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Lie Algebras which have Few Derivations

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Introduction

Let L be a Lie algebra over a field of characteristic 0 and let $D(L)$ be the derivation algebra of L. Let $I(L)$ and $C(L)$ be respectively the sets of all inner derivations and of all central derivations of L. In the paper [10], we studied the relationship between the structures of $D(L)$ and L and among other results we showed some results on Lie algebras L which have as few derivations as possible, that is, such that $D(L)=I(L)+C(L)$. It is furthermore natural to make a search for the properties of Lie algebras L such that $D(L)=I(L)^*+C(L)$ where $I(L)^*$ is the algebraic hull of $I(L)$, that is, Lie algebras which have few derivations. The purpose of this paper is to study such a type of Lie algebras.

There actually exists a Lie algebra L such that $D(L)=I(L)^*+C(L)$ but $D(L)\neq I(L)+C(L)$, as will be shown in Section 5. Owing to Lemma 1 in [10] which states that $I(L)^*=\mathrm{ad}_L L^*$ for a linear Lie algebra L, for such Lie algebras we can show the results analogous to those for Lie algebras which have as few derivations as possible.

In [10], generalizing a result of G. Leger [7], we showed that if $D(L)=I(L)+C(L)$ then the radical of L is not quasi-cyclic or is the center of L. We shall give the corresponding results with sharper assertions. Namely, we shall show that, if $D(L)=I(L)^*+C(L)$, then the radical R of L is the direct sum of a central ideal of L and of an ideal R_1 which has no abelian direct summands and all semisimple elements of the radical of $D(R_1)$ are contained in $I(R_1)^*$, and that, if R is further nilpotent, the radical of $D(R_1)$ consists precisely of the nilpotent elements (Theorem 2). It will also be shown that for a Lie algebra L such that $D(L)=I(L)^*$ we have similar statements with $R=R_1$ (Theorem 3). As one of the applications of these results we shall show that any non-abelian nilpotent Lie algebra which is quasi-cyclic or whose dimension is less than 6 cannot be the radical of a Lie algebra L such that $D(L)=I(L)+C(L)$ (Corollary 2 to Theorems 2 and 3).

We shall further prove that $D(L)=I(L)^*+C(L)$ if and only if this is the case for every direct summand of L (Theorem 1) and clarify the structure of Lie algebras whose radicals have few derivations (Theorem 4).

1. Preliminaries

Throughout the paper we shall use the same terminologies and notations as in the paper [10].

Let L be a Lie algebra over a field K of characteristic 0. Let $D(L)$ be the algebra of all derivations of L and let $I(L)$ be the ideal of $D(L)$ consisting of all inner derivations of L. Let $C(L)$ be the subalgebra of all central derivations of L, that is, of all derivations mapping L into the center Z of L. Then $C(L)$ is the set of all endomorphisms of L which map L into Z and $[L, L]$ into (0).

We first show that $C(L)$ is an algebraic subalgebra of $\mathfrak{gl}(L)$. Let D be any element of $C(L)$ and let D' be any replica of D. Then D' can be expressed in the form of polynomial in D with coefficients in K whose constant term is zero ([2], p. 181). Therefore

$$D'L \subset Z \qquad \text{and} \qquad D'[L, L] = (0),$$

whence D' is contained in $C(L)$. Thus we see that $C(L)$ is algebraic.

Contrary to this fact, $I(L)$ is not an algebraic subalgebra of $\mathfrak{gl}(L)$ in general.

By considering the fact that $D(L)$ necessarily contains $I(L)$ and $C(L)$, in [10] we said that L has as few derivations as possible, provided that $D(L) = I(L) + C(L)$. Since $D(L)$ is an algebraic subalgebra of $\mathfrak{gl}(L)$, $D(L)$ also contains $I(L)^*$. Thus we may say that L has few derivations, provided that $D(L) = I(L)^* + C(L)$.

2. Lemmas

We start with

Lemma 1. *Any Lie algebra is the direct sum of a central ideal and an ideal which has no abelian direct summands.*

Proof. Let L be a Lie algebra and let Z be the center of L. Suppose that L has an abelian direct summand. Then Z is not contained in $[L, L]$. We first take a subspace L_1 of Z such that

$$Z = L_1 + Z \cap [L, L], \qquad L_1 \cap (Z \cap [L, L]) = (0).$$

We next choose a subspace L_2 of L such that

$$L = L_1 + L_2, \qquad L_1 \cap L_2 = (0), \qquad L_2 \supset [L, L].$$

Then L_1 is a central ideal of L and L_2 is an ideal of L. If L_2 has an abelian direct summand, the summand is contained in Z and does not intersect $Z\cap[L, L]$, which contradicts the choice of L_1. Thus we see that L_2 has no abelian direct summands, completing the proof.

LEMMA 2. *Let L be a Lie algebra with radical R and center Z. Assume that $D(L)=I(L)^*+C(L)$.*

(1) *If R is abelian, then $R=Z$.*

(2) *If L has no abelian direct summands, then R has no abelian direct summands.*

PROOF. (1): If R is nilpotent, then all elements of $\mathrm{ad}_L R$ are nilpotent, whence the radical $\mathrm{ad}_L R$ of $I(L)$ is algebraic and therefore $I(L)$ is algebraic. If R is abelian and $R\neq Z$, then there exists a derivation of L which is not contained in $I(L)+C(L)$, as was shown in the proof of Theorem 5 in [10].

(2): Since L has a faithful representation, we may assume that L is a subalgebra of $\mathfrak{gl}(V)$ where V is a finite dimensional vector space over K. Suppose that L has no abelian direct summands but R has. Then by (1) we see that R is not abelian. Denoting by $Z(R)$ the center of R, we have

$$Z(R)\not\subset[R, R].$$

Let $L=S+R$ be a Levi decomposition of L. Then $\mathrm{ad}_L S$ is completely reducible, and $Z(R)$ and $[R, R]$ are stable under $\mathrm{ad}_L S$. Therefore there exists a subspace $Z_1\neq(0)$ of $Z(R)$ such that

$$Z(R)=Z_1+Z(R)\cap[R, R], \qquad Z_1\cap(Z(R)\cap[R, R])=(0),$$
$$(\mathrm{ad}_L S)Z_1\subset Z_1.$$

We can then choose a subspace R_1 of R containing $[R, R]$ such that

$$R=Z_1+R_1, \qquad Z_1\cap R_1=(0), \qquad (\mathrm{ad}_L S)R_1\subset R_1.$$

Now define an endomorphism D of L as follows:

$$Dz=z \quad \text{for } z\in Z_1 \qquad \text{and} \qquad D(S+R_1)=(0).$$

Then it is easy to see that D is a derivation of L.

Since $L^*=S+R^*$ and $R^*=Z_1^*+R_1^*$, using Lemma 1 in [10] we see that D can be expressed in the form

$$D=\mathrm{ad}_L s+\mathrm{ad}_L z_1+\mathrm{ad}_L r_1+D_0$$

$$\text{with} \quad s\in S, \quad z_1\in Z_1^*, \quad r_1\in R_1^* \quad \text{and} \quad D_0\in C(L).$$

Since $DS=(0)$ and $D_0S=(0)$, for any element s' of S we have

$$Ds'=[s, s']+[z_1+r_1, s']=0,$$

whence

$$[s, s']\in S\cap R^*=(0).$$

It follows that $s=0$. Now for any element z of Z_1 we have

$$Dz=[z_1, z]+[r_1, z]+D_0z=D_0z.$$

Since D is the identity on Z_1, we see that $z\in Z$. Hence $Z_1\subset Z$. It follows that L is the direct sum of a central ideal Z_1 and an ideal $S+R_1$, which contradicts our supposition that L has no abelian direct summands.

Thus the proof is complete.

Let $L=S+R$ be a Levi decomposition of L. Following G. Hochschild [6], we denote by $\mathfrak{A}(S)$ the set of all derivations of L which map S into (0). Then it is known that

$$D(L)=I(L)+\mathfrak{A}(S).$$

By a toroidal subalgebra of $\mathfrak{gl}(V)$ we mean an abelian subalgebra of $\mathfrak{gl}(V)$ consisting of semisimple endomorphisms.

We now show the following

Lemma 3. *Let $L=S+R$ be a Levi decomposition of L. Then among maximal toroidal subalgebras of the radical of $D(R)$ there exists one which can be imbedded in $\mathfrak{A}(S)$.*

Proof. Since $D(R)$ is algebraic, for the ideal $\mathfrak{N}$ of all nilpotent elements of the radical $\mathfrak{R}$ of $D(R)$ there exist a maximal semisimple subalgebra $\mathfrak{S}_1$ and a maximal toroidal subalgebra $\mathfrak{B}_1$ of $\mathfrak{R}$ such that

$$D(R)=\mathfrak{S}_1+\mathfrak{R},\qquad \mathfrak{R}=\mathfrak{B}_1+\mathfrak{N},$$

$$\mathfrak{S}_1\cap\mathfrak{R}=(0),\qquad \mathfrak{B}_1\cap\mathfrak{N}=(0),\qquad [\mathfrak{S}_1, \mathfrak{B}_1]=(0)$$

([5], p. 40). Take a maximal semisimple subalgebra $\mathfrak{S}$ of $D(R)$ containing $\mathrm{ad}_R S$. Then $\mathfrak{S}$ is the image of $\mathfrak{S}_1$ under an inner automorphism σ of $D(R)$. Denote by $\mathfrak{B}$ the image of $\mathfrak{B}_1$ under σ. Then $\mathfrak{B}$ is a maximal toroidal subalgebra of $\mathfrak{R}$ and $[\mathfrak{S}, \mathfrak{B}]=(0)$. It follows that

$$[\mathrm{ad}_R S, \mathfrak{B}]=(0).$$

For any D in $\mathfrak{B}$, any s in S and any r in R, we have

$$D[s, r] = [s, Dr].$$

Therfore D can be extended to be a derivation of L by putting $DS=(0)$. Thus we see that $\mathfrak{B}$ can be imbedded in $\mathfrak{A}(S)$.

The proof is complete.

3. Main theorems

In this section we shall give some properties of Lie algebras which have few derivations. We first show the following theorem corresponding to Theorem 4 in [10].

THEOREM 1. *Let L be a Lie algebra over a field K of characteristic* 0 *and assume that L is the direct sum of the ideals L_i $(i=1, 2, \cdots, n)$. Then* $D(L)=I(L)^*+C(L)$ if and only if $D(L_i)=I(L_i)^*+C(L_i)$ *for every i.*

PROOF. By Lemma 1 in [9] we see that

$$D(L) = \sum_{i=1}^{n} D(L_i) + \sum_{i\neq j} D(L_i, L_j)$$

and that for $i\neq j$ $D(L_i, L_j)\subset C(L)$. Since $I(L_i)^*\subset I(L)^*$, if $D(L_i)=I(L_i)^*+C(L_i)$ for every i then $D(L)=I(L)^*+C(L)$.

To prove the converse, we may assume that V is the direct sum of finite dimensional vector spaces V_i $(i=1, 2, \cdots, n)$ over K, $L\subset \mathfrak{gl}(V)$ and $L_i\subset \mathfrak{gl}(V_i)$ for every i. Then L^* is the direct sum of the ideals L_i^*. Suppose that $D(L)=I(L)^*+C(L)$ and let D_i be any derivation of L_i. Then D_i is trivially extended to a derivation of L which we denote by D. By Lemma 1 in [10], D can be expressed in the form

$$D = \mathrm{ad}_L x + D_0 \qquad \text{with } x \in L^* \text{ and } D_0 \in C(L).$$

Let $x=\sum_{j=1}^{n} x_j$ with x_j in L_j^* and put $\bar{D}_0=D_i-\mathrm{ad}_{L_i}x_i$. Then

$$\bar{D}_0 L_i = D_0 L_i \subset L_i \cap Z = Z_i$$

where Z and Z_i are respectively the centers of L and L_i. Hence $\bar{D}_0$ is contained in $C(L_i)$. Thus we have $D(L_i)=I(L_i)^*+C(L_i)$.

The proof of the theorem is complete.

Now we can show the following main theorems.

THEOREM 2. *Let L be a Lie algebra over a field K of characteristic* 0 *and let R be the radical of L. Assume that $D(L)=I(L)^*+C(L)$. Then*:

(1) *R is the direct sum of a central ideal of L and of an ideal R_1 which has no abelian direct summands, and all semisimple derivations in the radical of $D(R_1)$ are contained in $I(R_1)^*$.*

(2) *If R is nilpotent, the radical of $D(R_1)$ consists precisely of the nilpotent elements.*

(3) *If L is solvable* (*resp. nilpotent*), *L is the direct sum of a central ideal and a characteristically solvable* (*resp. characteristically nilpotent*) *ideal.*

PROOF. (1): By Lemma 1 L is the direct sum of an ideal L_1 which has no abelian direct summands and of a central ideal L_2. By Theorem 1 we see that $D(L_1)=I(L_1)^*+D(L_1)$. Let R_1 be the radical of L_1. Then R is the direct sum of the ideals R_1 and L_2, and by Lemma 2 R_1 has no abelian direct summands.

Let $L_1=S+R_1$ be a Levi decomposition of L_1. By Lemma 3 we see that there exists a maximal toroidal subalgebra $\mathfrak{B}_1$ of the radical $\mathfrak{R}$ of $D(R_1)$ which can be imbedded in $\mathfrak{A}(S)$. We may assume that L_1 is a subalgebra of $\mathfrak{gl}(V)$ with V a finite dimensional vector space over K. Therefore by Lemma 1 in [10] we have

$$I(L_1)^* = \mathrm{ad}_{L_1}L_1{}^* = \mathrm{ad}_{L_1}S + \mathrm{ad}_{L_1}R_1{}^*.$$

For any element D of $\mathfrak{B}_1$ we denote by $\bar{D}$ the derivation of L_1 to which D is trivially extended. Then $\bar{D}$ is expressed in the form

$$\bar{D} = \mathrm{ad}_{L_1}s + \mathrm{ad}_{L_1}r + D_0 \qquad \text{with } s \in S,\ r \in R_1{}^* \text{ and } D_0 \in C(L_1).$$

Since $\bar{D}S=(0)$ and $D_0S=(0)$, for any element s' of S we have

$$[s, s'] + [r, s'] = 0,$$

whence

$$[s, s'] \in S \cap R_1{}^* = (0).$$

It follows that $s=0$. Therefore we have

$$\bar{D} = \mathrm{ad}_{L_1}r + D_0.$$

Since L_1 has no abelian direct summands, the center Z_1 of L_1 is contained in $[L_1, L_1]$ and therefore

$$D_0^2 L_1 \subset D_0 Z_1 \subset D_0 [L_1, L_1] = (0),$$

whence D_0 is nilpotent. And we have

$$\begin{aligned}[\mathrm{ad}_{L_1} r, D_0] L_1 &\subset [r, D_0 L_1] + D_0 [r, L_1] \\ &\subset [R_1^*, Z_1] + D_0 [R_1^*, L_1] \\ &= [R_1, Z_1] + D_0 [R_1, L_1] = (0),\end{aligned}$$

which shows that D_0 commutes with $\mathrm{ad}_{L_1} r$ and therefore with the semisimple and the nilpotent components of $\mathrm{ad}_{L_1} r$. It follows that $\bar{D}$ is the semisimple component of $\mathrm{ad}_{L_1} r$, which is contained in the algebraic hull of $\mathrm{ad}_{L_1} R_1^*$. Since $\mathrm{ad}_{L_1} R_1^*$ is algebraic, $\bar{D}$ is contained in $\mathrm{ad}_{L_1} R_1^*$. It follows from Lemma 1 in [10] that D is contained in $I(R_1)^*$. Thus we see that $\mathfrak{B}_1 \subset I(R_1)^*$.

Now let T be any semisimple element of $\mathfrak{R}$. Take a maximal toroidal subalgebra $\mathfrak{B}$ of $\mathfrak{R}$ containing T. Then $\mathfrak{B}$ is conjugate with $\mathfrak{B}_1$ under an inner automorphism σ of $\mathfrak{R}$ of the form $\exp(\mathrm{ad}_{\mathfrak{R}} N)$ where N is an element of the derived algebra of $\mathfrak{R}$ ([8], p. 209). Since $I(R_1)$ is stable under $\mathrm{ad}_{\mathfrak{R}} N$, $I(R_1)^*$ is also. It follows that $I(R_1)^*$ is stable under σ. Thus we see that $\mathfrak{B} \subset I(R_1)^*$ and therefore that T belongs to $I(R_1)^*$.

(2): If R is nilpotent, then R_1 is nilpotent. Hence $I(R_1)^*$ consists of only the nilpotent elements. By using (1) proved above we see that any semisimple element of the radical $\mathfrak{R}$ of $D(R_1)$ is 0. Since $\mathfrak{R}$ is algebraic and therefore splittable, we obtain that $\mathfrak{R}$ consists precisely of the nilpotent elements.

(3): If L is solvable (resp. nilpotent), then $L = R$ and therefore L is the direct sum of a central ideal and an ideal R_1. By Theorem 1

$$D(R_1) = I(R_1) + C(R_1).$$

From the fact that the center of R_1 is contained in $[R_1, R_1]$, it follows that $C(R_1)$ is a central ideal of $D(R_1)$. Therefore $D(R_1)$ is solvable (resp. nilpotent). By using (2) we see that R_1 is characteristically solvable (resp. characteristically nilpotent).

Thus the proof of the theorem is complete.

Theorem 3. *Let L be a Lie algebra over a field of characteristic* 0 *and let R be the radical of L. Assume that $D(L) = I(L)^*$. Then*:

(1) *R has no abelian direct summands and all semisimple elements of the radical of $D(R)$ are contained in $I(R)^*$.*

(2) *If R is nilpotent, L is not solvable and the radical of $D(R)$ consists precisely of the nilpotent elements.*

(3) *If L is solvable, L is characteristically solvable and not nilpotent.*

Proof. If L has an abelian direct summand L_1, then $D(L_1)$, considered as a subset of $D(L)$, is not contained in $I(L)^*$. Therefore L has no abelian direct summands. Hence by Lemma 2 we see that R has no abelian direct summands.

If L is nilpotent, L has an outer derivation, which contradicts the assumption since $I(L)$ is algebraic. Hence if R is nilpotent, L is not solvable. If L is solvable, L is not nilpotent.

All other assertions follow from Theorem 2.

As an immediate consequence of Theorems 2 and 3 we have first the following statement, which gives us a test for seeing whether a nilpotent Lie algebra can be the radical of a Lie algebra with few derivations.

Corollary 1. *Let L be a Lie algebra whose radical R is nilpotent. If either $D(L)=I(L)^*+C(L)$ and L has no abelian direct summands or $D(L)=I(L)^*$, then the trace of any derivation of R is* 0.

Proof. By (2) in Theorems 2 and 3 we have

$$D(R)=\mathfrak{S}+\mathfrak{N}$$

where $\mathfrak{S}$ is a maximal semisimple subalgebra of $D(R)$ and $\mathfrak{N}$ is the ideal of all nilpotent elements of the radical of $D(R)$. Since $\mathfrak{S}=[\mathfrak{S},\mathfrak{S}]$, we see that the trace of any element of $D(R)$ is 0, completing the proof.

As the second consequence of Theorem 2 we have

Corollary 2. *Any non-abelian nilpotent Lie algebra R satisfying each of the following conditions cannot be the radical of a Lie algebra L such that $D(L)=I(L)+C(L)$:*

(1) *R is quasi-cyclic.*

(2) *The dimension of R is less than* 6.

Proof. Assume that R is the radical of a Lie algebra L such that $D(L)=I(L)+C(L)$. Then by Theorem 2 we see that R is the direct sum of a central ideal of L and an ideal R_1 which has no abelian direct summands and such that

$$D(R_1)=\mathfrak{S}+\mathfrak{N}$$

where $\mathfrak{S}$ is a semisimple subalgebra and $\mathfrak{N}$ is an ideal consisting of nilpotent elements. Hence the traces of all derivations of R_1 are 0.

(1): Suppose that R is quasi-cyclic. Then there exists a subspace U of R such that

$$R=\sum_i U^i \qquad \text{and} \qquad U^i \cap U^j=(0) \quad \text{for } i \neq j$$

where $U^1=U$ and $U^{i+1}=[U, U^i]$ for $i \geq 1$. Denote by U_1 the projection of U onto R_1. Then we have

$$R_1=\sum_i U_1^i \qquad \text{and} \qquad U_1^i \cap U_1^j=(0) \quad \text{for } i \neq j.$$

The endomorphism D of R_1 defined by

$$Du=iu \qquad \text{for} \quad u \in U_1^i \quad (i=1, 2, \cdots)$$

is a derivation of R_1 whose trace is not 0, which is a contradiction.

(2): Suppose that $\dim R<6$. Then $\dim R_1<6$. In [3] J. Dixmier has given all nilpotent Lie algebras whose dimensions are <6. By using the result, we see that R_1 is quasi-cyclic or one of the Lie algebras described in terms of a basis x_1, x_2, x_3, x_4, x_5 by the following multiplication tables:

(a) $[x_1, x_2]=x_4, \qquad [x_1, x_4]=x_5, \qquad [x_2, x_3]=x_5.$

(b) $[x_1, x_2]=x_3, \qquad [x_1, x_3]=x_4, \qquad [x_1, x_4]=x_5, \qquad [x_2, x_3]=x_5.$

In addition $[x_i, x_j]=-[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the tables above. In the case where R_1 is given by the table (a), the endomorphism D of R_1 defined by

$$Dx_1=x_1, \qquad Dx_2=x_2, \qquad Dx_3=2x_3, \qquad Dx_4=2x_4, \qquad Dx_5=3x_5$$

is a derivation of R_1. In the case where R_1 is given by the table (b), the endomorphism D of R_1 defined by

$$Dx_i=ix_i \qquad (i=1, 2, \cdots, 5)$$

is a derivation of R_1. Therefore, together with the proof of (1), we see that R_1 has a derivation whose trace is not 0, which is a contradiction.

Thus the proof is complete.

As for the case where R is a non-nilpotent solvable Lie algebra we don't know any result corresponding to Corollary 2. However we may of course apply the statement (1) in Theorem 2 to concrete algebras. For example, we can apply the statement to conclude that the non-nilpotent solvable Lie algebra given in [9], p. 213, cannot be the radical of any Lie algebra which has few derivations, as will be shown in Example 4 of Section 5.

As another consequence of Theorems 2 and 3 we have

Corollary 3. *Any non-zero abelian Lie algebra R is the radical of a Lie algebra L such that $D(L)=I(L)^{*}+C(L)$ if and only if R is the center of L. R cannot be the radical of a Lie algebra L such that $D(L)=I(L)^{*}$.*

Proof. If R is the center of L, L is the direct sum of a semisimple ideal S and R. Since $D(S, R)=D(R, S)=(0)$, by Lemma 1 in [9] we have

$$D(L)=D(S)+D(R)=I(L)+C(L).$$

The other assertions are immediate from Theorems 2 and 3.

Remark. The Lie algebra which will be given in Example 3 of Section 5 is an example for the statements (1) in Theorems 2 and 3. The Lie algebra L given by J. Dixmier and W. G. Lister in [4] is an example for the statements (2) and (3) in Theorem 2. Namely, L is characteristically nilpotent and $D(L)=I(L)+C(L)$ (see Remark 3 in [10]). The converses of the statements (1) and (2) in Theorems 2 and 3 are not valid in general. An example for these is the characteristically nilpotent Lie algebra L which was given in [1], p. 123. Namely, $D(L)$ is nilpotent and consists of nilpotent elements, but $D(L)\neq I(L)^{*}+C(L)$ (see Remark 1 in [10]).

4. Lie algebras whose radicals have few derivations

In the last section of [10] we studied the properties of Lie algebras whose radicals have as few derivations as possible. In this section we shall show the corresponding properties for Lie algebras whose radicals have few derivations.

Theorem 4. *Let L be a Lie algebra over a field of characteristic 0 and let R be the radical of L. $D(R)=I(R)^{*}+C(R)$ if and only if L is the direct sum of the ideals L_1 and L_2 satisfying the following conditions:*

(1) *L_1 is the direct sum of a semisimple ideal, a central ideal and a characteristically solvable ideal R_1 with $D(R_1)=I(R_1)^{*}+C(R_1)$.*

(2) *The radical of L_2 is abelian, the center of L_2 is (0) and $L_2=[L_2, L_2]$.*

And then L_1 and L_2 are characteristic ideals of L and

$$D(L_1)=I(L_1)^{*}+C(L_1), \qquad D(L_2)\neq I(L_2)^{*}+C(L_2).$$

Proof. The statement corresponding to Lemma 7 in [10] is that if $D(R)=I(R)^{*}+C(R)$ then L is the direct sum of a characteristically solvable ideal L_1 with $D(L_1)=I(L_1)^{*}+C(L_1)$ and of an ideal L_2 whose radical is abelian. This can be shown in the same way as in the proof of Lemma 7 in [10] by using respectively Theorem 1 and (3) in Theorem 2 instead of Theorem 4 and

Corollary 2 to Theorem 5 in [10]. Now we can prove the theorem exactly as in the proof of Theorem 7 in [10] by using respectively the result stated above and Theorem 1 instead of Lemma 7 and Theorem 4 in [10]. Therefore we omit the detail.

COROLLARY. *Let L be a Lie algebra with radical R. If $D(R)=I(R)^*$, then L is the direct sum of a semisimple ideal and R, and $D(L)=I(L)^*$.*

PROOF. If $D(R)=I(R)^*$, then L is the direct sum of the ideals L_1 and L_2 which are stated in Theorem 4. Since R has no abelian direct summands, $L_2=(0)$ and L is the direct sum of a semisimple ideal S and R. Consequently by Lemma 1 in [9] we see that

$$D(L)=D(S)+D(R)=I(S)+I(R)^*=I(L)^*,$$

completing the proof.

5. Examples

EXAMPLE 1. Let L be a Lie algebra over the field of real numbers described in terms of a basis x_1, x_2, x_3 by the following multiplication table:

$$[x_1, x_2]=x_2, \qquad [x_1, x_3]=\tau x_3, \qquad [x_2, x_3]=0$$

where τ is an irrational number. In addition $[x_i, x_j]=-[x_j, x_i]$. Let D be a derivation of L and put

$$Dx_i=\sum_{j=1}^{3}\lambda_{ij}x_j \qquad (i=1, 2, 3).$$

Then after calculation we obtain

$$\lambda_{11}=\lambda_{21}=\lambda_{23}=\lambda_{31}=\lambda_{32}=0.$$

Therefore the matrix of D is

$$\begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{pmatrix}.$$

From this the matrix of an inner derivation of L is obtained by putting

$$\lambda_{33}=\tau\lambda_{22}.$$

Hence $D(L) \neq I(L)$. However, by using Proposition 2 in [2], p. 160, it is easily shown that $D(L)=I(L)^*$.

Example 2. Let L be the direct sum of the Lie algebra in Example 1 and a non-zero abelian Lie algebra over the field of real numbers. Then $C(L)\neq(0)$. After calculation we see that $D(L)=I(L)^*+C(L)$ but $D(L)\neq I(L)+C(L)$.

Example 3. Let L be a Lie algebra over a field of characteristic 0 described in terms of a basis $x_1, x_2, \cdots, x_6$ by the following multiplication table:

$$[x_1, x_2]=2x_2, \qquad [x_1, x_3]=-2x_3, \qquad [x_2, x_3]=x_1,$$
$$[x_1, x_5]=-x_5, \qquad [x_1, x_6]=x_6, \qquad [x_2, x_5]=-x_6,$$
$$[x_3, x_6]=-x_5, \qquad [x_4, x_5]=-x_5, \qquad [x_4, x_6]=-x_6.$$

In addition $[x_i, x_j]=-[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the table above. After calculation we see that $D(L)=I(L)$. The radical R of L is (x_4, x_5, x_6). For a derivation D of R we put

$$Dx_{i+3}=\sum_{j=1}^{3}\lambda_{ij}x_{j+3} \qquad (i=1, 2, 3).$$

Then after calculation we see that the matrix of any derivation in the radical $\mathfrak{R}$ of $D(R)$ is

$$\begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{22} \end{pmatrix}$$

and that $\mathfrak{R}=I(R)=I(R)^*$. Thus L is an example for the statements (1) in Theorems 2 and 3.

Example 4. Let R be a solvable Lie algebra over a field of characteristic 0 described in terms of a basis $x_1, x_2, \cdots, x_5$ by the following muitiplication table:

$$[x_1, x_2]=x_2, \qquad [x_1, x_3]=x_3, \qquad [x_1, x_4]=2x_4,$$
$$[x_1, x_5]=3x_5, \qquad [x_2, x_3]=x_4, \qquad [x_2, x_4]=x_5.$$

In addition $[x_i, x_j]=-[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the table above. This was given in [9], p. 213. Let D be a derivation of R and put

$$Dx_i = \sum_{j=1}^{5} \lambda_{ij} x_j \qquad (i=1, 2, \cdots, 5).$$

Then the matrix of D is

$$\begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ 0 & \lambda_{22} & \lambda_{23} & \lambda_{13} & \frac{1}{2}\lambda_{14} \\ 0 & 0 & \lambda_{33} & -\lambda_{12} & 0 \\ 0 & 0 & 0 & \lambda_{22}+\lambda_{33} & -\lambda_{12} \\ 0 & 0 & 0 & 0 & 2\lambda_{22}+\lambda_{33} \end{pmatrix}.$$

From this the matrix of an inner derivation of R is obtained by putting

$$\lambda_{23}=0, \qquad \lambda_{22}=\lambda_{33}.$$

$D(R)$ is solvable and $I(R)$ is algebraic. The semisimple derivation of R defined by

$$Dx_1 = Dx_2 = 0, \qquad Dx_3 = x_3, \qquad Dx_4 = x_4, \qquad Dx_5 = x_5$$

is not contained in $I(R)$. Therefore by using the statement (1) in Theorem 2 we see that R cannot be the radical of a Lie algebra L such that $D(L) = I(L)^* + C(L)$.

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