{\partial^2 F}{\partial z_j \partial z_k}(x, \nabla w(x) + \theta \nabla v(x))\dfrac{\partial v}{\partial x_j}(x)\dfrac{\partial v}{\partial x_k}(x) \\
&= F(x, \nabla w(x)) + \textstyle\sum_{j=1}^n f_j(x, \nabla w(x))\dfrac{\partial v}{\partial x_j}(x) \\
&\quad + \textstyle\sum_{j,k=1}^n a_{jk}(x, \nabla w(x) + \theta \nabla v(x))\dfrac{\partial v}{\partial x_j}(x)\dfrac{\partial v}{\partial x_k}(x) \\
&\geq F(x, \nabla w(x)) + \textstyle\sum_{j=1}^n f_j(x, \nabla w(x))\dfrac{\partial v}{\partial x_j}(x) + a(x)|\nabla v(x)|^2
\end{aligned}$$

for a.e. $x \in \Omega$, where $\theta = \theta(x, \nabla w, \nabla v) \in (0, 1)$. Hence (4.6) holds. In the same way we have (4.7) by using ($g1$).

Lemma 4.2. *Let Φ_1 and Φ_2 be as mentioned in Lemma 4.1. Then for each $v \in \mathfrak{D}(\Phi)$ one has*

(4.8) $\Phi_i(v)$, $i = 1, 2$, *are finitely valued,*

(4.9) $\Phi(v) = \Phi_1(v) + \Phi_2(v)$,

(4.10) $v \in W_{\mathrm{loc}}^{1,2}(\mathbf{R}^n)$,

$$\Phi(v) \geq \int_\Omega \{-\textstyle\sum_{j=1}^n (\partial/\partial x_j) f_j(x, 0) + g(x, 0)\} v(x)\, dx + \displaystyle\int_\Omega a(x)|\nabla v|^2\, dx. \tag{4.11}$$

Proof. First we show that

$$\Phi_2(v) > -\infty \qquad \text{for } v \in \mathfrak{D}(\Phi). \tag{4.12}$$

Let $v \in \mathfrak{D}(\Phi)$. Put $w = 0$ in (4.7). Since $G(x, 0) = 0$, one has

$$\Phi_2(v) \geq \int_\Omega g(x, 0)v(x)\, dx. \tag{4.13}$$

Since $v \in L^2(\mathbf{R}^n)$ and $g(.,0) \in L^2(\mathbf{R}^n)$ by (g1), the right side of this inequality is finite. Hence (4.12) holds.

Next we show that

$$\Phi_1(v) \in (-\infty, +\infty) \qquad \text{for } v \in \mathfrak{D}(\Phi). \tag{4.14}$$

By (4.12) one has

$$\Phi_1(v) < +\infty \qquad \text{for } v \in \mathfrak{D}(\Phi).$$

To show that

$$\Phi_1(v) > -\infty \qquad \text{for } v \in \mathfrak{D}(\Phi),$$

we put $w = 0$ in (4.6), and recall that $F(.,0) \equiv 0$ and $f_j(.,0) \in H^1(\Omega)$ for $j = 1, 2 \ldots, n$. Then one has

$$\begin{aligned} \Phi_1(v) &\geq \int_\Omega \left\{ \textstyle\sum_{j=1}^n f_j(x,0) \dfrac{\partial v}{\partial x_j}(x) + a(x) |\nabla v|^2 \right\} dx \\ &= \int_\Omega \{ -\textstyle\sum_{j=1}^n (\partial/\partial x_j) f_j(x,0) \} v(x)\, dx + \int_\Omega a(x) |\nabla v|^2 \}\, dx, \end{aligned} \tag{4.15}$$

and the first term on the right side of (4.15) is finite by condition (f1). Hence (4.14) holds.

Now by (4.12) and (4.14) we have

$$\Phi(v) = \Phi_1(v) + \Phi_2(v), \qquad v \in \mathfrak{D}(\Phi).$$

Hence $\Phi_i(v)$, $i = 1, 2$, lie in $(-\infty, +\infty)$ for each $v \in \mathfrak{D}(\Phi)$. Therefore (4.8) and (4.9) hold.

We next verify (4.10). For each compact subdomain K of Ω there is a positive constant c such that

$$a(x) \geq c, \qquad x \in K,$$

since $a(.) \in C(\mathbf{R}^n)$ and $a(.) > 0$ by (f3). Hence by (4.8) and (4.15) one has $|\nabla v|^2 \in L^1_{\mathrm{loc}}(\Omega)$. This means that (4.10) holds.

Finally, (4.13) and (4.15) together imply (4.11). This completes the proof of Lemma 4.2.

Lemma 4.3. *Φ is l.s.c. on $L^2(\mathbf{R}^n)$.*

Proof. Let $\{v^m\}$ be a sequence in $\mathfrak{D}(\Phi)$ such that

$$v^m \longrightarrow v^\infty \qquad \text{in } L^2(\mathbf{R}^n), \tag{4.16}$$

$$\Phi(v^m) \leq r \qquad \text{for } m \in \mathbf{N} \text{ and some } r > 0. \tag{4.17}$$

In view of the definition of $\mathfrak{D}(\Phi)$, we see from (4.16) that

$$v^{\infty}(x)=0, \qquad \text{a.e. } x\in \mathbf{R}^n\backslash\Omega. \tag{4.18}$$

On the other hand, since $-\sum_{j=1}^{n}(\partial/\partial x_j)f_j(x, 0)+g(x, 0)\in L^2(\Omega)$ by $(f1)$ and $(g1)$, we infer from (4.16), (4.17) and (4.11) that

$$\int_{\Omega} a(x)|\nabla v^m(x)|^2\,dx \le c_1, \qquad m\in \mathbf{N} \tag{4.19}$$

with some constant c_1.

From (4.16) and (4.19) it follows that $v^{\infty}\in W^{1,2}_{\mathrm{loc}}(\Omega)$, and that there is a sequence $\{w^k\}$ of convex combinations of $\{v^m; m\in\mathbf{N}\}$ such that

$$\|w^k - v^{\infty}\|_{L^2(\Omega)} + \int_{\Omega} a(x)|\nabla w^k(x) - \nabla v^{\infty}(x)|^2\,dx \longrightarrow 0 \qquad \text{as } k\longrightarrow\infty;$$

we recall at this point that $a(.)\in C(\mathbf{R}^n)$ and $a(.)>0$ by $(f3)$. Choosing a subsequence $\{w^{k'}\}$ of $\{w^k\}$, we have

$$\lim_{k'\to\infty} w^{k'}(x) = v^{\infty}(x), \quad \lim_{k'\to\infty} \nabla w^{k'}(x) = \nabla v^{\infty}(x) \qquad \text{a.e. } x\in\Omega. \tag{4.20}$$

Since w^k, $k\in\mathbf{N}$, are convex combinations of $\{v^m; m\in\mathbf{N}\}$, the convexity of Φ as well as (4.17) implies that

$$\Phi(w^k)\le r, \qquad k\in\mathbf{N}. \tag{4.21}$$

Now, using (4.20), (4.21) and Fatou's lemma, we have

$$\begin{aligned}&\int_{\Omega}\{F(x, \nabla v^{\infty}(x)) + G(x, v^{\infty}(x))\}\,dx\\ &\quad\le \liminf_{k'\to\infty}\int_{\Omega}\{F(x, \nabla w^{k'}(x)) + G(x, w^{k'}(x))\}\,dx\\ &\quad\le r.\end{aligned}$$

Therefore $v^{\infty}\in\mathfrak{D}(\Phi)$ and $\Phi(v^{\infty})\le r$. Hence Φ is l.s.c. on $L^2(\mathbf{R}^n)$.

Finally, we see that $\mathscr{A}=\partial\Phi$ in the following two lemmas.

LEMMA 4.4. *For $u\in\mathfrak{D}(\mathscr{A})$, the integrals $\int_{\Omega}\sum_{j=1}^{n}f_j(x, \nabla u)(\partial u/\partial x_j)\,dx$ and $\int_{\Omega}g(x, u)u(x)\,dx$ are finite.*

PROOF. Condition $(f3)$ and condition $(\partial/\partial s)g(x, s)\ge 0$ in $(g1)$ yield that

$$\sum_{j=1}^{n}\{f_j(x, \nabla u(x)) - f_j(x, 0)\}\frac{\partial u}{\partial x_j}(x)\ge 0,$$

$$\{g(x, u(x)) - g(x, 0)\}\,u(x)\ge 0$$

for a.e. $x \in \Omega$, respectively. Noting by $(f1)$ and $(g1)$ that $f_j(.,0) \in H^1(\Omega)$ and $g(.,0) \in L^2(\Omega)$, and by (4.1) that $-\sum_{j=1}^n (\partial/\partial x_j) f_j(., \nabla u(.)) + g(.,u(.))|_\Omega \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$, we get Lemma 4.4.

LEMMA 4.5. $\mathscr{A} = \partial\Phi$.

PROOF. Let $w, w + v \in \mathfrak{D}(\Phi)$. Then by (4.6), (4.7) and (4.8), the integrals $\int_\Omega \sum_{j=1}^n f_j(x, \nabla w)(\partial/\partial x_j)v(x)\,dx$ and $\int_\Omega g(x, w)v(x)\,dx$ are finite. By the smoothness of $f_j(x, .)$ and $g(x, .)$,

$$\lim_{h\to 0} h^{-1}\{\Phi(w + hv) - \Phi(w)\}$$
$$= \int_\Omega \sum_{j=1}^n f_j(x, \nabla w)(\partial/\partial x_j)v(x)\,dx + \int_\Omega g(x, w)v(x)\,dx.$$

Since $v|_{\partial\Omega} = 0$ by (4.4), we have

$$\mathfrak{D}(\partial\Phi) = \{w \in \mathfrak{D}(\Phi):\ -\sum_{j=1}^n (\partial/\partial x_j) f_j(., \nabla w(.)) + g(., w(.))|_\Omega \in L^2(\Omega)\},$$
$$\partial\Phi(w) = \{h \in L^2(\mathbf{R}^n):\ h(x) = -\sum_{j=1}^n (\partial/\partial x_j) f_j(x, \nabla w) + g(x, w), \text{ a.e. } x \in \Omega\}.$$

To show that $\mathfrak{D}(\mathscr{A}) = \mathfrak{D}(\partial\Phi)$, we have only to see that $\mathfrak{D}(\mathscr{A}) \subset \mathfrak{D}(\Phi)$. Let $u \in \mathfrak{D}(\mathscr{A})$. Since the convexity of the potential function F implies that

$$F(x, y) = F(x, y) - F(x, 0) \le \sum_{j=1}^n \frac{\partial F}{\partial z_j}(x, y)y_j = \sum_{j=1}^n f_j(x, y)y_j$$

for $y \in \mathbf{R}^n$, we see from Lemma 4.4 that

$$\int_\Omega F(x, \nabla u)\,dx \le \int_{\mathbf{R}^n} -\sum_{j=1}^n f_j(x, \nabla u)\frac{\partial u}{\partial x_j}(x)\,dx < +\infty.$$

In the same way, we have

$$\int_\Omega G(x, u(x))\,dx \le \int_{\mathbf{R}^n} g(x, u(x))u(x)\,dx < +\infty.$$

Hence $\mathfrak{D}(\mathscr{A}) \subset \mathfrak{D}(\Phi)$ holds, and Lemma 4.5 is proved.

5. An application of Theorem 2.1

In this section we make an attempt to apply Theorem 2.1 and some other results given in [4] to the following equation:

$$(5.1)\qquad \begin{cases} (\partial/\partial t)v(t, x) - \sum_{j=1}(\partial/\partial x_j) f_j(t, x, \nabla v) + g(t, x, v) = 0, \\ \qquad\qquad\qquad (t, x) \in \bigcup_{t\ge 0}[\{t\} \times Q(t)]\ (= Q), \\ v(t, x) = 0, \qquad (t, x) \in \bigcup_{t\ge 0}[\{t\} \times \Gamma(t)]\ (= \Gamma). \end{cases}$$

Here $Q(t)$, $t \geq 0$, is an exterior or interior domain of $\mathbf{R}^n$ with smooth compact boundary $\Gamma(t)$. The functions f_j, $j = 1, 2, \dots, n$, and g satisfy the following conditions $(\bar{f}1)$–$(\bar{f}3)$ and $(\bar{g}1)$, which are parallel to the conditions $(f1)$–$(f3)$ and $(g1)$ of the t-independent case discussed in Section 4.

$(\bar{f}1)$ $f_j(t, x, .) \in C^1(\mathbf{R}^n)$, $f_j(t, ., 0) \in H^1(Q(t))$ and $f_j(t, ., z)$ are measurable function on Ω for $t \geq 0$, $x \in Q(t)$, $z \in \mathbf{R}^n$ and $j = 1, 2, \dots, n$.

$(\bar{f}2)$ For each $(t, x) \in Q$, *the family* $\{f_j(t, x, .)\}$ is completely integrable, that is, the equation

$$(\partial/\partial z_k) f_j(t, x, z) = (\partial/\partial z_j) f_k(t, x, z) \qquad (\equiv a_{jk}(t, x, z))$$

holds for $z \in \mathbf{R}^n$ and $j, k = 1, 2, \dots, n$.

$(\bar{f}3)$ There is a function $a(.) \in C(\mathbf{R}^n)$ with $a(x) > 0$ for $x \in \mathbf{R}^n$ and a function $r(.)$ on $[0, +\infty)$ with $r(t) > 0$ for $t \geq 0$ such that

$$r(t) a(x) |\xi|^2 \leq \textstyle\sum_{j,k=1}^n a_{jk}(t, x, z) \xi_j \xi_k, \quad (t, x, z) \in Q \times \mathbf{R}^n, \ (\xi_j) \in \mathbf{R}^n. \tag{5.2}$$

$(\bar{g}1)$ $g(t, x, .) \in C^1(\mathbf{R})$ and $(\partial/\partial s) g(t, x, s) \geq 0$ for $t \geq 0$ and $x \in \Omega$. $g(t, ., 0) \in L^2(Q(t))$ and $g(t, ., r)$ is measurable for each $t \geq 0$ and $r \in \mathbf{R}$.

Hence, by Proposition 4.1, equation (5.1) considered on $[0, +\infty) \times \mathbf{R}^n$ through the zero-extension is written in the subdifferential form (E) in the real Hilbert space $L^2(\mathbf{R}^n)$.

To apply Theorem 2.1 to the problem (5.1), we assume the following conditions (H1) and (H2):

(H1) The function $r(.)$ in $(\bar{f}3)$ satisfies

$$\int_0^\infty r(t)\, dt = \infty. \tag{5.3}$$

(H2) (cf. (2.6) and (2.7)) There is a domain Ω_1 of $\mathbf{R}^n$ such that

$$\Omega_1 \text{ is bounded and } \textstyle\bigcup_{t \geq 0} \Gamma(t) \subset \Omega_1, \tag{5.4}$$

$$r(t) a(x) |\xi|^2 \leq \textstyle\sum_{j,k=1}^n a_{jk}(t, x, z) \xi_j \xi_k \leq c_0 r(t) a(x) |\xi|^2, \quad x \in \mathbf{R}^n \setminus \Omega_1, \ t \geq 0, \ z, \ (\xi_j) \in \mathbf{R}^n, \tag{5.5}$$

$$r(t) b(x) \leq (\partial/\partial s) g(t, x, s) \leq c_0 r(t) b(x), \quad x \in \mathbf{R}^n \setminus \Omega_1, \ t \geq 0, \ s \in \mathbf{R}, \tag{5.6}$$

where $c_0 \geq 1$, $b(.)$ is a nonnegative measurable function on $\mathbf{R}^n \setminus \Omega_1$ and $a(.)$ and $r(.)$ are the functions stated in $(\bar{f}3)$.

Then the application of Theorem 2.1. implies the following theorem.

THEOREM 5.1. *Suppose* (H1) *and* (H2). *Then, for any two solutions* $u_1, u_2 \in W^{1,1}_{\text{loc}}((0, +\infty); L^2(\mathbf{R}^n))$ *of* (5.1),

$$\lim_{t \to +\infty} \| u_1(t, .) - u_2(t, .) \|_{L^2(\mathbf{R}^n)} = 0. \tag{5.7}$$

To prove Theorem 5.1 we note that for any fixed $t \geq 0$ the functions $f_j(t, ., .)$, $j = 1, 2, ., n$, and $g(t, ., .)$ satisfy the assumptions $(f1)$–$(f3)$ and $(g1)$ of Proposition 4.1. Hence, in the same way as in the proof of Proposition 4.1, we can define the functionals Φ^t, $t \geq 0$, by the following:

$$\mathfrak{D}(\Phi^t) = \{ w \in L^2(\mathbf{R}^n) \colon w \in W^{1,1}_{\text{loc}}(\mathbf{R}^n),\ w(x) = 0, \text{ a.e. } x \in \mathbf{R}^n \backslash Q(t), \text{ and} \\ \int_{Q(t)} \{F(t, x, \nabla w(x)) + G(t, x, w(x))\}\, dx < +\infty\}. \tag{5.8}$$

$$\Phi^t(w) = \begin{cases} \displaystyle\int_{Q(t)} \{F(t, x, \nabla w(x)) + G(t, x, w(x))\}\, dx, & w(x) \in \mathfrak{D}(\Phi^t), \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.9}$$

Then equation (5.1) formulated in $L^2(\mathbf{R}^n)$ by the zero-extension is equivalent to the evolution equation

$$u'(t) + \partial\Phi^t(u(t)) \ni 0, \qquad t \geq 0. \tag{5.10}$$

Let u_2 be any solution in $W^{1,1}_{\text{loc}}((0, +\infty); L^2(\mathbf{R}^n))$ of (5.10). We then define a new family of functionals $\{\varphi^t; \text{a.e. } t \geq 0\}$ by

$$\begin{aligned} &\mathfrak{D}(\varphi^t) = \mathfrak{D}(\Phi^t) - u_2(t), \\ &\varphi^t(w) = \Phi^t(w + u_2(t)) + (u_2'(t), w)_{L^2(\mathbf{R}^n)} - \Phi^t(u_2(t)), \end{aligned} \tag{5.11}$$

for all t at which (5.10) makes sense for u_2. Then it follows from Lemma 1.1 that the proof of Theorem 5.1 is reduced to the proof of the statement that

$$\text{s} - \lim_{t \to +\infty} u(t) = 0 \tag{5.12}$$

for any solution u of (1.8).

We derive (5.12) by verifying that the new family $\{\varphi^t\}$ defined by (5.11) satisfies $F(\varphi^t) = \{0\}$ for $t \geq 0$, and that all of the assumptions of Theorem 2.1.

The following lemma follows from definition (5.11) and Lemma 4.4.

LEMMA 5.1. $\varphi^t(w) = \int_{Q(t)} \gamma(t, x, \nabla w(x), w(x))\, dx$, a.e. $t \geq 0$ *holds with*

$$\gamma(t, x, \nabla w(x), w(x)) = \textstyle\sum_{j,k=1}^{n} a_{j,k}(t, x, \nabla u_2(t, x) + \theta \nabla w(x)) \dfrac{\partial w}{\partial x_j} \dfrac{\partial w}{\partial x_k}$$
$$+ \frac{\partial g}{\partial s}(t, x, u_2(t, x) + \bar{\theta} u(x)) w(x)^2\} \, dx,$$
$$\text{a.e. } t \geq 0, \text{ a.e. } x \in \Omega,$$

where θ *and* $\bar{\theta}$ *denote some numbers of* $(0, 1)$ *depending on* $(t, x, \nabla u_2(t, x), \nabla w(x))$ *and* $(t, x, u(t, x), w(x))$, *respectively.*

Now we define the functionals ψ_i, $i = 1, 2$, and the operator $A: L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ employed in Theorem 2.1 by the following:

$$\mathfrak{D}(\psi_1) = \mathfrak{D}(\psi_2) = \{u \in L^2(\mathbf{R}^n) \cap W_{\mathrm{loc}}^{1,2}(\mathbf{R}^n) : \int_{\mathbf{R}^n} \{a |\nabla u|^2 + bu^2\} \, dx < +\infty\},$$

$$\psi_1(u) = c_0 \int_{\mathbf{R}^n} \{a(x) |\nabla u(x)|^2 + b(x) u(x)^2\} \, dx, \qquad u \in \mathfrak{D}(\psi_1),$$

$$\psi_2(u) = (1/c_0) \psi_1(u), \qquad u \in \mathfrak{D}(\psi_2),$$

where c_0, $a(.)$ and $b(.)$ are the constant and the functions, stated in (H2), respectively.

$$(Au)(x) = \alpha(x) u(x), \tag{5.13}$$

with a fixed $\alpha \in C_0^\infty(\mathbf{R}^n)$ satisfying $\alpha|_{\Omega_1} \equiv 1$, where Ω_1 is the domain stated in (H2).

Since each functional is strictly convex, we have the folloeing lemma.

LEMMA 5.2. $F(\varphi^t) = F(\psi_i) = \{0\}$, $\varphi^t(0) = \psi_i(0) = 0$, $t \geq 0$, $i = 1, 2$.

To choose the constants a, b and c in Theorem 2.1, we need the following lemma.

LEMMA 5.3. *Let* α *be a function appearing in* (5.13). *There is a constant* $c_1 > 0$ *such that*

$$\int_{\mathbf{R}^n} a(x) |\nabla (u - \alpha u)|^2 \, dx \leq c_1 \int_{\mathbf{R}^n} a(x) |\nabla u|^2 \, dx \qquad \text{for } u \in \textstyle\bigcup_{t \geq 0} \mathfrak{D}(\varphi^t), \tag{5.14}$$

$$\int_{\mathbf{R}^n} a(x) |\nabla (\alpha u)|^2 \, dx \leq c_1 \int_{\mathbf{R}^n} a(x) |\nabla u|^2 \, dx \qquad \text{for } u \in \textstyle\bigcup_{t \geq 0} \mathfrak{D}(\varphi^t). \tag{5.15}$$

PROOF. Let $\hat{\Omega} = \mathrm{supp}(\alpha)$. Since Ω_1 is an open subset of $\hat{\Omega}$ by $\alpha|_{\Omega_1} \equiv 1$, there is a constant $c > 0$ such that

$$\|u\|_{H^1(\hat{\Omega})} \leq c\{\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\hat{\Omega})}\}, \qquad u \in H^1(\hat{\Omega}).$$

(See [8; Théorème 7.4 in Chapitre 2].) On the other hand by the definition of φ^t we have

$$u|_{\Gamma(t)} = 0 \qquad \text{for } u \in \mathfrak{D}(\varphi^t).$$

Since $\Gamma(t) \subset \Omega_1$ for $t \geq 0$ and Ω_1 is bounded, one has

$$\|u\|_{L^2(\Omega_1)} \leq c' \|\nabla u\|_{L^2(\Omega_1)} \qquad \text{for } u \in \textstyle\bigcup_{t \geq 0} \mathfrak{D}(\varphi^t).$$

Hence

$$\|u\|_{H^1(\hat{\Omega})} \leq c'' \|\nabla u\|_{L^2(\Omega_1)} \qquad \text{for } u \in \textstyle\bigcup_{t \geq 0} \mathfrak{D}(\varphi^t).$$

Since $a \in C(\mathbf{R}^n)$ and $a > 0$, we have

$$\begin{aligned}\int_{\mathbf{R}^n} a(x)|\nabla(u - \alpha u)|^2\, dx &= \int_{\hat{\Omega} \setminus \Omega_1} a(x)|u\nabla(1-\alpha) + (1-\alpha)\nabla u|^2\, dx \\ &\leq c_2\{\|u\|_{L^2(\hat{\Omega}\setminus\Omega_1)} + \|\nabla u\|_{L^2(\hat{\Omega}\setminus\Omega_1)}\} \leq c_3 \|\nabla u\|_{L^2(\hat{\Omega})} \\ &\leq c_4 \int_{\hat{\Omega}} a(x)|\nabla u|^2\, dx, \qquad u \in \textstyle\bigcup_{t \geq 0} \mathfrak{D}(\varphi^t).\end{aligned}$$

This is the aimed estimate (5.14). In the same way, we have (5.15).

Finally, we show the following lemma.

Lemma 5.4. *Put* $a = b = c = c_1$, *for the constant* c_1 *found in Lemma* 5.3. *Then all of* (D2), (D3) *and* (D4) *in Theorem* 2.1 *are satisfied.*

Proof. (D2) and (D3) are direct consequences of the definitions. Condition (D4) follows from

$$\{\mathrm{dist}\,(Au, F)\}^2 = \|\alpha u\|^2_{L^2(\mathbf{R}^n)} \leq \text{const. } \|\nabla(\alpha u)\|^2_{L^2(\mathbf{R}^n)} \leq \text{const. } \psi_1(Au).$$

Consequently, Theorem 2.1 can be applied and Theorem 5.1 is proved.

To end this section, we make three remarks on the application to equation (5.1) of the results [4; Theorem 2.1], [4; Theorem 2.2] and [4; Theorem 7.1].

Remark 5.1. Suppose all the assumptions imposed in this section except (H2). Suppose either (H3) or (H4):

(H3) The set $\bigcup_{t \geq 0} Q(t)$ is bounded in $\mathbf{R}^n$.

(H4) There is a positive constant c such that $(\partial/\partial s) g(t, x, s) \geq c$ for $(t, x, s) \in Q \times \mathbf{R}$.

Then, by Lemma 5.1, the family $\{\varphi^t\}$ satisfies the assumptions of [4; Theorem 2.1], that is,

$$\text{(A1)} \quad F(\varphi^t) \equiv F \neq \phi,\ t \geq 0, \qquad \text{and } \min \varphi^t = 0.$$

(A2) There is a nonnegative measurable function $r(.)$ satisfying (i) $\int_0^\infty r(t)\,dt = +\infty$ and (ii) for each $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{dist}(u, F) \leq \varepsilon$ for $u \in \bigcup_{t \geq 0} \{u \in \mathfrak{D}(\varphi^t) : \varphi^t(u) \leq \delta r(t)\}$.

Hence, by [4; Theorem 2.1], (5.7) holds for any two solutions u_1 and u_2 of (5.1).

Remark 5.2. Suppose the following condition (H5) instead of (H2):

(H5) Both (5.5) and (5.6) hold on the whole space $\bigcup_{t \geq 0} \{t\} \times Q(t)$ and $Q(t) \subset Q(s)$ holds for each case of $s < t$.

Then the family $\{\varphi^t\}$ satisfies the assumptions of [4; Theorem 2.2]. Hence (5.7) holds for all two solutions u_1 and u_2 of (5.1).

Remark 5.3. Suppose all the assumptions imposed in this section except for (H2). Then, by Lemma 5.1, the family $\{\varphi^t\}$ satisfies the assumptions of [4; Theorem 7.1], that is,

there is a proper l.s.c. convex functional ψ on $L^2(\mathbf{R}^n)$ such that $F(\varphi^t) = F(\psi) \neq \phi$ and $\min \varphi^t = \min \psi = 0$ for $t \geq 0$, and such that $\psi(u) \leq \varphi^t(u)$ for $u \in \mathfrak{D}(\varphi^t)$ and $t \geq 0$.

Hence, by [4; Theorem 7.1], for any two solutions u_1 and u_2 of (5.1) we have the following weaker result:

There is a measurable set Λ in $[0, +\infty)$ with $\lim_{t \to +\infty} \operatorname{meas}(\Lambda \cap [t, t+1]) = 1$ such that for any $w \in L^2(\mathbf{R}^n)$,

$$\lim_{\lambda \to +\infty, \lambda \in \Lambda} (u_1(\lambda) - u_2(\lambda), w)_{L^2(\mathbf{R}^n)} = 0.$$

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