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A Unique Imbedding of a Torus Homotopic to 0 in a 3-manifold

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In 1958, R. H. Bing [1] proved among other things the following: Let M_1 be the 3-manifold obtained by removing a polyhedral tubular neighbourhood of a trifolium (i.e. Kleeblattschlinge, [3] p. 2, Fig. 2) from S^3, that is, by piercing a hole along the knot and let M_2 be a torus. If M is a simply connected 3-manifold such that $M=M_1\cup M_2$ and $M_1\cap M_2=\mathrm{Bd}\ M_1=\mathrm{Bd}\ M_2$, then M is topologically S^3. Furthermore he stated in [1], p. 36, "It has been pointed out to me by C. D. Papakyriakopoulos that, by using a combination of the methods of homology and homotopy, it can be shown that M is topologically S^3 no matter how the hole is knotted."

Let T be a torus in a 3-manifold M. T is called to be homotopic to 0 in M, provided that there exists a continuous mapping f of the product space $T\times I$ into M such that $f(x, 0)=x$ and $f(x, 1)=p$ for each x in M, where I is the unit interval and p a fixed point in M. In this paper we shall prove the fact in a more general form, which is as follows:

Theorem. *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty. Let T be a polyhedral torus homotopic to 0 in M and T^* a torus. If M^* is a 3-manifold such that $M^*=(M-T)\cup T^*$, $(M-\mathit{Int}\ T)\cap T^*=BdT=Bd\ T^*$ and T^* is homotopic to 0 in M^*, then M^* is topologically M.*

Proof. We may suppose without loss of generality that $T\cap \mathrm{Bd}\ M=\phi$. For if not, it is sufficient to thicken Bd M. Since $T(T^*)$ is homotopic to 0 in M (in M^*) and Bd T (Bd T^*) is of genus 1, there exists a longitude l (l^*) of $T(T^*)$ such that

$$l\ (l^*)\sim 0^{(1)} \text{ in } M-\mathrm{Int}\ T \tag{1}$$

(cf. [2] pp. 153-154). If m is a meridian of T conjugate to l, we can find integers a and b such that $l^*\simeq al+bm^{(2)}$ on $\mathrm{Bd}(M-\mathrm{Int}\ T)$. Hence by (1) we have

$$al+bm\sim 0 \text{ in } M-\mathrm{Int}\ T. \tag{2}$$

Since l^* not~ 0 in T^*, $|a|+|b|\neq 0$. Furthermore $a\neq 0$. For if not, $bm\sim 0$ in $M-\mathrm{Int}\ T$ and $bm=\partial Z_1$, where Z_1 is a 2-chain in $M-\mathrm{Int}\ T$. On the other hand

(1) $\sim$ means homologous to.
(2) $\simeq$ means homotopic to.

$bm \sim 0$ in T and $bm = \partial Z_2$, where Z_2 is a 2-chain in T. Let K be a core (Seele) of T. Then the linking number of K and the 2-cycle $Z_1 - Z_2$ is equal to $b \neq 0$, which contradicts the fact K is homotopic to 0 in M.

From (1) and (2), we conclude $b=0$ for the same reason as above. Since l^* is a simple closed curve, $|a|=1$. Therefore l^* circles T once and hence by the standard method it follows that M^* is topologically M.

COROLLARY. *Let K be a knot in S^3, T a closed, polyhedral, tubular neighbourhood of K and T^* a torus. If M is a simply connected 3-manifold, such that $M = (S^3 - T) \cup T^*$ and $(S^3 - T) \cap T^* = Bd\ T = Bd\ T^*$. Then M is topologically S^3.*

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