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On Local Loops in Affine Manifolds

Michihiko Kikkawa

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1. Introduction.

Local loops[1] have been treated, in the di-associative case, by Malcev [4], and general properties of topological loops have been studied by Hofmann [2] and others.

In the present paper, we shall show that a differentiable manifold with an affine connection forms a local loop in a neighbourhood of each of its points, if a product operation of two points on it is defined by means of parallel displacement of geodesics.

Next, we shall lead the fact that in the manifold with symmetric affine connection, if the local loop constituted at a point is left di-associative, the curvature tensor vanishes at the point.

Moreover, we shall refer to a sufficient condition for the group of linear transformations of tangent space induced by right inner transformations of the local loop to coincide with the local holonomy group, at the unit element of the local loop.

In particular, both of them really coincide with each other in reductive homogeneous space with canonical affine connection.

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2. Local Loops in Affinely Connected Spaces.

Definition 1. A *local loop* $\mathfrak{L}(U, f)$ is a pair formed by a topological space U and a continuous mapping f of an open subset S of $U\times U$ into U, satisfying the following conditions:

(a) On each of subsets $S_x^1=S\cap(\{x\}\times U)$ and $S_y^2=S\cap(U\times\{y\})$, the mapping f is a local homeomorphism.

The image $f(x, y)$ of $(x, y)\in S$ is called the product of x and y, and denoted by xy.

(b) There exists an element e of U such that $(e, e)\in S$ and that $ey=y$ on

(1) See Definition 1.

S_e^1 and $xe=x$ on S_e^2. The element e is called the unit element of $\mathfrak{L}(U, f)$.

Let M be an n-dimensional differentiable(2) manifold with a differentiable affine connection ∇.

Let us now define the product of two points of M by using geodesics in M, and show that a local loop is constructed in a neighbourhood of any point. Let T_p denote the tangent space of M at a point p. We can choose a star-shaped open neighbourhood N_o of the origin in T_p which is mapped diffeomorphically onto some open neighbourhood of p by the geodesic exponential mapping Exp of T_p into M.(3) For such N_o, the open neighbourhood $U_p=Exp\, N_o$ of p is called a normal neighbourhood of p.

DEFINITION 2. A normal neighbourhood U_p is said to be *restricted* if it is a normal neighbourhood of each of its points.

In a restricted normal neighbourhood, any two points of it can be joined by exactly one geodesic arc in it, and at each point p of M, a restricted normal neighbourhood of p can be found.

In a normal neighbourhood U_p of p, let $\tau_{p,q}$ denote the parallel displacement of tangent vectors at p along the geodesic arc joining p to the point q of U_p. Then, for an element X_p of T_p, the mapping

$$X: q \rightarrow \tau_{p,q}(X_p), \quad q \in U_p$$

defines a differentiable vector field X on U_p, which is said to be *adapted* to the tangent vector $X_p \in T_p$.

Now, we adopt normal coordinates $(u^1, u^2, \cdots, u^n)$ with origin p on a restricted normal neighbourhood U_p in M. Let Γ^i_{jk} $(i, j, k=1, 2, \cdots, n)$ be the coefficients of the affine connection ∇ with respect to this local coordinate system. For any two points $x=(x^1, x^2, \cdots, x^n)$ and $y=(y^1, y^2, \cdots, y^n)$ in U_p different from p, let $x(t)$ $(0 \leqslant t \leqslant t_0)$ and $y(s)$ $(0 \leqslant s \leqslant s_0)$ be geodesic arcs in U_p joining p to the points x and y respectively, where the parameters t and s are both affine parameters. If we put $\frac{dx^i(t)}{dt}\big|_{t=0}=X^i_p$ $(i=1, 2, \cdots, n)$, the coordinates of point $x(t)$ are given by $x^i(t)=tX^i_p$. At first, it is seen that the components of vectors $X(s)=\tau_{p,y(s)}(X_p)$ formed by parallel displacement of the tangent vector $X_p=(X^i_p)$ along the geodesic $y(s)$ are uniquely determined as the solutions $X^i(s)$ $(i=1, 2, \cdots, n)$ of the following system of differential equations:

$$\frac{dX^i(s)}{ds}+\Gamma^i_{jk}(y(s))\frac{dy^j(s)}{ds}X^k(s)=0, \qquad (i, j, k=1, 2, \cdots, n)$$

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(2) In the rest of the paper, we mean by "differentiable" always C^∞-differentiable.

(3) Helgason [1] p. 33

satisfying the initial conditions $X^i(0)=X^i_p$ $(i=1, 2, \cdots, n)$. These solutions $X^i(s)$ are differentiable with respect to the initial values $X^1_p, X^2_p, \cdots, X^n_p$ and hence they can be regarded as differentiable functions with arguments $x^1, x^2, \cdots, x^n$.

Next, let $z^i(t)$ $(i=1, 2, \cdots, n)$ denote the coordinates of points on the geodesic arc $z(t)$ $(z(0)=y(s_0))$ through the point y and tangent to the vector $X(s_0)$.

Then, for some $\delta>0$, they are uniquely determined in $0\leqq t<\delta$ as the solutions of the following system of differential equations:

$$(*)\quad \frac{d^2z^i(t)}{dt^2}+\Gamma^i_{jk}\big(z(t)\big)\frac{dz^j(t)}{dt}\frac{dz^k(t)}{dt}=0 \qquad (i, j, k=1, 2, \cdots, n)$$

satisfying the initial conditions $z^i(0)=y^i(s_0)$ and $\frac{dz^i}{dt}\big|_{t=0}=X^i(s_0)$ $(i=1, 2, \cdots, n)$. By extending this geodesic arc in U_p, we provide that the geodesic $z(t)$ $(0\leqq t<\delta)$ is maximal in U_p in the positive sense of t. Since U_p is a restricted normal neighbourhood, the geodesic $z(t)$ can be extended as long as it is contained in U_p.

For the pair (x, y) (where $x=x(t_0)$), we assign the point $z=z(t_0)$ if $t_0<\delta$, and denote this correspondence by f_p. (See Fig.) In the case when $x=p$ or $y=p$, we define f_p in the natural way as $f_p(p, y)=y$ and $f_p(x, p)=x$ respectively. Thus we have the mapping f_p of a subset S of $U_p\times U_p$ into U_p.

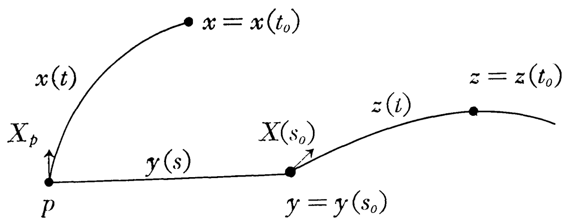

Fig.

Definition 3. For an ordered pair (x, y) of points of a restricted normal neighbourhood U_p, we define the *product* of x and y by the point $z=f_p(x, y)$ if it is determined by the mapping f_p introduced above, that is, if (x, y) belongs to S, and we denote it as $z=xy$.

The apparatus at p which consists of a normal neighbourhood U_p and the mapping f_p defining the product of two points of U_p is denoted by $\mathscr{L}(U_p, f_p)$.

From the definition of f_p, p plays a role of the unit element of $\mathscr{L}(U_p, f_p)$, that is, $S^1_p=\{p\}\times U_p$, $S^2_p=U_p\times\{p\}$ and $px=xp=x$ for all $x\in U_p$.

Moreover, the above definition of the product of two points depends not on the choice of affine parameters t and s on those geodesics, but on parallel displacement of geodesics. (Since the affine parameters on a geodesic are transformed to each other by a linear transformation.)

DEFINITION 4. A local loop $\mathfrak{L}(U, f)$ is called a *differentiable* local loop if U is a differentiable manifold and if the mapping f is differentiable.

THEOREM 1. *Let M be a differentiable manifold with an affine connection ∇. Then, at each point p of M, $\mathfrak{L}(U_p, f_p)$ forms a differentiable local loop by suitable choice of restricted normal neighbourhood U_p.*

PROOF. Let U'_p be a restricted normal neighbourhood of p and introduce in U'_p the product mapping f_p of DEFINITION 3.

Now, let $(u^1, u^2, \cdots, u^n)$ be a system of normal coordinates on U'_p. Since the coefficients Γ^i_{jk} of ∇ are differentiable functions on U'_p, for two points x and y of U'_p, the solutions $z^i(t)$ of the equations (*) are differentiable with respect to the affine parameter t and initial values $y^1, y^2, \cdots, y^n$; $X^1(s_o)$, $X^2(s_o)$, $\cdots$, $X^n(s_o)$, and we have a uniquely determined set of differentiable functions $f^i_p(x^1, x^2, \cdots, x^n; y^1, y^2, \cdots, y^n)$ $(i=1, 2, \cdots, n)$ such as we have the expressions

$$z^i = f^i_p(x^1, x^2, \cdots, x^n; y^1, y^2, \cdots, y^n) \ (i=1, 2, \cdots, n)$$

whenever z belongs to U'_p. Since U'_p is open, we see that the domain S on which f_p is defined is open in $U'_p \times U'_p$.

In the same way as above, it is seen that the element z satisfying $zx=y$ is uniquely determined by the elements x, $y \in U'_p$, if such element z exists in U'_p. The point z depends differentiably on x and y in U'_p. Thus, it is seen that the mapping f_p is a local homeomorphism on S^2_y.

Moreover, if the relation $xz=y$ holds for three points x, y and z in U'_p, then, in the local coordinates, it is expressed as $y^i = f^i_p(x^1, x^2, \cdots, x^n; z^1, z^2, \cdots, z^n)$, and $y^i (i=1, 2, \cdots, n)$ are differentiable functions of $x^1, x^2, \cdots, x^n; z^1, z^2, \cdots, z^n$ satisfying $f^i_p(0, 0, \cdots, 0; z^1, z^2, \cdots, z^n) = z^i$ $(i=1, 2, \cdots, \text{n})$. Hence we have

$$\left.\frac{\partial y^i}{\partial z^j}\right|_{x=p} = \frac{\partial f^i_p(0, 0, \cdots, 0; z^1, z^2, \cdots, z^n)}{\partial z^j}$$
$$= \delta^i_j.$$

It follows that the Jacobian does not vanish when x belongs to some neighbourhood U_p of p contained in U'_p. This implies that the point z, if it exists, is uniquely determined by arbitrarily given points x and y in U_p and its coordinates z^i are differentiable with respect to $x^1, x^2, \cdots, x^n; y^1, y^2, \cdots, y^n$.

It follows that the mapping f_p is a local homeomorphism on $S_x^1 \cap (\{x\} \times U_p)$.

Taking a restricted normal neighbourhood as U_p and restricting f_p onto it, we have a differentiable local loop $\mathscr{L}(U_p, f_p)$. Q.E.D.

In this manner, a differentiable local loop $\mathscr{L}(U_p, f_p)$ is formed around each point p of a differentiable manifold with an affine connection.

REMARK. In the space of distant parallelism, the differentiable local loop $\mathscr{L}(U_p, f_p)$ obtained above forms a local group. But, in general, $\mathscr{L}(U_p, f_p)$ is not a local group. In fact, if the curvature tensor of the affine connection does not vanish at p, $a(ab)=(aa)b$ does not always hold for two points a and b in U_p not contained in the same geodesic arc through p, even if $a(ab)$ and $(aa)b$ are defined. (See THEOREM 2.)

However, if contained in U_p, each geodesic arc through p forms a 1-parameter local subgroup of the local loop $\mathscr{L}(U_p, f_p)$, that is, the relation $f_p(x(t^1), x(t^2))=x(t^1+t^2)$ holds if $x(t^1)$, $x(t^2)$ and $x(t^1+t^2)$ belong to U_p.

Furthermore, if $y(t)$ is a geodesic arc through p, then for any x in U_p the relation $f_{y(t_1)}(f_p(x, y(t_1)), y(t_2))=f_p(x, y(t_1+t_2))$ holds whenever both sides are defined.

Now, we consider the case when the local loop $\mathscr{L}(U_p, f_p)$ is *left di-associative*, i.e., for two elements $x, y \in U_p$, $x(xy)=(xx)y$ holds so far as both of $x(xy)$ and $(xx)y$ are defined. In this case, we have the following theorem.

THEOREM 2. *Let M be a differentiable manifold with a symmetric affine connection ∇.*

If a differentiable local loop $\mathscr{L}(U_p, f_p)$ around a point p of M is left di-associative, the curvature tensor R vanishes at p.

PROOF. Let X_p be a tangent vector at p and let X be the vector field on U_p adapted to X_p which is defined at each $q \in U_p$ by the parallel displacement $\tau_{p,q}(X_p)$ of X_p. The trajectory $x(t)$ of X through p is a geodesic, and the geodesic arc through a point y in U_p and tangent to X_y at y is given by $y(t)=x(t)y$.

From the assumption of the theorem, we have the relation $y(u+t)=x(u+t)y=x(u)\big(x(t)y\big)$. Hence we see that $\dot{y}(t)=X_{y(t)}$, that is, the geodesic arc $y(t)$ is the trajectory of X through y. Since y is an arbitrary point in U_p, it is seen that all trajectories of the vector field X are geodesic arcs.

Therefore we have

$$\nabla_X X = 0 \tag{1}$$

on U_p for any adapted vector field X.

Let X and Y be vector fields on U_p adapted to the tangent vectors X_p and Y_p, respectively, then the vector field $X+Y$ is adapted to the tangent vector X_p+Y_p and hence satisfies (1).

Then we get

$$\nabla_X Y + \nabla_Y X = 0 \qquad \text{on} \quad U_p. \tag{2}$$

This implies the following modification of the relation of curvature tensor R:

$$\begin{aligned} R(X, Y)X &= \nabla_X\nabla_Y X - \nabla_Y\nabla_X X - \nabla_{[X, Y]}X \\ &= -\nabla_X\nabla_X Y - \nabla_Y\nabla_X X - \nabla_{[X, Y]}X. \end{aligned} \tag{3}$$

Considering the assumption $T(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y]=0$ and the relation (2), we have

$$\nabla_X Y = \tfrac{1}{2}[X, Y].$$

Since Y is adapted to Y_p, $\nabla_X Y=0$ holds along the trajectory $x(t)$ of X through p, and then we have

$$[X, Y]_p = 0 \tag{4}$$

and

$$(\nabla_X\nabla_X Y)_p = 0. \tag{5}$$

By means of the relations (1), (4) and (5), we get the value of (3) at p as follows:

$$\begin{aligned} R_p(X_p, Y_p)X_p &= (R(X, Y)X)_p \\ &= -(\nabla_X\nabla_X Y)_p - (\nabla_Y\nabla_X X)_p - (\nabla_{[X,Y]}X)_p \\ &= 0, \qquad \text{for all } X_p,\ Y_p \text{ in } T_p. \end{aligned} \tag{6}$$

Hence

$$R_p(X_p, Y_p)Z_p + R_p(Z_p, Y_p)X_p = 0 \qquad \text{for all } X_p,\ Y_p \text{ and } Z_p \text{ in } T_p. \tag{7}$$

In the space of symmetric connection, we have the identity:

$$R_p(X_p, Y_p)Z_p + R_p(Z_p, X_p)Y_p + R_p(Y_p, Z_p)X_p = 0 \tag{8}$$

for all X_p, Y_p and Z_p in T_p.

By (7) and (8), we have

$$R_p(X_p, Y_p)Z_p = 0 \qquad \text{for all } X_p, Y_p \text{ and } Z_p \text{ in } T_p.$$

Therefore, $R_p=0$ whcih is the conclusion of our theorem.

3. Group of Inner Transformations.

In a differentiable local loop $\mathcal{L}(U_p, f_p)$ which is formed around a point p of a differentiable manifold M with an affine connection, the mapping $R_a: x \to xa$ $(a \in U_p)$ is a local homeomorphism of U_p, which we shall call a right transformation of $\mathcal{L}(U_p, f_p)$.

If a, b and ab belong to U_p, we have $R_{ab}^{-1}R_bR_a(p)=p$ and we see that the mapping $R_{ab}^{-1}R_bR_a$ is a homeomorphism of an open neighbourhood of p onto an open neighbourhood of p. Therefore, if all elements a_i, b_i and a_ib_i $(i=1, 2, \cdots, m)$ belong to U_p, we can define a finite product $R_{a_1b_1}^{-1}R_{b_1}R_{a_1}R_{a_2b_2}^{-1}R_{b_2}R_{a_2}\cdots$, $R_{a_mb_m}^{-1}R_{b_m}R_{a_m}$ as a local homeomorphism around p.

Definition 5. Let $\mathcal{L}(U_p, f_p)$ be a differentiable local loop around p. The *group of right inner transformations* $\mathcal{J}(\mathcal{L}_p)$ of $\mathcal{L}(U_p, f_p)$ is defined as the group generated by the set of local homeomorphisms $\{R_{ab}^{-1}R_bR_a;\ a,\ b \text{ and } ab \in U_p\}$.

In a differentiable local loop $\mathcal{L}(U_p, f_p)$, any right transformation $R_a(a \in U_p)$ is regular at each point where it is defined. Hence, if two points x and xa belong to U_p, R_a induces a linear mapping dR_a of the tangent space T_x onto the tangent space T_{xa}.

Each element of the group $\mathcal{J}(\mathcal{L}_p)$ of right inner transformations of $\mathcal{L}(U_p, f_p)$ leaves the point p invariant and hence induces a linear transformation of tangent space T_p at p.

Let $I(\mathcal{L}_p)$ denote the group of linear transformations of T_p induced from $\mathcal{J}(\mathcal{L}_p)$ in the above way.

In the local loop $\mathcal{L}(U_p, f_p)$, the mapping $dR_b^{-1}\cdot\tau_{b,ab}\cdot dR_b$ is an isomorphism of the tangent space T_p at p onto the tangent space T_a at any fixed point a, whenever b and ab belong to U_p. Concerning these isomorphisms we have the following theorem.

Theorem 3. *Let $R_x(x \in U_p)$ denote a right transformation of $\mathcal{L}(U_p, f_p)$*

around p of M, and let $\tau_{x,y}$ $(x, y \in U_p)$ be the parallel displacement of tangent vectors at x along the geodesic arc joining x to y.

Suppose that, for any point a of U_p, the isomorphism $dR_b^{-1}\circ\tau_{b,ab}\circ dR_b$ (where b and $ab \in U_p$) of the tangent space T_p onto the tangent space T_a does not depend on b.

Then, the group $I(\mathcal{L}_p)$ of linear transformations of T_p induced by right inner transformations coincides with the local holonomy group $\sigma_p(U_p)$ at p of the connected open neighbourhood U_p.

PROOF. Since the mapping $dR_b^{-1}\circ\tau_{b,ab}\circ dR_b$ of T_p onto T_a does not depend on b, putting $b=p$, we have

$$dR_b\circ\tau_{p,a} = \tau_{b,ab}\circ dR_b.$$

On the other hand, for each x in U_p, the equality $dR_x=\tau_{p,x}$ always holds.
Hence we have

$$\begin{aligned} dR_b\circ dR_a &= dR_b\circ\tau_{p,a} \\ &= \tau_{b,ab}\circ dR_b \\ &= \tau_{b,ab}\circ\tau_{p,b}. \end{aligned}$$

Therefore,

$$\begin{aligned} dR_{ab}^{-1}\circ dR_b\circ dR_a &= \tau_{p,ab}^{-1}\circ\tau_{b,ab}\circ\tau_{p,b} \\ &= \tau_{ab,p}\circ\tau_{b,ab}\circ\tau_{p,b}. \end{aligned}$$

Thus, we find that an element $d(R_{ab}^{-1}R_bR_a)$ of $I(\mathcal{L}_p)$ is also an element of the holonomy group $\sigma_p(U_p)$ at p determined by the geodesic triangle in U_p joining points p, b and ab in this order.

But, arbitrary closed curve in U_p at p is divided into a set of geodesic triangles figured above. Therefore, we see that any element of the holonomy group $\sigma_p(U_p)$ is generated by elements of $I(\mathcal{L}_p)$ of the form $dR_{ab}^{-1}\, dR_b\, dR_a$ such that a, b and ab are contained in U_p. Q.E.D.

we consider a reductive homogeneous space $M=G/H$ with canonical connection ∇[4].

At the point $\pi(e)=p$ (where e is the identity of the Lie group G and π is the natural projection), let us constitute a differentiable local loop $\mathcal{L}(U_p, f_p)$ associated to ∇. Then all right transformations $R_a (a \in U_p)$ of $\mathcal{L}(U_p, f_p)$ are

(4) Lichnerowicz [3] p. 48

affine transformations with respect to ∇. Hence we have

$$dR_b \circ \tau_{p,a} = \tau_{b,ab} \circ dR_b \quad (b \in U_p)$$

which satisfies the assumption of the above theorem.

Therefore, we have the following corollary.

COROLLARY. *At the point $p=\pi(e)$ of a reductive homogeneous space $M=G/H$ with canonical affine connection, the group $I(\mathfrak{L}_p)$ of linear transformations of T_p induced by right inner transformations of a differentiable local loop $\mathfrak{L}(U_p, f_p)$ coincides with the local holonomy group $\sigma_p(U_p)$ defined on a restricted normal neighbourhood U_p of p.*

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Hiroshima Commercial College