Note on Formal Lie Groups (II)

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1. Let $K$ be an algebraically closed field. For any algebraic subgroup of the general linear group $GL(n, K)$ we can associate a formal Lie group, and for any subgroup $G$ of the formal Lie group $GL^*(n, K)$ associated with $GL(n, K)$ the algebraic hull $\mathcal{A}(G)$ can be defined in $GL(n, K)$. On the base of such connection with algebraic linear groups, non-commutative formal Lie groups were investigated in [3] by making use of the properties of algebraic linear groups in [1]. In [4], we settled some questions raised in [3] on maximal solvable subgroups, maximal tori etc. of a subgroup of $GL^*(n, K)$.

The purpose of this note is to show some properties of formal Lie groups which follow from the results in [4].

The following theorem was proved by J. Dieudonné in [3]: In order that a formal Lie group $G$ over an algebraically closed field $K$ of characteristic $p > 0$ be nilpotent, it is necessary and sufficient that it contain a unique maximal torus. We shall give another condition for $G$ to be nilpotent and give another proof of the sufficiency part of the theorem by using [4, Th. 2], which allows us to make use of the corresponding theorem of algebraic linear groups. We shall also show some properties of maximal unipotent subgroups of a subgroup $G$ of $GL^*(n, K)$. E.g., if a maximal torus and a maximal unipotent subgroup are associated with algebraic subgroups of $GL(n, K)$, then so is $G$.

2. We shall recall some definitions, results and notations on formal Lie groups in [3, Chap. III]. We denote by $H^*$ the formal Lie group associated with an algebraic subgroup $H$ of $GL(n, K)$. Let $f$ be a rational homomorphism of $H$ into an algebraic linear group $H_1$. Then there exists a corresponding homomorphism $f'$ of $H^*$ into $H_1^*$ and $f'(H^*) = f(H)^*$. If $N$ is the kernel of $f$, then $N^*$ is the kernel of $f'$. Given an element $s$ of $GL(n, K)$, we denote by $a_s$ the automorphism of $GL^*(n, K)$ corresponding to the inner automorphism of $GL(n, K)$ induced by $s$. If $H$ is connected, then $H^*$ is solvable (resp. nilpotent, commutative) if and only if $H$ is solvable (resp. nilpotent, commutative). A formal Lie group over $K$ is called representable provided it is isogenous to a subgroup of the formal Lie group $GL^*(n, K)$. The quotient group of a formal Lie group by its center is always representable. For a subgroup $G$ of $GL^*(n, K)$, the algebraic hull $\mathcal{A}(G)$ is solvable (resp. nilpotent, commutative) if and only if $G$ is solvable (resp. nilpotent, commutative). $\mathcal{A}(G)^*$ is denoted by $\mathcal{A}^*(G)$. It is known that $DG = \mathcal{A}^*(DG) = D(\mathcal{A}(G))^*$. For a connected algebraic linear group $H$, we have $\mathcal{A}(H^*) = H$. The subgroups of any formal Lie group form
a complete lattice. For its subgroups $G_1$ and $G_2$, we denote by $G_1 \wedge G_2$, $G_1 \vee G_2$ the g.l.b. and the l.u.b. of $G_1$ and $G_2$. If, for connected algebraic subgroups $H_1$ and $H_2$ of $GL(n, K)$, we denote by $H_1 \vee H_2$ the smallest algebraic subgroup of $GL(n, K)$ containing $H_1$ and $H_2$, then we have $(H_1 \vee H_2)^* = H_1^* \vee H_2^*$.

3. We first write the following results in [4], on which we essentially depend in developing our theorems.

Let $K$ be an algebraically closed field and let $G$ be a subgroup of $GL^*(n, K)$. Then:

(A) If $S_1$ and $S_2$ are maximal solvable subgroups (resp. maximal tori, Cartan subgroups) of $G$, then there exists an element $s$ of $\mathcal{A}(DG)$ such that $a_s(S_1) = S_2$.

(B) The algebraic hull of any maximal solvable subgroup (resp. any maximal torus, any Cartan subgroup, the radical) of $G$ is a maximal solvable connected subgroup (resp. a maximal torus, a Cartan subgroup, the radical) of $\mathcal{A}(G)$ and conversely.

(C) $G$ is associated with an algebraic subgroup of $GL(n, K)$ if and only if so is a maximal solvable subgroup (resp. a Cartan subgroup, the radical).

These results were proved by the author in [4, Th. 1, Th. 2 and Cor. 2, Th. 4 and Cor. 1], where $G$ should obviously be a subgroup of $GL^*(n, K)$ as above although it was assumed to be a representable formal Lie group.

**Theorem 1.** Let $G$ be a formal Lie group over an algebraically closed field $K$ of characteristic $p > 0$. Then $G$ is nilpotent

1. if and only if it has a unique maximal torus;
2. if and only if a maximal solvable subgroup is nilpotent.

The first statement is a theorem of J. Dieudonné [3, Th. 6]. We here give another proof of “if” part by using the result (B), which allows us to make use of the corresponding result of algebraic linear groups [2, Exposé 6, Cor. 2 to Th. 4]. Suppose that $G$ has a unique maximal torus $T$. Put $G' = G/Z(G)$, where $Z(G)$ is the center of $G$. If $f$ is the natural epimorphism of $G$ onto $G'$, then any maximal torus of $G'$ is the image of a maximal torus of $G$ by $f$ [3, p. 379]. Therefore $f(T)$ is the unique maximal torus $T'$ of $G'$. Since $G'$ is representable and since maximal tori are preserved by an isogeny, we may suppose that $G'$ is a subgroup of $GL^*(n, K)$. Then, by (B) for maximal tori, we see that $\mathcal{A}(T')$ is the unique maximal torus of $\mathcal{A}(G')$. Therefore it follows that $\mathcal{A}(G')$ is nilpotent. Hence $G'$ is nilpotent and therefore $G$ is nilpotent.

To prove the second statement, suppose that a maximal solvable subgroup $R$ of $G$ is nilpotent. Put $R' = f(R)$. Then it is easy to see that $R'$ is a maximal solvable subgroup of $G'$. We may suppose that $G'$ is a subgroup of $GL^*(n, K)$. Then, by virtue of (B), $\mathcal{A}(R')$ is a maximal solvable connected subgroup of $\mathcal{A}(G')$. Since $\mathcal{A}(R')$ is nilpotent, it follows from the result of algebraic linear groups corresponding to (2) [2, Exposé 6, Cor. 2 to Th. 4] that $\mathcal{A}(G')$ is
nilpotent. Hence $G'$ and therefore $G$ is nilpotent.

**Theorem 2.** Let $K$ be an algebraically closed field of characteristic $p > 0$ and let $G$ be a subgroup of $GL^*(n, K)$. Then:

1. If $H_1$ and $H_2$ are maximal unipotent subgroups of $G$, then there exists an element $s$ of $A(DG)$ such that $a_s(H_1) = H_2$.

2. The algebraic hull of any maximal unipotent subgroup $H$ of $G$ is a maximal unipotent subgroup of $A(G)$ and conversely. And we have $H = G \cap A^*(H)$.

3. $G$ is associated with an algebraic subgroup of $GL(n, K)$ if and only if a maximal torus and a maximal unipotent subgroup are associated with algebraic subgroups of $GL(n, K)$.

Let $R$ be a maximal solvable subgroup of $G$. Then it is known that $R$ has a largest unipotent subgroup $R_u$, which is normal in $R$ and $R = T \cup R_u$ for any maximal torus $T$ of $R$ [3, Prop. 38]. If $H$ is a unipotent subgroup of $G$ containing $R_u$, take a maximal solvable subgroup $R'$ of $G$ containing $H$. Then $R_u \subset H \subset R'_u$. By (A) there exists an element $s$ of $A(DG)$ such that $a_s(R) = R'$, whence $a_s(R_u) = R'_u$. Hence $R_u$ and $R'_u$ have the same dimension and therefore $R_u = H = R'_u$. Thus $R_u$ is a maximal unipotent subgroup of $G$. The converse is easy and we have the following statement:

$\alpha$ Any maximal unipotent subgroup of $G$ is the largest unipotent subgroup of $R$ of a maximal solvable subgroup of $G$ and conversely.

We can similarly prove the corresponding result for maximal unipotent subgroups of a connected algebraic subgroup of $GL(n, K)$, which we denote by $\alpha'$.

Further we know the following fact [3, Cor. to Prop. 38]:

$\beta$ If $G$ is a solvable subgroup of $GL^*(n, K)$, then $\mathcal{A}(G_u)$ is the largest unipotent subgroup of $\mathcal{A}(G)$.

Now we have all the statements of the theorem as follows. (1) is immediate from $\alpha$ and the conjugation theorem (A) for maximal solvable subgroups. The first part of (2) follows from $\alpha$, $\alpha'$, $\beta$ and (B) for maximal solvable subgroups. The second part of (2) is immediate from the first part. As for (3), let $R$ be a maximal solvable subgroup of $G$. Then $R = T \cup R_u$ with $T$ a maximal torus of $R$. If $T$ and $R_u$ are associated with algebraic linear groups, then we have

$$R = T \cup R_u = \mathcal{A}^*(T) \cup \mathcal{A}^*(R_u) = (\mathcal{A}(T) \cup \mathcal{A}(R_u))^*,$$

whence $R$ is associated with an algebraic linear group. Since $T$ is a maximal torus of $G$ [3, Prop. 34], (3) now follows immediately from (1), $\alpha$, the conjugation theorem (A) for maximal tori and (C) for maximal solvable subgroups.
References


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