Let $f(x)$ be any locally summable and positive-valued function defined almost everywhere on $\mathbb{R}^n$, Euclidean $n$-space. Let $H^f$ be a Hilbert space obtained by completing the space $\mathcal{S}E$, the linear space of the inverse Fourier transforms of $\mathcal{S}E$, with norm $\|\xi\|^2 = \int |\hat{\xi}(x)|^2 f(x) dx$, where $\hat{\xi}$ denotes the Fourier transform of $\xi \in \mathcal{S}E$. Under more special conditions on $f$, the space $H^f$ has been investigated by J. Deny [1] and B. Malgrange [5] in connection with the study of the potential theory and the theory of partial differential equations respectively. In connection with this situation, we say that $f$ is of Deny type (simply type $D$) if it satisfies the condition:

\[(D) \quad f(x), \frac{1}{f(x)} \in (1 + |x|^2)^m \times L^1 \quad \text{for an integer } m,\]

where $L^1$ denotes the space of the summable functions on $\mathbb{R}^n$.

We also say that $f$ is of Malgrange type (simple type $M$) if it satisfies the condition:

\[(M) \quad f(x), \frac{1}{f(x)} \leq C(1 + |x|^2)^m \quad \text{a.e. for a constant } C \text{ and an integer } m.\]

Actually Malgrange was concerned with the continuous $f$ of type $M$.

The purpose of our investigation is to characterize these types of $f$ by means of the properties of $H^f$ and its related spaces.

In Section 1 we show that $f$ is of type $D$ if and only if $H^f$ is a normal space of distributions. If $\mu$ is a positive measure with which we define the space $H^\mu$ in the same way as before, we can show that $\mu$ must be of the form $f(x) dx$ when $H^\mu$ is a space of distributions.

In the following sections we shall only be concerned with normal $H^f$. Section 2 begins with the definition of the space $H_{f,\infty} = \bigcap_s H^{f,s}$ (resp. $H_{f,\infty} = \bigcup_s H^{f,s}$) with the topology of projective limit (resp. of inductive limit). $H^{f,s}$ stands for $H^f \setminus \{0\}$, where $f(x) = (1 + |x|^2)^s f(x)$ and $s$ is a real number. Then $H_{f,\infty}$ will be a reflexive space of type $(F)$ consisting of the distinguished elements of $H^f \setminus \{0\}$, and $H_{f,\infty}$ the anti-dual of $H_{f,\infty}$. We show that $f$ is of type $M$ if and only if $H_{f,\infty} = \mathcal{D}_L^s$ or $H_{f,\infty} = \mathcal{D}_L^s$.

In Section 3 we show that $H^f$ is of local type if and only if, for some integer $m$, \[\frac{1}{(1 + |x|^2)^m} \sqrt{\frac{f(y)}{f(x)}}\] is a kernel of a continuous linear application of $L^p_\gamma$ into $L^p_\gamma$. The condition is shown to be satisfied if, for a $k(x)$ such that
\( k(x) \in (1 + |x|^2)^m \times L^1 \) for an integer \( m \), the inequality \( f(x + y) \leq k(x)f(y) \) holds almost everywhere for \( |y| \geq c \). On the other hand, \( H_{f,\infty} \) and \( H_{f,\infty}' \) are of local type without any restriction on \( f \).

Section 4 is devoted to studying convolution \( \xi \ast \eta \) between elements of \( H_{f,\infty} \) and \( H_{f,\infty}' \). We show that \( f \) is of type \( M \) if and only if any \( \xi \in H_f \) is composable with every \( \eta \in H_{f,\infty}' \).

However it is to be noticed that by means of the Fourier transformation we can as usual define a convolution \( \hat{\xi} \ast \hat{\eta} \) in such a way that the operation \( \ast \) is separately continuous on \( H' \times H_{f,\infty}' \) and coincides with the usual convolution \( * \) on \( \mathcal{D} \times \mathcal{D} \). We show also that \( f \) is of type \( M \) if and only if \( \mathcal{A} H_{f,\infty} \subseteq H_{f,\infty} \) or \( \mathcal{A} H_{f,\infty}' \subseteq H_{f,\infty}' \). But we could not succeed in giving the conditions on \( f \) under which \( \mathcal{A} H_{f,\infty} \) is a part of \( H_f \).

1. **The Space of** \( H_f \). In what follows by \( x, y, \ldots \) we denote respectively points \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n), \ldots \) of the \( n \)-dimensional Euclidean space \( R^n \). We use the notations \( |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}, \) \( x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n, \) and if \( p = (p_1, p_2, \ldots, p_n) \), where the \( p_i \) are non-negative integers, we will write \( |p| = p_1 + p_2 + \ldots + p_n, \) \( D^p = \left( \frac{\partial}{\partial x_1} \right)^{p_1} \left( \frac{\partial}{\partial x_2} \right)^{p_2} \ldots \left( \frac{\partial}{\partial x_n} \right)^{p_n}. \)

Let \( f(x) \) be a locally summable and positive-valued function defined almost everywhere on \( R^n \). Following F. Trèves [18] we denote by \( \mathcal{S}E \) the family of the functions whose inverse Fourier transforms lie in \( \mathcal{S} \), the space of the indefinitely differentiable functions with compact support in \( R^n \). We consider \( \mathcal{S}E \) as a prehilbert space with inner product \( (\hat{\xi}, \hat{\eta})_f = \int \hat{\xi}(x) \hat{\eta}(x) f(x) dx, \) and hence with norm \( \|\xi\|_f = \left( \int |\hat{\xi}(x)|^2 f(x) dx \right)^{1/2}, \) where \( \hat{\xi} \) denotes the Fourier transform of \( \xi \). We shall denote by \( H_f \) a completion of \( \mathcal{S}E \) with respect to this norm. According to the usual notations, we shall write \( H^c \) if \( f(x) = (1 + |x|^2)^c \) and \( H^{c,s} \) if \( f(x) = (1 + |x|^2)^s f_1(x) \). Following L. Schwartz ([9], p. 7) we say that a locally convex space \( F \) is a space of distributions if it is algebraically a subspace of \( \mathcal{D}' \) and the injection \( F \rightarrow \mathcal{D}' \) is continuous, and that \( F \) is normal if, in addition, it contains \( \mathcal{D} \), the injection \( \mathcal{D} \rightarrow F \) is continuous, and \( \mathcal{D} \) is dense in \( F \). In general \( H_f \), as shown by Trèves ([13], p. 184), is not a space of distributions.

We shall first show

**Proposition 1.** The space \( H_f \) is a space of distributions if and only if the following condition (A) is satisfied:

\[
\text{A) } \quad \frac{1}{\int f} \in (1 + |x|^2)^m \times L^1 \text{ for some integer } m.
\]

Before proving the statement we remark that \( H_f \) is a space of distributions if and only if the following conditions are satisfied:

(i) The injection of \( \mathcal{S}E \) into \( \mathcal{D}' \) is continuous, that is, for any \( \varphi \in \mathcal{D}, \xi \rightarrow \langle \xi, \varphi \rangle \) is a continuous linear form on \( \mathcal{S}E \).

When this condition is satisfied, the linear forms \( \xi \rightarrow \langle \xi, \varphi \rangle \) can be extended
On a Space $H^f$

to the forms continuous on the whole space $H^f$.

(ii) If we are given an $\eta \in H^f$ with $\langle \eta, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$, then $\eta = 0$.

**Proof of Proposition 1.** *Necessity.* For any $\phi \in \mathcal{D}$ and any $\xi \in \mathcal{S}^c$ we have

$$\langle \xi, \phi \rangle = \int \hat{\xi}(x) \hat{\phi}(x) dx = \int \hat{\xi}(x) \sqrt{f(x)} \hat{\phi}(x) \sqrt{g(x)} dx, \text{ where } g = \frac{1}{f}.$$  

Since by (i) the form $\xi \rightarrow \langle \xi, \phi \rangle$ is continuous and the set $\{\sqrt{f} \xi; \xi \in \mathcal{S}^c\}$ is dense in $L^2$, it follows that $\sqrt{g} \phi \in L^2$. Then, by the closed graph theorem, the application $\phi \rightarrow \sqrt{g} \phi$ of $\mathcal{D}$ into $L^2$ becomes continuous. This implies that the application is continuous in $\mathcal{D}$, $B$ being the unit ball of $\mathbb{R}^n$ with center 0, with the topology induced by $\mathcal{D}'_B$ for some integer $k$ ($\mathcal{D}'_B$ is a Banach space of the $k$-times continuously differentiable functions with support in $B$). We can take a positive integer $l$ such that an $\alpha \in \mathcal{D}'_B$ is a parametrix of an iterated Laplacian $\Delta^l (J = x^2)$:

$$\delta = \Delta^l \alpha + \beta, \quad \beta \in \mathcal{D}_B.$$  

Now we can choose a sequence $\{\alpha_j\}, \alpha_j \in \mathcal{D}_B$, such that $\alpha_j \rightarrow \alpha$ in $\mathcal{D}'_B$ as $j \rightarrow \infty$. This together with (2) yields that $\sqrt{g} \phi \in L^2$, and $\sqrt{g} = |x|^2 \sqrt{\gamma} + \hat{\beta} \sqrt{g} \in (1 + |x|^2)^l \times L^2$, hence it follows that $g(x) \in (1 + |x|^2)^m \times L^1$ for $m=2l$.

**Sufficiency.** To complete the proof it is enough to establish the statements (i) and (ii). By the condition (A), $\phi \sqrt{g} \in L^2$ for any $\phi \in \mathcal{D}$. As for (i), since $\sqrt{f} \xi \in L^2$ for any $\xi \in \mathcal{S}^c$, the relation (1) shows that the application $\xi \rightarrow \langle \xi, \phi \rangle$ is continuous. Let $\{\xi_j\}$ be a sequence from $\mathcal{S}^c$ such that $\xi_j \rightarrow \eta$ in $H^f$, $\eta$ being any given element of $H^f$. Since $L_{f\Delta^l}^2 \subset L^2 \subset \langle 1 + |x|^2 \rangle^m \times L^1 \subset L^2 \subset \langle 1 + |x|^2 \rangle^m \times L^1 \subset \mathcal{S}'$, the injection $L_{f\Delta^l}^2 \rightarrow \mathcal{S}'$ becomes continuous by the closed graph theorem, so that $\{\xi_j\}$ converges in both $L_{f\Delta^l}^2$ and $\mathcal{S}'$ to the same element which we shall denote by $\hat{\eta}$. Then the relation (1) gives that $\langle \eta, \phi \rangle = \int \hat{\eta}(x) \sqrt{f(x)} \hat{\phi}(x) \sqrt{g(x)} dx$. Therefore, if $\langle \eta, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$, then $\eta = 0$ since $\{\phi \sqrt{g}; \phi \in \mathcal{D}\}$ is dense in $L^2$. (ii) is thus established.

**Remark.** If $H^f$ is a space of distributions, its elements are characterized as temperate distributions $\xi$ whose Fourier transforms $\hat{\xi}$ lie in $L_{f\Delta^l}^2$.

In general, even if $H^f$ is a space of distributions, it does not contain $\mathcal{D}$.

**Proposition 2.** Let $H^f$ be a space of distributions. Then the following three conditions are equivalent:

(B) $f \in (1 + |x|^2)^m \times L^1$ for some integer $m$.

(i) $\mathcal{D} \subset H^f$.

(ii) $H^f$ is normal.
PROOF. Ad (i) $\mapsto$ (B). (i) implies that we have $\hat{\phi}\sqrt{f} \in L^2$ for every $\phi \in \mathcal{D}$. Then, by the closed graph theorem, the application $\phi \mapsto \hat{\phi}\sqrt{f}$ of $\mathcal{D}$ into $L^2$ becomes continuous. Hence, as shown in the proof of the proposition 1, we can use a parametrix of an iterated Laplacian to conclude that $f \in (1 + |x|^2)^m \times L^1$ for some integer $m$.

Ad (B) $\mapsto$ (ii). Let $\phi$ be any function of $\mathcal{S}$. Since $\hat{\phi} \in \mathcal{S}$ and $\hat{\phi}\sqrt{f} \in \hat{\phi}(1 + |x|^2)^m \times L^2 \subset L^2$, we have $\mathcal{S} \subset H'$. $\mathcal{S}$ is dense in $H'$ as an immediate consequence of the definition of $H'$. On the other hand, $\mathcal{D}$ is dense in $\mathcal{S}$ and the injection $\mathcal{S} \to H'$ is continuous, so that $\mathcal{D}$ is dense in $H'$, that is, $H'$ is normal.

The implication (ii) $\mapsto$ (i) is almost evident.

Thus the proof is complete.

The following theorem is an immediate consequence of Proposition 1 and Proposition 2.

THEOREM 1. $H'$ is a normal space of distributions if and only if the following condition is satisfied:

\[(D) \quad f, \frac{1}{f} \in (1 + |x|^2)^m \times L^1 \text{ for some integer } m.\]

We say that $f$ is of Deny type (simply type $D$) if $f$ satisfies the condition (D), which is the same as Deny called Hypothesis (A) in his thesis ([1], p. 119). The condition (D) shows that if $H'$ is a normal space of distributions then $H^{1/2}$ is so also.

PROPOSITION 3. (i) If $f(x)$ is of type $D$, so is $f(x)(1 + |x|^2)^s$ for any real $s$.
(ii) If $f_1(x), f_2(x)$ are of type $D$, so is $f_1^{-\theta}(x)f_2^\theta(x)$ for any $0 < \theta < 1$.

PROOF. Ad (i). Setting $h(x) = f(x)(1 + |x|^2)^s$, we have $h(x), \frac{1}{h(x)} \in (1 + |x|^2)^{m+|s|} \times L^1$.

Ad (ii). As we may assume that integer $m$ in (D) is the same for $f_1(x)$ and $f_2(x)$, so we have

$$f_1^{-\theta}(x)f_2^\theta(x) \leq (1 - \theta)f_1(x) + \theta f_2(x) \in (1 + |x|^2)^m \times L^1$$

and also

$$\frac{1}{f_1^{-\theta}(x)} \frac{1}{f_2^\theta(x)} \leq (1 - \theta)\frac{1}{f_1(x)} + \theta \frac{1}{f_2(x)} \in (1 + |x|^2)^m \times L^1.$$
On a Space $H^f$

so that there is a constant $C$ such that $\int |\hat{\xi}(x)|^2 f_j(x) dx \leq C \int |\hat{\xi}(x)|^2 f_j(x) dx$ for any $\xi \in H^f$. Setting $\sigma(x) = \xi(\chi) \chi(x)$, we have $\int |\sigma(x)|^2 f_j(x) dx \leq C \int |\sigma(x)|^2 dx$ for any $\sigma \in L^2$, whence $\int_{\mathbb{R}^n} |\hat{\sigma}(x)|^2 dx \leq C$ a.e., which concludes the proof.

We can define $H^\mu$ for any positive measure $\mu$ in the similar way as $H^f$ is defined. We remark that in order that $H^\mu$ may be a space of distributions it is necessary for $\mu$ to be absolutely continuous with respect to the ordinary Lebesgue measure. For the proof of this fact, we consider a characteristic function of any compact subset $K$ of $\mathbb{R}^n$ and a pointwise convergent sequence $\{\xi_j\}$, $\xi_j \in \mathcal{D}$, to $\chi$ such that $|\xi_j| \leq 1$ for any $j$ and the supports of $\xi_j$ are contained in a fixed compact subset of $\mathbb{R}^n$. Then $\xi_j \rightarrow \chi$ in $L^2$, as $j \rightarrow \infty$. When $H^\mu$ is a space of distributions, $\{\xi_j\}$ converges to a distribution $T$ in $H^\mu$, and a fortiori in $\mathcal{D}'$. Then for any $\varphi \in \mathcal{D}$ we have

$$<T, \varphi> = \lim_j <\xi_j, \varphi> = \lim_j \int_{\mathbb{R}^n} \hat{\xi}_j \hat{\varphi} dx = \int_{\mathbb{R}^n} \hat{\varphi} dx.$$

Hence if $K$ is a null set in the Lebesgue measure, then $\int_K \hat{\varphi} dx = 0$, so that $T=0$. This means that $\int |\chi|^2 d\mu = 0$, and therefore $\mu(K) = 0$. Thus $\mu$ is absolutely continuous with respect to the Lebesgue measure.

2. $H_{f,\infty}$ and $H'_{f,\infty}$. Throughout the following discussions in this paper we shall be concerned only with normal spaces of distributions $H^f$. Then, as we see in the preceding section, $H^{f,s}$ is also a normal space of distributions for any real $s$. Let $H_{f,\infty}$ be the space $\bigcup H^{f,s}$ with the topology of the inductive limit of $\{H^{f,s}\}$, and $H_{f,\infty}$ the space $\bigcap H^{f,s}$ with the topology of the projective limit of $\{H^{f,s}\}$. Clearly $H_{f,\infty}$ also is a normal space of distributions and of type $(F)$. It is easy to see that any bounded subset of $H_{f,\infty}$ is weakly relatively compact, so that $H_{f,\infty}$ is reflexive and the strong anti-dual $(H_{f,\infty})'$ of $H_{f,\infty}$ is a complete bornological, barrelled space. $H_{g,\infty}$, where $g = \frac{1}{f}$, consists of the same elements as the anti-dual $(H_{f,\infty})'$. Both $(H_{f,\infty})'$ and $H_{g,\infty}$ are bornological and their anti-duals coincide with $H_{f,\infty}$. It follows since any bornological space has Mackey topology that $(H_{f,\infty})' = H_{g,\infty}$ also holds topologically. In a similar way we have $(H_{g,\infty})' = H_{f,\infty}$.

We first note that $H_{f,\infty} \subset \mathcal{B}$. Let $m$ be a positive integer such that $\frac{1}{(1 + |x|^2)^m \sqrt{f(x)}} \in L^2$. Consider any element $\xi$ of $H_{f,\infty}$. By definition we have $\xi \in H^{f,2l}$ for any integer $l$, that is, $\hat{\xi}(x)(1 + |x|^2)^l \in L^1$, so that $(1 - \Delta)^{-m} \xi$ is a continuous function tending to $0$ at infinity. This implies that $\hat{\xi}$ is an element of $\mathcal{B}$.
An element \( \xi \in H^f \) belongs to \( H_{f,\infty} \) if and only if \( D^p \xi \in H^f \) for every \( p \). This is clear from the definition of \( H_{f,\infty} \). An element \( \xi \in H_{f,\infty} \) belongs to \( H_{f,\infty} \) if and only if there exists a bounded subset \( B \) of \( H_{f,\infty} \) such that every \( D^p \xi \) is absorbed by \( B \), that is, \( \xi \) is a distinguished element of \( H_{f,\infty} \) [6]. In fact, necessity is evident. Sufficiency follows from the fact that any bounded subset \( B \) is contained in an \( H^{p,\infty} \).

A distribution \( \xi \) belongs to \( H_{f,\infty} \) if and only if \( \xi \star \varphi \in H^f \) (or \( H''_{f,\infty} \)) for any \( \varphi \in \mathcal{D} \). This is shown by means of a parametrix of an iterated Laplacian as in the preceding section.

**Proposition 5.** Let \( \mathcal{K} \) be a space of distributions contained in \( H'_{f,\infty} \). If \( \mathcal{K} \) is of type \((F)\) and closed for differentiation, then \( \mathcal{K} \subset H_{f,\infty} \). Thus \( H_{f,\infty} \) is the maximal one among such \( \mathcal{K} \).

**Proof.** Let \( \xi \in \mathcal{K} \). As \( \mathcal{K} \) is of type \((F)\), there exists a bounded subset \( B \) by which each \( D^p \xi \) is absorbed. By the closed graph theorem, the injection \( \mathcal{K} \to H'_{f,\infty} \) is continuous, so that \( B \) also is bounded in \( H'_{f,\infty} \), which implies \( \xi \in H_{f,\infty} \). The proof is complete.

**Proposition 6.** The following conditions are equivalent to each other:

(i) \( H_{f,1,\infty} \subset H_{f,2,\infty} \).

(ii) \( H'_{f,1,\infty} \subset H'_{f,2,\infty} \).

(iii) There exist a constant \( C \) and an integer \( l \) such that

\[
\frac{f_2}{f_1} \leq C(1 + |x|^2)^l \quad \text{a.e.}
\]

**Proof.** Ad (i) \( \to \) (ii). For any \( \xi \in H_{f,1,\infty} \), we have that \( \xi \star \varphi \in H_{f,2,\infty} \subset H'_{f,s,\infty} \), whence \( \xi \in H'_{f,s,\infty} \).

Ad (ii) \( \to \) (iii). (ii) implies that \( H^{1,s} \subset H'_{f,s,\infty} \) for some \( s \) ([2], Théorème A p. 16). Consequently, by Proposition 4, we have (iii).

(iii) \( \to \) (i) follows from Proposition 4.

Thus the proof is complete.

As an immediate consequence of Proposition 6 we have

**Corollary.** The following conditions are equivalent to each other:

(M) \( f, \frac{1}{f} \leq C(1 + |x|^2)^l \) a.e. for some constant \( C \) and an integer \( l \).

(i) \( H_{f,\infty} = \mathcal{D}_L^s \).

(ii) \( H'_{f,\infty} = \mathcal{D}'_L^s \).

(iii) \( H_{f,\infty} = H^{1,f,\infty} \).

(iv) \( H'_{f,\infty} = H'_{1,f,\infty} \).

We say that \( f \) is of Malgrange type (simply type \( M \)) if it satisfies the condition (M). Malgrange called a continuous function of type \( M \) “fonction-poids” ([5], p. 284).
3. Spaces of local type. A space of distributions $\mathcal{H}$ is said to be of local type if $\mathcal{D} \subset \mathcal{H}$. $H^f$ is not necessarily of local type even when $f$ is of type $M$. Let $\xi \in H^f$. Setting $\sigma = \xi \sqrt{f} \in L^2$, we have for any $\alpha \in \mathcal{D}$

$$\int |\widehat{\alpha \xi}(x)|^2 f(x) dx = \int |\alpha(y) (1 + |y|^2)^m \frac{\sigma(x-y)}{(1 + |y|^2)^m} \sqrt{\frac{f(x)}{f(x-y)}} dy|^2 dx.$$  

We first show

**Proposition 7.** $H^f$ is of local type if and only if, for some integer $m$,

$$\frac{1}{(1 + |x-y|^2)^m} \sqrt{\frac{f(x)}{f(y)}}$$

is a kernel of continuous linear application of $L^2_y$ into $L^2_x$.

**Proof.** Sufficiency. Let $\xi \in H^f$ and $\alpha \in \mathcal{D}$. As $\alpha(y) (1 + |y|^2)^m$ is bounded, it follows from (1) that there is a constant $C$ such that

$$\int |\widehat{\alpha \xi}(x)|^2 f(x) dx \leq C \int \frac{1}{(1 + |x-y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \sigma(y) dy dx < +\infty,$$

since, by hypothesis, the linear operator generated by the kernel $\frac{1}{(1 + |x-y|^2)^m} \sqrt{\frac{f(x)}{f(y)}}$ is continuous on $L^2$.

Necessity. Let $\xi \in H^f$, $\eta \in H^{1/f}$ and $\alpha \in \mathcal{D}$. We first assume that $\hat{\xi}$, $\hat{\eta}$, and $\hat{\alpha}$ are non-negative. As $H^f$ is of local type, we have $\alpha \xi \in H^f$, so that $(\widehat{\alpha \xi}) \hat{\eta} \in L^1$, i.e. $(\hat{\alpha} \hat{\xi}) \hat{\eta} \in L^1$. Then, by Fubini theorem,

$$\int (\hat{\alpha} \hat{\xi}) \hat{\eta} dx = \int (\hat{\alpha} \hat{\xi} \hat{\eta}) dx,$$

which implies that $\hat{\alpha} (\hat{\xi} * \hat{\eta}) \in L^1$, and hence for any $\beta \in \mathcal{D}$

$$|\hat{\beta}|^2 (\hat{\xi} * \hat{\eta}) \in L^1.$$

For any $\alpha$, $\beta \in \mathcal{D}$, on account of the inequality $|\hat{\alpha} \hat{\beta}| \leq |\hat{\alpha}|^2 + |\hat{\beta}|^2$ it follows from the above relation that

$$\hat{\alpha} \hat{\beta} (\hat{\xi} * \hat{\eta}) \in L^1,$$

whence by making use of a parametrix of an iterated Laplacian as in Section 1 we have

$$\hat{\xi} * \hat{\eta} \in (1 + |x|^2)^m \times L^1$$

for some integer $m \geq 0$.

Let $\xi$ (resp. $\eta$) be any element of $H^f$ (resp. $H^{1/f}$). Then $\mathcal{F}^{-1}(|\hat{\xi}|) \in H^f$ and $\mathcal{F}^{-1}(|\hat{\eta}|) \in H^{1/f}$. It follows that

$$\hat{\xi} * \hat{\eta} \in (1 + |x|^2)^m \times L^1$$

for some integer $m \geq 0$, where $m$ may depend on $\xi$ and $\eta$. But we can show that $m$ may be chosen independent of $\xi$ and $\eta$. In fact, the application $(\hat{\xi}, \hat{\eta}) \rightarrow \hat{\xi} * \hat{\eta}$ of $L^2_y \times L^2_{1/f} \rightarrow L^2_x$ into $\mathcal{D}'$ is continuous, each $(1 + |x|^2)^m \times L^1$, $m = 1, 2, \ldots$, is a Banach space, and the
injection \((1 + |x|^2)^m \times L^1 \to \mathcal{D}'\) is continuous, whence, by a theorem of Yoshinaga-Ogata ([14], p. 16), we can choose \(m\) as desired.

By a change of variables we have

\[
\frac{\hat{\xi}(x) \hat{\eta}(y)}{(1 + |x-y|^2)^m} \in L^1 \quad \text{for any } \xi \in H^f \text{ and any } \eta \in H^{1/f},
\]

from which, by setting \(\sigma = \hat{\xi} \sqrt{f} \in L^2\) and \(\tau = \frac{1}{\sqrt{f}} \in L^2\) we have

\[
\int \int \frac{1}{(1 + |x-y|^2)^m} \sqrt{f(y)} \frac{f(x)}{f(x)} |\sigma(x)| |\tau(y)| dx dy < + \infty \quad \text{for any } \sigma, \tau \in L^2,
\]

which concludes the proof.

**Remark.** (i) From the proof of Proposition 7 it is clear that for any \(\xi \in H^f\), \(\eta \in H^{1/f}\) and \(\alpha \in \mathcal{S}\), if \(H^f\) is of local type, we have

\[
<\alpha \xi, \eta> = \int \alpha(\xi^* \eta) dx,
\]

where \(\xi^* \eta \in (1 + |x|^2)^m \times L^1\) for an integer \(m\) independent of \(\xi\) and \(\eta\). As a consequence we see that if \(H^f\) is of local type, then \(\mathcal{S} H^f \subset H^f\), and the equation (3) also holds for any \(\alpha \in \mathcal{S}\). Consequently we can define multiplicative product \(\xi \eta\) for any \(\xi \in H^f\) and \(\eta \in H^{1/f}\) in the sense of [3]. In fact, \(\hat{\xi}, \hat{\eta}\) have \(\mathcal{S}\)-convolution, since \((\hat{\xi} \hat{\eta}) \in L^1 \subset \mathcal{D}'\) for any \(\phi \in \mathcal{S}\) ([12], p. 151).

(ii) Owing to the relation (2), \(H^f\) is of local type if and only if there exists a positive integer \(m\) such that \((1 + |x|^2)^{-m} \xi \in L^1_{\beta ds}\) for any \(\xi \in L^1_{\beta ds}\), i.e., \(L^1_{\beta ds} \in H^f\) for any \(\xi \in H^f\), where \(L^1_{\beta} \) denotes the Fourier transform of \((1 + |x|^2)^{-m}\) ([8], p. 116).

(iii) Let \(\{\beta_{\varepsilon}\}_{0 < \varepsilon < 1}\) be a family of functions of \(\mathcal{D}\) such that the support of \(\beta_{\varepsilon}\) is contained in \(B_{\varepsilon} = \{x : |x| \leq \varepsilon\}, \beta_{\varepsilon} \geq 0\), and \(\int \beta_{\varepsilon}(x) dx = 1\). Further we assume that \(\beta_{\varepsilon} \leq \frac{M}{\varepsilon^n}, \left| \frac{\partial \beta_{\varepsilon}}{\partial x_j} \right| \leq \frac{M}{\varepsilon^{n+1}}\) for some constant \(M\). Schwartz ([11], p. 28) has shown that the following inequality holds for some constant \(C\):

\[
|\beta_{\varepsilon}(x - y) - \beta_{\varepsilon}(x)| (1 + |y|^2)^{\frac{1}{2}} \leq C(1 + |y|^2)^{\frac{1}{2}},
\]

Then, using this inequality and noting that the application \(\hat{\xi} \rightarrow (1 + |x|^2)^{-m} \hat{\xi}\) of \(L^1_{\beta ds}\) into itself is continuous for large \(m\), we can show that Friedrichs' lemma ([11], p. 27) holds: Let \(H^f\) be of local type, then, for any \(\xi \in H^f\), \(\beta_{\varepsilon}(\alpha \xi) - \alpha(\beta_{\varepsilon} \xi)\) tends to zero in \(H^{1/f}\) as \(\varepsilon \rightarrow 0\), where \(\alpha\) is any element of \(\mathcal{S}\).

It is easy to see from Proposition 7 that \(H^{1/f}\) is of local type if so is \(H^f\).

**Corollary 1.** If \(H^f\) is of local type, then so is \(H^{1/f}\) for any real \(s\).

**Proof.** Setting \(h(x) = (1 + |x|^2)^s f(x)\), and using the inequality \((1 + |x|^2)^s \leq C(1 + |y|^2)^s (1 + |x-y|^2)^{\frac{1}{2}}\), \(C\) being a constant, we have
On a Space $H^f$

\[ \sqrt{h(x)} \leq C \sqrt{\frac{f(x)}{f(y)}} (1 + |x - y|^2)^{1/2}, \]

whence we can choose an integer $m$ such that

\[ \frac{1}{(1 + |x - y|^2)^m} \sqrt{h(x)} \] is a kernel of a continuous linear operator in $L^2$.

**Corollary 2.** If $H^{f_1}$ and $H^{f_2}$ is of local type, then $H^f$, where $f = f_1^{1-\theta} f_2^\theta$ and $0 < \theta < 1$, also is of local type.

**Proof.** It follows from the inequality

\[ \sqrt{\frac{f(x)}{f(y)}} \leq (1 - \theta) \sqrt{\frac{f_1(x)}{f_1(y)}} + \theta \sqrt{\frac{f_2(x)}{f_2(y)}}. \]

We shall give a sufficient condition for $H^f$ to be of local type.

**Proposition 8.** $H^f$ is of local type if the following condition is satisfied:

\[ f(x + y) \leq k(x)f(y) \quad \text{a.e. for } |y| \geq c, \]

where $k(x) \in (1 + |x|^2)^l \times L^1$, $l$ being an integer, and $c$ is a constant.

**Proof.** If we put $k(x) = (1 + |x|^2)^l h(x)$, then $h \in L^1$. As $f(x + y) \leq k(x)f(y)$, we have

\[ \frac{f(x)}{f(y)} \leq (1 + |x - y|^2)^l h(x - y) \quad \text{a.e. for } |y| \geq c. \]

Choose a positive integer $m$ such that

\[ \frac{\sqrt{f(x)}}{(1 + |x|^2)^m} \in L^2 \quad \text{and} \quad \frac{\sqrt{h(x)}}{(1 + |x|^2)^m-1/2} \in L^1. \]

Then, by our hypothesis, we have

\[ \frac{1}{(1 + |x - y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \leq \frac{\sqrt{h(x - y)}}{(1 + |x - y|^2)^m-1/2} \quad \text{a.e. for } |y| \geq c. \]

Now using the inequality $(1 + |x - y|^2)^{-m} \leq C_1(1 + |y|^2)^m(1 + |x|^2)^{-m}$, where $C_1$ is a constant, we have, for any $\sigma \in L^2$,

\[ I = \int_{|y| \leq c} \left| \int_{|y| \leq c} \frac{1}{(1 + |x - y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \sigma(x) dx \right|^2 dy \]

\[ = \int_{|y| \leq c} \left| \int_{|x| \leq c} \frac{1}{(1 + |x - y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \sigma(x) dx \right|^2 dy \]

\[ + \int_{|y| \geq c} \left| \int_{|x| \leq c} \frac{1}{(1 + |x - y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \sigma(x) dx \right|^2 dy = I + J, \]

but

\[ I \leq C_1 \int_{|y| \leq c} \left( \frac{1 + |y|^2}{f(y)} \right)^{2m} \sigma(y) dy \left( \frac{\sqrt{f(x)}}{(1 + |x|^2)^m} \sigma(x) dx \right)^2 \]
\[ \sum \leq C_2 \left( \frac{\sqrt{f(x)}}{1 + |x|^2} \right)^m \sum \leq \| \tilde{f} \|_{\tilde{L}^2}^2 \]

and

\[ J \leq \left\{ \left\{ \frac{\sqrt{h(x-y)}}{(1 + |x-y|)^{\gamma-m+i/2}} |\sigma(x)\, dx \right\}^2 \right\}^2 \sum \leq C_3 \left( \frac{\sqrt{h(x)}}{(1 + |x|^2)^{\gamma-m+i/2}} \right)^2 \sum \leq \| \tilde{f} \|_{\tilde{L}^2}^2, \]

where \( C_2, C_3 \) are some constants. It follows that

\[ I + J \leq C_4 \left( \frac{\sqrt{f(x)}}{(1 + |x-y|^2)^m} \right)^2 \sum + \left( \frac{\sqrt{h(x)}}{(1 + |x|^2)^{\gamma-m+i/2}} \right)^2 \sum \leq \| \tilde{f} \|_{\tilde{L}^2}^2, \]

where \( C_4 \) is a constant. This yields that

\[ \frac{1}{(1 + |x-y|^2)^m} \sqrt{\frac{f(x)}{f(y)}} \]

is a kernel of a continuous linear application \( L_2^2 \rightarrow L_2^2 \), so that, by Proposition 7, the space \( H^f \) is of local type. The proof is complete.

**Example.** If \( f(x) = |x|^\lambda, 0 < \lambda < n \), then the space \( H^f \) is of local type. Indeed, since the inequalities \( |x+y|^\lambda \leq 2^\lambda (|x|^{\lambda} + |y|^{\lambda}) \leq 2^\lambda |y|^{\lambda} (1 + |x|^{\lambda}) \) hold for \( |y| \geq 1 \), it follows by setting \( k(x) = 2^\lambda (1 + |x|^{\lambda}) \), that \( f(x+y) \leq k(x)f(y) \) and \( k(x) = 2^\lambda (1 + |x|^{\lambda}) (1 + |x|^{\lambda})^{-m} \times (1 + |x|^{\lambda}) \in (1 + |x|^{\lambda})^{-m} L_1 \), and hence \( H^f \) is of local type.

**Remark.** Consider the condition (essentially due to Malgrange ([5], p. 289)):

\[ (M') \quad f(x+y) \leq C(1 + |x|^{2^m}) f(y) \quad \text{a.e.}, \]

where \( C \) is a constant and \( m \) is a positive integer.

If \( f \) satisfies \( (M') \), the equation (1) gives

\[ \|\alpha \xi\|_2^2 \leq C \|\sigma\|_{\tilde{L}^2} \left\{ \int |\hat{\alpha}(x)| (1 + |x|^{2^m}) \, dx \right\}^2. \]

Let \( \alpha_j(x) = \alpha \left( \frac{x}{j} \right) \), and suppose that \( \alpha \) is 1 near the origin. \( \{\alpha_j(x)\} \) is a sequence of multiplicators. \( \hat{\alpha}_j(x) = j^n \hat{\alpha}(jx) \).

Hence

\[ \int \left| \hat{\alpha}(x) \right| (1 + |x|^{2^m}) \, dx = \int \left| \hat{\alpha}(x) \right| (1 + \left| \frac{x}{j} \right|^{2^m}) \, dx, \]

whence \( \{\|\alpha_j \xi\|_j\} \) is a bounded sequence. Then, by a theorem of Banach-Steinhaus, we see that \( H^f \) has the approximation property by truncation, i.e. \( \alpha_j \xi \rightarrow \xi \) uniformly in \( H^f \) when \( \xi \) runs through any compact subset of \( H^f \). On the other hand, the approximation property by regularization is possessed by any \( H^f \).

**Proposition 9.** \( H_{L,m} \) and \( H_{F,m} \) are of local type.
To complete the proof of this proposition it is enough to establish the following proposition.

**Proposition 10.** There exists a real number $s_0$ such that $\mathcal{D} H^f \subset H^f \cdot \subset \mathcal{D} \cdot$ for every $s$. But $s_0$ may depend on $f$.

**Proof.** First we note that $\mathcal{D} H^f \subset H^f \cdot$ if and only if there exists an integer $m$ such that

$$
\frac{(1 + |x|^2)^{s/2}}{(1 + |x-y|^2)^m} \frac{f(x)}{f(y)}
$$

is a kernel of a continuous linear application of $L^2_{\mathcal{D}}$ in $L^2_{\mathcal{D}}$. The proof is very similar to that of Proposition 7 and will not be supplied here.

As $H^f$ is normal, there exists an integer $l$ such that $\frac{1}{f} \in (1 + |x|^2)^{2l} \times L^1$. Setting $s_0 = -4l$, $m = \left| \frac{s}{2} - l \right|$, we have

$$
\frac{(1 + |x|^2)^{s/2}}{(1 + |x-y|^2)^m} \frac{f(x)}{f(y)} \leq C \frac{\sqrt{f(x)}}{(1 + |x|^2)^l} \frac{1}{(1 + |y|^2)^{l/2}}
$$

where $C$ is a constant such that $(1 + |x|^2)^{l/2} \leq C(1 + |x-y|^2)^l (1 + |y|^2)^{l/2}$. The right hand side of the above inequality is clearly a kernel of a continuous linear operator in $L^2_f$, which concludes the proof.

**Remark.** $\mathcal{D} H_{f,\infty} \subset H_{f,\infty}$, $\mathcal{D} H_{f,\infty} \subset H_{f,\infty}$. In fact, $\mathcal{D} H^f \subset H^f \cdot \subset \mathcal{D} \cdot$ implies $\mathcal{D} H^f \subset H^f \cdot \subset \mathcal{D} \cdot$. This can be shown as in Remark (i) after Proposition 7.

**Proposition 11.** Let $\mathcal{D} H^f \subset H^f$. Then, setting $g = \frac{1}{f}$, we have

(i) $\xi \eta \in \mathcal{D} L^1$ for every $\xi \in H^f$ and $\eta \in H^f$,

(ii) $\mathcal{D} H^f \subset H^f$,

(iii) $\mathcal{D} H^{f,s} \subset H^{f,s}$ for any real $s$,

(iv) $\mathcal{D} H^{f,s} \subset H^{f,s}$ for any real $s$.

**Proof.** Ad (i). Let $\xi$ (resp. $\eta$) be any element of $H^f$ (resp. $H^f$). For any $\alpha \in \mathcal{D}$, we have

$$
<\alpha \xi, \eta> = \int (\alpha \ast \xi) \eta dx = \int (\hat{\alpha} \hat{\xi})^* \eta dx = <\alpha, \xi \eta>.
$$

Since the application $\beta \mapsto \beta \xi$ of $\mathcal{D}$ into $H^f$ is continuous by the closed graph theorem, the above relations show that $<\alpha, \xi \eta>$ is a continuous form of $\alpha \in \mathcal{D}$ even when we impose on $\mathcal{D}$ the topology of $\mathcal{D}$. Since $\mathcal{D} L^1$ is the dual space of $\mathcal{D}$, it follows that $\xi \eta \in \mathcal{D} L^1$.

Ad (ii). Let $\gamma$ be any element of $\mathcal{D}$. Let $\{\alpha_k\}$ be a sequence from $\mathcal{D}$ with $\alpha_k \gamma \to \gamma$ in $\mathcal{D}$. Then, as $\alpha_k \gamma \in \mathcal{D}$, we have $<\alpha_k \gamma \xi, \eta> = <\alpha_k \gamma, \xi \eta>$. Therefore it follows since $\xi \eta \in \mathcal{D} L^1$, that $\{\alpha_k \gamma \xi\}$ converges weakly to a $\xi' \in H^f$ and $<\xi, \eta> = <\gamma, \xi \eta>$. On the other hand, $\alpha_k \gamma \xi \to \gamma \xi$ in $\mathcal{D}$, which implies $\xi' = \gamma \xi$. Thus
we have $\mathcal{B}H^f \subset H^f$.

Ad (iii). We first consider the case $0 < s < 1$. For any $\xi \in H^f_{s'}$ we put
\[ \|\xi\|_{f, s} = \left\{ \int \frac{\|\xi_a - \xi\|_f^2}{|a|^{n + 2s}} \, da \right\}^{\frac{1}{2}}, \]
where $\xi_a(x) = \xi(x + a)$.

As in J. Peetre ([7], p.17), we have after some calculations
\[ \|\xi\|_{f, s} = J(s) \left\{ \int |y|^{2s} |\xi(y)|^2 f(y) \, dy \right\}^{\frac{1}{2}}, \]
where $J(s)$ is a constant depending only on $s$. Therefore in $H^f_{s'}$ the norm $\|\cdot\|_{f, s}$ is equivalent to $\{\|\cdot\|^2_{f} + \|\cdot\|^2_{f, s}\}^{\frac{1}{2}}$. Let $\beta$ be any element of $\mathcal{B}$. Then for any element $\xi$ of $H^f_{s'}$ we have
\[ \|\beta \xi\|_{f, s}^2 = \int \frac{\|\beta_a \xi_a - \beta \xi\|^2}{|a|^{n + 2s}} \, da \]
\[ \leq 2 \int \frac{\|\beta_a \xi_a - \xi\|^2}{|a|^{n + 2s}} \, da + 2 \int \frac{\|\beta_a - \beta\|_f^2}{|a|^{n + 2s}} \, da \]
\[ = I_1 + I_2. \]

Now, as the application $(\beta, \xi) \rightarrow \beta \xi$ of $\mathcal{B} \times H^f$ into $H^f$ is continuous by the closed graph theorem, there exists a constant $C$ such that
\[ \|\beta_a (\xi_a - \xi)\|^2 \leq C \|\xi_a - \xi\|^2 \]
and
\[ \|\beta_a - \beta\|_f^2 \leq C \min \{|a|^2, 1\} \|\xi\|^2, \]
since $\{\beta_a - \beta\}$ is bounded in $\mathcal{B}$ and we can write $\beta_a - \beta = \sum_{i=1}^{\eta} \gamma_{i,a}$ with bounded $\gamma_{i,a} \in \mathcal{B}$. Hence we have
\[ I_1 \leq C \int \frac{\|\xi_a - \xi\|^2}{|a|^{n + 2s}} \, da = C \|\xi\|_{f, s}^2 < +\infty, \]
and
\[ I_2 \leq C \left\{ \int_{|a| \leq 1} \frac{da}{|a|^{n + 2s - 2}} + \int_{|a| \leq 1} \frac{da}{|a|^{n + 2s}} \right\} \|\xi\|^2 < +\infty. \]

On account of these inequalities we see that $\|\beta \xi\|_{f, s} < +\infty$. We also have that $\|\xi\|_{f} \leq \|\xi\|_{f, s} < +\infty$. Hence $\|\beta \xi\|_f^2 + \|\beta \xi\|_{f, s}^2 < +\infty$. From the remark just given with respect to the equivalent norms of $H^f_{s'}$, we see that $\beta \xi \in H^f_{s'}$ for any $\xi \in H^f_{s'}$.

Next consider the case $s > 0$. We choose a positive integer $N$ such that $0 < \frac{s}{N} < 1$. Then repeating the above process $N$-times we can conclude that
$\beta \xi \in H^{t,s}$ for any $\xi \in H^{t,s}$.

Finally consider the case $s < 0$. As $H^s$ is the anti-dual of $H^t$, then the adjoint application of $\xi \mapsto \beta \xi$ of $H^t$ into $H^t$ yields $\mathcal{A} H^s \subset H^t$. Then, from the preceding discussions, we have $\mathcal{A} H^{t,-s} \subset H^{t,-s}$, and therefore $\mathcal{A} H^{t,s} \subset H^{t,s}$.

Ad (iv). From (iii) we have $\mathcal{A} H^{t,s} \subset H^{t,s}$ for any real $s$. Then, by considering the adjoint application as in the proof of (iii), we see that $\mathcal{A} H^{t,s} \subset H^{t,s}$ for any real $s$.

The proof is complete.

Remark. The proof of (iii) can also be carried out by the aid of the interpolation theorem (e.g. [4]). As clear from the proof of the case (iii), it suffices to show that $\mathcal{A} H^{t,s} \subset H^{t,s}$ for any positive $s$. For any temperate distribution $\xi$, $\xi \in H^{t,1}$ is equivalent to that $\xi$, $\frac{\partial \xi}{\partial x_1}, \ldots, \frac{\partial \xi}{\partial x_n} \in H^t$. Suppose that $\mathcal{A} H^t \subset H^t$, for any $\beta \in \mathcal{A}$ and any $\xi \in H^{t,1}$ we have $\frac{\partial}{\partial x_j} (\beta \xi) = \frac{\partial \beta}{\partial x_j} \xi + \beta \frac{\partial \xi}{\partial x_j} \in H^t$, so that $\beta \xi \in H^{t,1}$. By repeating this process we see that if $\mathcal{A} H^t \subset H^t$, then $\mathcal{A} H^{t,m} \subset H^{t,m}$ for any positive integer $m$. Now we can make use of the interpolation theorem cited above to conclude our assertion.

4. Convolution. We shall first recall the definition of convolution concerning two distributions $S$, $T$. We shall say that $S$, $T$ are composable provided

$$S(T^* \phi) \in \mathcal{D}', \text{ for every } \phi \in \mathcal{D}. \quad (1)$$

If this is the case, the convolution $S * T$ is defined by the equation

$$\langle S * T, \phi \rangle = \int S(T^* \phi) dx.$$ 

This is the usual convolution due to L. Schwartz [9]. Various conditions equivalent to (1) have been discussed by Shiraishi [12]. However, when convolution is considered as an application, another definition is possible. Let $\mathcal{K}$ and $\mathcal{K}'$ be normal spaces of distributions and let $\mathcal{L}$ be a space of distributions. We shall follow Schwartz ([10], p.151) in saying that a bilinear application of $\mathcal{K} \times \mathcal{K}'$ into $\mathcal{L}$ is a convolution of $\mathcal{K} \times \mathcal{K}'$ into $\mathcal{L}$ if the application is separately continuous and coincides with the usual convolution on $\mathcal{D} \times \mathcal{D}$. For our temporary purpose such convolution will be denoted by $\otimes$ and we shall take $\mathcal{L}$ for $\mathcal{D}'$, in the following discussions.

If we are given a subset $E$ of $\mathcal{D}'$, we shall denote by $E^*$ the set of the distributions composable with every element of $E$. It follows from (1) that $E^*$ is a linear space stable for differentiation, and $\mathcal{A} E^* \subset E^*$. In the following we shall write $g = \frac{1}{f}$.

Proposition 12. (i) $(H^{t})^* = (H_{f,\omega})^* = (H_{f,\omega})^*$. (ii) $(H^{t})^* \subset H_{e,\omega}^*$. 


PROOF. (i) is clear from the fact that $S$, $T$ are composable if and only if $S \ast \varphi$, $T$ are composable for any $\varphi \in \mathcal{D}$. As for (ii), let $\eta$ be any element of $(H')^*$. Then by (1) we have $\xi(\bar{\eta} \ast \varphi) \in \mathcal{D}_{1,1}$ for every $\xi \in H'$ and every $\varphi \in \mathcal{D}$. Hence $\bar{\eta} \ast \varphi \in H'$ since $\xi \mapsto \xi(\bar{\eta} \ast \varphi)$ is a continuous application of $H'$ into $\mathcal{D}_{1,1}$, so that $\bar{\eta} \in H_{g, \infty}$ and in turn $\eta \in H_{g, \infty}$, as desired.

Now we shall show

**Theorem 2.** $f$ is of type $M$ if and only if any of the following equivalent conditions holds:

(i) $(H')^* \supset H^f$.

(ii) $(H')^* = H_{g, \infty}$.

(iii) $\mathcal{B} H_{f, \infty} \subseteq H_{f, \infty}$.

(iv) $\mathcal{B} H_{g, \infty} \subseteq H_{g, \infty}$.

(v) $\mathcal{B} H^f \subseteq H_{f, s_0}$ for some real $s_0$.

(v') $\mathcal{B} H^g \subseteq H_{g, s_0}$ for some real $s_0$.

**Proof.** Ad (i) $\rightarrow$ (ii). This follows from the fact that $\eta$ lies in $H_{g, \infty}$ if and only if $\eta \ast \varphi \in H^f$ for any $\varphi \in \mathcal{D}$.

Ad (ii) $\rightarrow$ (iii). This is clear, because, for any $E \subseteq \mathcal{D}'$, $E^*$ is stable for multiplication by any element of $\mathcal{B}$.

Ad (iii) $\supseteq$ (iv). This equivalence is obtained by considering the adjoint application of the multiplications by elements of $\mathcal{B}$.

Ad (iii) $\supseteq$ (v). (iii) implies that $\mathcal{B} H^f \subseteq H_{g, \infty}$. As $\mathcal{B}$, $H^f$ are spaces of type $(F)$ and $H_{g, \infty}$ is a space of type $(LF)$, so we have $\mathcal{B} H^f \subseteq H_{g, s_0}$ for some real $s_0$.

Assume that (v) holds. By the closed graph theorem the application $(\beta, \xi) \mapsto \beta \xi$ of $\mathcal{B} \times H^f$ into $H_{g, s_0}$ is continuous, and therefore there exist a constant $C_1$ and a positive integer $m$ such that for any $\xi \in H^g$

$$\|\beta \xi\|_{s_0, s_0} \leq C_1 \|\xi\|_g^2 \max\|D^p \beta\|_{L^\infty}, \ |p| \leq 2m.$$

Consider the set $\Phi$ of the functions $\left\{\frac{e^{2\pi ixt}}{(1 + |t|^m)^m} \right\}$, where $t$ is a parameter running through $\mathbb{R}$.

Then the set of functions

$$\left\{ D^p \frac{e^{2\pi ixt}}{(1 + |t|^m)^m}; |p| \leq 2m, \ t \in \mathbb{R} \right\}$$

is uniformly bounded, whence for a constant $C_2$

$$\| \frac{e^{2\pi ixt}}{(1 + |t|^m)^m} \xi \|_{s_0, s_0}^2 \leq C_2 \|\xi\|_g^2$$
for any $\xi \in H^s$ and any $t \in \mathbb{R}^n$.

Consequently,

$$\int \frac{|\frac{\partial^s}{\partial x^s}(x-t)|^2(1+|x|^2)^{2m}g(x)}{(1+|t|^2)^{2m}}dx \leq C_2 \int |\frac{\partial^s}{\partial x^s}(x)|^2g(x)dx,$$

which implies for every $t \in \mathbb{R}^n$

$$\frac{(1+|x+t|^2)^{2m}g(x+t)}{(1+|t|^2)^{2m}g(x)} \leq C_2 \quad \text{a.e.}$$

Therefore for any $x_0$ with $g(x_0) \neq 0$, $\infty$, we have

$$g(x_0+t) \leq Cg(x_0)\frac{(1+|t|^2)^{2m}}{(1+|x_0+t|^2)^{2m}} \quad \text{a.e.}$$

If we put $x=x_0+t$, then for some constant $C'$ and a positive integer $l'$

$$g(x) \leq C'(1+|x|^2)^{l'} \quad \text{a.e.}$$

As $\mathcal{S}H^s \subset H^{s+s_0}$, then $\mathcal{B}H^{l''-s_0} \subset H^{l'}$. By repeating a similar reasoning as above, we have

$$f(x) \leq C''(1+|x|^2)^{l''} \quad \text{a.e.},$$

for some constant $C''$ and a positive integer $l''$. Thus we see that $f$ is of type $M$.

If $f$ is of type $M$, then $H_{f,\infty} = H_{g,\infty} = \mathcal{D}'_{L^2}$, and $H'_{f,\infty} = H'_{g,\infty} = \mathcal{D}'_{L^2}$, so that $(H')^* = (\mathcal{D}'_{L^2})^* = \mathcal{D}'_{L^2} = H'_{g,\infty} \supset H^s_{f,\infty}$.

Now, by definition, $f$ is of type $M$ if and only if $g$ is of type $M$. Hence the substitution of $f$ by $g$ in the above discussions will complete the proof of the theorem.

**Remark.** The condition (v) of the theorem implies that $\mathcal{B}H^{l'} \subset H^{l'+s_0}$ for every real $s$. This can be shown by an interpolation theorem as indicated in the remark after Proposition 11. In general, $s_0$ cannot be chosen to be zero. For, suppose the contrary. Every $H^l$, $f$ being of type $M$, would be of local type. However, this is not the case.

**Proposition 13.** $(H')^* \supset H^s$ if and only if there exist a constant $C$ and a real $s_0$ such that

$$f(x) \geq C(1+|x|^2)^{s_0} \quad \text{a.e.}$$

**Proof. Necessity.** $H^l \subset (H')^* \subset H_{g,\infty}$. Hence $H_{f,\infty} \subset H_{g,\infty}$ by Proposition 5, and we obtain by Proposition 6

$$\frac{g(x)}{f(x)} \leq C_1(1+|x|^2)^s \quad \text{a.e.}$$

for a constant $C_1$ and a real $s$. Thus we have (2).

**Sufficiency.** (2) implies $H^f \subset H^0 \subset \mathcal{D}'_{L^2}$. Since any two distributions of $\mathcal{D}'_{L^2}$
are composable, so we have \( H' \subset (H')^* \), and our proof is complete.

**Corollary.** \( (H')^* = H'_{f,\infty} \) if and only if \( f \) is of type \( M \).

**Proof.** It is enough to show the "only if" part. By Proposition 13 we have \( f \geq C(1+|x|^3)^{s_0} \) for a real \( s_0 \), hence by Proposition 6 \( H' \subset H'_{f,\infty} \subset \mathcal{D}'_{L^1} \), which implies that \( H'_{f,\infty} = (H')^* \cap (\mathcal{D}'_{L^1})^* \). Consequently we have \( H_{f,\infty} = \mathcal{D}'_{L^1} \). Then, by the Corollary to Proposition 6, we see that \( f \) is of type \( M \).

If we define \( \xi \odot \eta = \mathcal{F}^{-1}(\xi \hat{\eta}) \), where \( \xi \in H'_{f,\infty} \) and \( \eta \in H'_{g,\infty} \), then it is not difficult to see that \( \odot \) is a convolution of \( H'_{f,\infty} \times H'_{g,\infty} \) into \( \mathcal{D}' \). However, as Theorem 2 shows, the application \( \odot \) coincides with the usual convolution \( * \) if and only if \( f \) is of type \( M \).

Finally we shall conclude this section by stating a sufficient condition for a convolution of \( \mathcal{H} \times \mathcal{H} \) into \( \mathcal{D}' \) to be well defined, which will also be applied to the case where \( \mathcal{H} = H'_{f,\infty} \) and \( \mathcal{X} = H'_{g,\infty} \).

**Proposition 14.** Let \( \mathcal{H}, \mathcal{X} \) be normal spaces of distributions. Let \( \mathcal{H} \) be barrelled. Assume that the application \( (T,\varphi) \rightarrow T^* \varphi \) of \( \mathcal{H} \times \mathcal{D} \) into \( \mathcal{H}' \) (the strong dual of \( \mathcal{H} \)) is hypocontinuous. Then the application \( \odot \) defined by the following relation is a convolution of \( \mathcal{H} \times \mathcal{X} \) into \( \mathcal{D}' \):

\[
<S \odot T, \varphi> = <S, T^* \varphi>, \quad S \in \mathcal{H}, \quad T \in \mathcal{X} \quad \text{and} \quad \varphi \in \mathcal{D}.
\]

Furthermore if \( \mathcal{H} \) possesses the approximation property by truncation, then \( S \ast T, \) if it exists, coincides with \( S \odot T \).

**Proof.** It is evident that \( \odot \) coincides with \( * \) on \( \mathcal{D} \times \mathcal{D} \). Let \( C \) be any compact disk of \( \mathcal{D} \). If \( T \rightarrow 0 \) in \( \mathcal{H} \), then \( T^* C \rightarrow 0 \) in \( \mathcal{H}' \) since the application \( (T, \varphi) \rightarrow T^* \varphi \) is hypocontinuous. Hence \( <S, T^* C> \rightarrow 0 \) for any \( S \in \mathcal{H} \). If \( S \rightarrow 0 \) in \( \mathcal{H} \) and \( T \) is a fixed element of \( \mathcal{X} \), then \( T^* C \) is a compact disk of \( \mathcal{H}' \), and hence an equicontinuous subset of \( \mathcal{H}' \), so that \( <S, T^* C> \rightarrow 0 \) as \( S \rightarrow 0 \). Thus we have shown that \( \odot \) is separately continuous.

For the proof of the last part of the statements we use the notations \( <, >_{\mathcal{H}},\alpha,>_{\mathcal{D}},>_{\mathcal{D}}, \) to make clear the duality between the spaces of distributions under question. Suppose \( S \ast T \) exists, that is, \( S(T^* \varphi) \in \mathcal{D}'_{L^1} \) for any \( \varphi \in \mathcal{D} \). Let \( \{\alpha_k\} \) be a sequence of multiplicators such that \( \alpha_k \rightarrow 1 \) in \( \mathcal{D} \) and \( \alpha_k S \rightarrow S \) in \( \mathcal{H} \) as \( k \rightarrow \infty \). Then

\[
<S \odot T, \varphi>_{\mathcal{D},\alpha} = <S, T^* \varphi>_{\mathcal{H},\alpha} = \lim_k <\alpha_k S \ast T^* \varphi>_{\mathcal{H},\alpha} = \lim_k \int \alpha_k S(T^* \varphi) dx = \int (S(T^* \varphi)) dx = <S \ast T, \varphi>_{\mathcal{D},\alpha}.
\]

Therefore \( S \odot T = S \ast T \), as desired.

**References**

On a Space $H^f$


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