J. Sci. Hiroshima Univ. Ser. A-I
26 (1962), 21-25

On a Space of Distributions with Support in a Closed Subest

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(Received February 20, 1962)

For any closed subset F of R^n, the n-dimensional Euclidean space, $\mathscr{D}'_F$ denotes the space of all the distributions with support in F. As well known, the space $\mathscr{D}'$ of the distributions on R^n is complete and bornological, hence barrelled, and the space $\mathscr{D}'_F$ is the closed subspace of $\mathscr{D}'$. In general, a closed subspace of a bornological (resp. barrelled) space is not always bornological (resp. barrelled). In this paper it is shown that $\mathscr{D}'_F$ is bornological and barrelled for any closed subset F. Spaces of this type are often encountered in the applications of the theory of distributions. We shall also be concerned with constructive structure of $\mathscr{D}'_{\Gamma_0}$, Γ_0 being the first quadrant of R^n, because of its importance in symbolic calculus [9].

In this paper we shall use the notations of L. Schwartz [6] without any further reference.

1. Preliminaries. For our later purpose we need the following lemma of Hirata [4]: Let E be the projective limit of a sequence of bornological spaces E_j with epijective continuous mappings $\pi_{j,j+1}: E_{j+1} \rightarrow E_j,$ then E is bornological under the condition that for any bouned subset B_j of E_j there exists a bounded subset B of E such that $\pi_j(B)=\mathrm{B}_j$, where π_j is the mapping of E onto E_j determined by $\{\pi_{j,j+1}\}$. It is to be noticed that this condition is satisfied if each E_j is a Silva space in the sense of Yoshinaga [10], that is, the dual space of a Schwartz (F) space. To see this, we first show

Proposition 1. *If ϕ is an epijective continuous mapping of a Silva space H onto another G, then ϕ is epimorphic and any bounded subset of G is an image of a bounded subset of H.*

Prooe. Any Silva space is reflexive and is the dual of a Schwartz (F) space. Therefore we may consider H (resp. G) as the dual of a Schwartz (F) space K (resp. L). As ϕ is onto, it becomes the dual mapping of a monomor phism ψ of L into K [1]. Then any equicontinuous subset of L' $(=G)$ is a ϕ-image of an equicontinuous subset of $K'(=H)$ [2]. Since the space of type (F) is a barrelled space, any equicontinuous subset of K' (resp. L') is a bounded subset of H (resp. G) and vice versa. As ϕ is onto, it is epimorphic [10]. The proof is complete.

Let B_j be any bounded subset of E_j. We put $B_i=\pi_{i,i+1}\circ\cdots\circ\pi_{j-1,j}(B_j)$ for

$i<j$. Then B_i is a bounded subset of E_i since it is a continuous image of a bounded subset. By Proposition 1, B_j is the $\pi_{j,j+1}$-image of a bounded subset B_{j+1} of E_{j+1}. By repeating this process, we obtain a sequence $\{B_l\}_{1\leqq l<\infty}$, where B_l is a bounded subset of E_l. This sequence defines a bounded subset B of E such that $\pi_j(B)=B_j$.

We note that any closed subspace of a Silva space is a Silva space [10].

2. Main theorem. As well known, the space $\mathscr{D}$ of the indefinitely differentiable functions defined on R^n with compact support is the inductive limit of the spaces $\mathscr{D}_{\bar{Q}_j}$ of the functions of $\mathscr{D}$ whose supports are contained in $\bar{Q}_j$, where Q_j is a cube $\{x \in R^n: |x_1|<\lambda_j, \cdots, |x_n|<\lambda_j\}$ with $\lambda_j \uparrow \infty$. Therefore the space $\mathscr{D}'$ of the distributions is the projective limit of the spaces $\mathscr{D}'_{\bar{Q}_j}$ (dual of $\mathscr{D}_{\bar{Q}_j}$) with projections $\pi_{j,j+1}$: $\mathscr{D}'_{\bar{Q}_{j+1}} \rightarrow \mathscr{D}'_{\bar{Q}_j}$ and π_j: $\mathscr{D}' \rightarrow \mathscr{D}'_{\bar{Q}_j}$ $(j=1, 2, \cdots)$. Any element ξ of $\mathscr{D}'_{\bar{Q}_j}$ is, by definition, a continuous linear form on $\mathscr{D}_{\bar{Q}_j}$, and therefore a restriction $T|Q_j$ of a distribution T to Q_j owing to the extension theorem of Hahn-Banach. Therefore ξ is identified with a distribution on Q_j of the form $\mathbb{D}^p f$, f being an element of the space $C(\bar{Q}_j)$ of the contiunous functions on $\bar{Q}_j$, $\mathbb{D}=\dfrac{\partial^n}{\partial x_1 \cdots \partial x_n}$ and p a non-negative integer. Here we have for any $\phi \in \mathscr{D}_{\bar{Q}_j}$

$$\xi(\phi)=(-1)^{np}\int_{\bar{Q}_j} f(x)\ \mathbb{D}^p \phi(x) dx. \tag{1}$$

Conversely any such $\mathbb{D}^p f$ defines a continuous linear form ξ according to the equation (1). Therefore we can identify $\mathscr{D}'_{\bar{Q}_j}$ with the set of all such restrictions $T|Q_j$. Pietsch has shown that f may be taken to be equal to zero on the boundary $\dot{Q}_j$ [5]. As well known, since the space $\mathscr{D}_{\bar{Q}_j}$ is a Schwartz (F) space, $\mathscr{D}'_{\bar{Q}_j}$ is a Silva space. On account of the remark made in Section 1 we see that $\mathscr{D}'$ is bornological and barrelled.

It is almost clear that $\mathscr{D}'_F$ becomes the projective limit of the sequence $\{\pi_j(\mathscr{D}'_F): \pi_{j,j+1}\}$ with projection $\pi_{j,j+1}$: $\pi_{j+1}(\mathscr{D}'_F) \rightarrow \pi_j(\mathscr{D}'_F)$. We shall denote by $\mathfrak{C}(Q_j)$ the subspace $\{T|Q_j: T \in \mathscr{D}'$, supp. $(T|Q_j) \subset F \cap Q_j\}$. Then $\mathfrak{C}(Q_j)$ is a closed subspace of $\mathscr{D}'_{\bar{Q}_j}$, whence it is a Silva space. It is clear that $\pi_j(\mathscr{D}'_F) \subset \mathfrak{C}(Q_j)$. Therefore from the discussions made in Section 1 we obtain

PROPOSITION 2. *$\mathscr{D}'_F$ is bornological and barrelled, when the following condition is satisfied for any j:*

(*) If $T \in \mathscr{D}'$ is any distribution such that the support of $T|Q_j$ is contained in $F \cap Q_j$, then there exists an element $S \in \mathscr{D}'_F$ such that $S|Q_j=T|Q_j$.

The condition (*) represents the fact that $\pi_j(\mathscr{D}'_F)$ coincides with $\mathfrak{C}(Q_j)$, so that $\pi_j(\mathscr{D}'_F)$ becomes a Silva space.

We can show that if $F \cap Q_j$ is compact for any j the condition (*) is satisfied. In fact, let $U \subset Q_j$ be a compact neighbourhood of $F \cap Q_j$, and let α be a function of $\mathscr{D}$ such that it is equal to 1 on a neighbourhood $W \subset U^0$ (the interior of U) of $F \cap Q_j$ and vanishes outside U. Let T be any distribution with supp.

$(T|Q_j)\subset F\cap Q_j$. If we take S as αT then it is clear that S is a distribution of $\mathscr{D}'_F$ which satisfies the condition $(*)$.

By making use of Proposition 2 we shall show

THEOREM. *Let F be any closed subset of R^n, then the space $\mathscr{D}'_F$* is bornological and barrelled.

PROOF. We first assume that there exist two closed subsets F_1, F_2 with the properties:

(i) $F=F_1\cup F_2$.

(ii) There exists a partition $\alpha+\beta=1$ of the unity such that (supp. $\alpha)\cap F\subset F_1$ and (supp. $\beta)\cap F\subset F_2$.

(iii) There exist $\{Q'_j\}$, $\{Q''_j\}$ such that $Q'_j\cap F_1$, $Q''_j\cap F_2$ are compact for each j, where $\{Q'_j\}$, $\{Q''_j\}$ are increasing sequences of such cubes as defined above in this section.

Then by the remark after Propostion 2, $\mathscr{D}'_{F_1}$ and $\mathscr{D}'_{F_2}$ are bornological and barrelled. Now consider the continuous linear mapping θ: $(T_1, T_2)\to T_1+T_2$ of $\mathscr{D}'_{F_1}\times\mathscr{D}'_{F_2}$ into $\mathscr{D}'_F$. We first show that θ is onto. We put $T_1=\alpha T$, $T_2=\beta T$ for any $T\in\mathscr{D}'_F$, then by the assumption (ii) we see that $(T_1, T_2)\in\mathscr{D}'_{F_1}\times\mathscr{D}'_{F_2}$, and $T_1+T_2=T$. Furthermore if $T\to O$ in $\mathscr{D}'_F$, then T_1 (resp. $T_2)\to O$ in $\mathscr{D}'_{F_1}$ (resp. $\mathscr{D}'_{F_2}$). Therefore the mapping θ is epimorphic. As $\mathscr{D}'_{F_1}\times\mathscr{D}'_{F_2}$ is bornological and barrelled, so is also $\mathscr{D}'_F$.

It remains to show the existence of F_1 and F_2 with the properties described above. Let C_j be the subset $\left\{x\in R^n:\ |x_i|<j+\frac{1}{3},\ i=1, 2,\cdots, n, \text{ but} |x_i|>j-1-\frac{1}{3} \text{ for some } i\right\}$. Then $\{C_j\}$ $(j=1, 2,\cdots)$ defines an open covering of R^n. Let $\{\alpha_j\}$ be a partition of the unity subordinate to this covering. Now put

$$F_1=F\cap(\textstyle\bigcup_{j=1}^{\infty}\bar{C}_{2j-1}),\quad F_2=F\cap(\textstyle\bigcup_{j=1}^{\infty}\bar{C}_{2j}),$$

$$\alpha=\textstyle\sum_{j=1}^{\infty}\alpha_{2j-1},\qquad \beta=\textstyle\sum_{j=1}^{\infty}\alpha_{2j}.$$

Then we obtain

$$F_1\cup F_2=F\cap(\textstyle\bigcup_j\bar{C}_j)=F,$$

$$(\text{supp. }\alpha)\cap F\subset(\textstyle\bigcup_{j=1}^{\infty}\bar{C}_{2j-1})\cap F=F_1,$$

$$(\text{supp. }\beta)\cap F\subset(\textstyle\bigcup_{j=1}^{\infty}\bar{C}_{2j})\cap F=F_2.$$

Let Q'_j and Q''_j be the cubes $\{x\in R^n:\ |x_i|<2j-1+\frac{1}{2},\ i=1, 2,\cdots, n\}$ and $\{x\in R^n:\ |x_i|<2j+\frac{1}{2},\ i=1, 2,\cdots, n\}$ respectively, then it is easy to see that $\{Q'_j\}$, $\{Q''_j\}$ satisfy (iii). This completes the proof.

It is remarked that the similar arguments with necessary modifications can be applied to establish that $\mathscr{D}'_F(\Omega)$ is bornological and barrelled, where $\mathscr{D}'(\Omega)$ is the space of the distributions defined on an open subset $\Omega\subset R^n$ and $\mathscr{D}'_F(\Omega)$ is its subspace consisting of the distributions on Ω with support contained in a closed subset $F\subset\Omega$.

By the way we can see the following properties of $\mathscr{D}'_F$. A locally convex topological vector space E is called a Montel space, if it is barrelled and any bounded subset of E is relatively compact. The space $\mathscr{D}'_F$ is barrelled and a closed subspace of $\mathscr{D}'$ which is a Montel space, so that $\mathscr{D}'_F$ is also a Montel space, hence reflexive. Since $\mathscr{D}'$ is nuclear $\mathscr{D}'_F$ is also a nuclear space [2].

3. Space $\mathscr{D}'_{\Gamma_0}$. As shown in the preceding theorem, $\mathscr{D}'_F$ is bornological and barrelled for any closed subset F of R^n. However, as shown below, the condition of Proposition 2 can be directly verified in most practical cases considered in the applications of the Schwartz theory of distributions.

Let F be the closed subset $\Gamma_0=\{x: x_1\geqq O,\cdots, x_n\geqq O\}$ and Q_j be any open cube $\{x: |x_1|<\lambda_j\cdots, |x_n|<\lambda_j\}$. As stated in Section 2, for any element $T\in\mathscr{D}'$, the restriction $T|Q_j$ can be written in the form $\mathbb{D}^p f$ with $f\in C(\bar{Q}_j)$. Put $Q'_j=\Gamma_0\cap Q_j$. We first show that if the support of $T|Q_j$ is contained in Q'_j, we can take f to vanish outside $\bar{Q}'_j$. To this end we can make use of a projective operator $P^{(p)}$ of $C(\bar{Q}_j)$ into the kernel $N(\mathbb{D}^p)$ of $\mathbb{D}^p$ devised by Silva in his axiomatic construction of the theory of distributions [8], where the projective operator $P^{(p)}$ has been defined as follows. Choose any distinct p points $\alpha_{i1}, \alpha_{i2},\cdots, \alpha_{ip}$ in each x_i-axis such that $-\lambda_j<\alpha_{i1}<O$, $-\lambda_j<\alpha_{i2}<O,\cdots$, $-\lambda_j<\alpha_{ip}<O$. Using the projective operators $P_i^{(p)}: C(\bar{Q}_j)\to N(\mathbb{D}^p_{x_i})$ defined by

$$P_i^{(p)}g=\sum_{k=1}^{p}\frac{(x_i-\alpha_{i1})\cdots(x_i-\alpha_{i,k-1})(x_i-\alpha_{i,k+1})\cdots(x_i-\alpha_{ip})}{(\alpha_{ik}-\alpha_{i1})\cdots(\alpha_{ik}-\alpha_{i,k-1})(\alpha_{ik}-\alpha_{i,k+1})\cdots(\alpha_{ik}-\alpha_{ip})}\times g(x_1,\cdots, x_{i-1},\alpha_{ik}, x_{i+1},\cdots,x_n)$$

for any $g\in C(\bar{Q}_j)$, where $N(\mathbb{D}^p_{x_i})$ is the kernel of $\mathbb{D}^p_{x_i}=\left(\frac{\partial}{\partial x_i}\right)^p$, $P^{(p)}$ is given by the equation $P^{(p)}=1-(1-P_1^{(p)})(1-P_2^{(p)})\cdots(1-P_n^{(p)})$. From this construction of $P^{(p)}$, it can be easily seen that $f-P^{(p)}f=O$ outside Q'_j. Then it is sufficient to consider $f-P^{(p)}f$ instead of f. Let $\tilde{f}$ be equal to f on $\bar{Q}_j$ and zero elsewhere. If we put $S=\mathbb{D}^p\tilde{f}$, then it is clear that $S\in\mathscr{D}'_{\Gamma_0}$ and $T|Q_j=S|Q_j$, so that the condition of Proposition 2 is verified.

Let $\mathfrak{C}_0(Q_j)$ be the space of the continuous functions f on $\bar{Q}_j$ with support in $\bar{Q}'_j$ and norm $\|f\|=\max_{x\in\bar{Q}_j}|f(x)|$. Let $\mathfrak{C}_p(Q_j)$ be the space of the distributions $\mathbb{D}^p f$ with $f\in\mathfrak{C}_0(Q_j)$. When $\mathbb{D}^p f=O$, f must be identically zero. This follows from the consideration of the projective operator $P^{(p)}$ stated above. In fact, then f coincides with $P^{(p)}f$ which will be identically zero as seen from the expression of $P_i^{(p)}f$. Setting $\|\mathbb{D}^p f\|=\|f\|$, the space $\mathfrak{C}_p(Q_j)$ becomes a Banach space with norm $\|\mathbb{D}^p f\|$. Now consider the mapping $h_{p+1,p}: \mathfrak{C}_p(Q_j)\to\mathfrak{C}_{p+1}(Q_j)$ defined by $\mathbb{D}^p f\to\mathbb{D}^{p+1}g$, where $g(x)=\int_0^{x_n}\cdots\int_0^{x_1}f(t_1,\cdots, t_n)dt_1\cdots dt_n$. It is clear that $\mathbb{D}^p f$ and

$\mathbb{D}^{p+1}g$ coincide as distributions on $\bar{Q}_j$. Since the mapping $f \to g$ is compact, it follows from the definition of the norms in these two spaces that $h_{p+1,p}$ is also compact. Therefore the inductive limit $\mathfrak{E}(Q_j)$ of $\{\mathfrak{E}_p(Q_j): h_{p+1,p}\}$, $p = 0, 1, 2, \cdots$, is a Silva space. The space $\mathfrak{E}(Q_j)$ just obtained is the same space $\mathfrak{E}(Q_j)$ as considered in Section 2 for the case $F = \Gamma_0$. Thus $\mathscr{D}'_{\Gamma_0}$ is the projective limit of $\mathfrak{E}(Q_j)$ with the projections $\mathfrak{E}(Q_{j+1}) \to \mathfrak{E}(Q_j)$.

From these considerations together with the fact that any Silva space possesses the strict condition of convergence of Mackey [3], we see that, if we are given a bounded subset of $\mathscr{D}'_{\Gamma_0}$, then its restriction on Q_j can be represented algebraically as well as topologically by a bounded subset of $\mathfrak{E}_p(Q_j)$ for some integer p.

Vasilach [9] considered the space $\lim_j \operatorname{ind} \mathscr{D}'_{\Gamma_j}$ where Γ_j is the displacement of Γ_0 by the vector $(j, \cdots, j)$, j being an integer. And he tried to show that the space $\lim_j \operatorname{ind} \mathscr{D}'_{\Gamma_j}$ is barrelled for $n \geqq 1$. His method of proof, however, is only applicable to the case $n = 1$.

The same arguments as above can be applied to the case where F is the semi-space $\{x \in R^n : x_n \geqq 0\}$.

References

[1] J. Dieudonné et L. Schwartz, *La dualité dans les espaces (F) et (LF)*, Ann. Inst. Fourier, Grenoble, 1 (1950), 61–101.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., No. 16 (1955).

[3] ——————, *Sur les espaces (F) et (DF)*, Summa Brasil. Math., 3 (1954).

[4] Y. Hirata, *On a theorem in an (LF) space*, This Journal, 20 (1957), 111–113.

[5] A. Pietsch, *Ein elementarer Beweis des Darstellungssatzes für Distributionen*, Math. Nachr., 22 (1960), 47–50.

[6] L. Schwartz, *Théorie des distributions*, t. I et II, Hermann, Paris, 1950–1951.

[7] ——————, *Théorie des distributions à valeurs vectorielles*, Ann. Inst. Fourier, 8 (1958), 1–209.

[8] J. S. E. Silva, *Sur une construction axiomatique de la théorie des distributions*, Rev. Fac. Ciencias Lisboa, 4 (1954–55), 79–186.

[9] S. Vasilach, *Calcul opérationnel algébrique des distributions à support dans R^n_+, $n \geqq 1$*, Rev. Math. Pures App., 4 (1959), 495–531.

[10] K. Yoshinaga, *On a locally convex space introduced by J.S.E. Silva*, This Journal, 21 (1957), 89–98.

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