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A Remark on a Certain G-structure

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Introduction

Since the notion of G-structures on a differentiable manifold M was introduced by S. S. Chern [2][1] in 1953, a number of papers on this subject have been published by many writers, such as D. Bernard, R. S. Clark and M. Bruckheimer. Many structures which appear in differential geometry are closely related to the G-structures defined by certain special tensor fields whose components relative to some covering of M by moving frames are constants. Especially, among the G-structures defined by special vector 1-forms one finds the almost product, the almost complex and the almost tangent structures, etc..

As is well known, for such a G-structure we can define two tensors, that is, the Chern invariant and the Nijenhuis tensor. These two tensors play an impotant role in the theory of connections and the integrability of the G-structures. So far, however, we have known of the relation between them only in some special cases. For example, the Chern invariant vanishes if and only if the Nijenhuis tensor vanishes for almost product, almost complex and almost tangent structures [3]. The main purpose of this paper is to investigate how such a relation will be generalized in the case of the real G-structure defined by any special vector 1-form whose eigenvalues are all real.

As usual, we assume that all the objects we encounter in this paper are of class C^∞.

§ 1. Preliminaries.

In this section we introduce some general notions and symbols that will be used later on, and then state a main theorem.

1) *Chern invariant.* Let us assume that an m-dimensional differentiable manifold M has a G-structure, that is, the frame bundle over M admits a sub-bundle H with structure group G. Suppose Σ to be the torsion tensor of some structure connection, then the components $t\Sigma$ relative to any adapted frame are defined on H and P-valued, where $P=R^m\otimes R_m\wedge R_m$. Let e_i be the vectors of the natural basis for R^m, then $e_i^{jk}=e_i\otimes e^j\wedge e^k$ is a basis for P. We denote

(1) The number in bracket refers to the references at the end of the paper.

by W the subspace consisting of elements $A^i_{jk}(\xi)e_i^{jk}$, being

$$A^i_{jk}(\xi)=A^i_{\rho k}\xi^\rho_j-A^i_{\rho j}\xi^\rho_k$$

where A_ρ is a basis for Lie algebra $\mathfrak{G}$ of G and each ξ^ρ_j is an arbitrary real number. Let Z be some complementary subspace of W and ∂ be the natural projection of P onto Z. Then, following D. Bernard, $C=\partial(t\Sigma)$ is defined on H, Z-valued and independent of the choice of structure connection. We call C the Chern invariant associated with the G-structure. Further, he has proved that, given the covering of M by adapted coframes θ^*, we can calculate locally the Chern invariant C by the following equation:

$$\partial(t\,d\theta^*)=C. \tag{1.1}$$

2) *The G-structure defined by a special vector 1-form.* A vector 1-form h on M is said to be special if for all points $x\in M$, the Jordan canonical form of h_x is equal to a constant matrix $\Omega=\|\Omega^i_j\|$. Then $\bigcup_{x\in M}\{\text{frame } \theta \text{ at } x;\ \theta^{-1}h_x\theta=\Omega\}$ defines a subbundle H of the frame bundle over M, with structure group $G=\{T\in Gl(m;R);\ T\Omega=\Omega T\}$. The subbundle H is called the G-structure defined by h. Now we assume that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of Ω are all real and $\lambda_p\neq\lambda_q$ for $p\neq q$, and each λ_p has the multiplicity m_p respectively. Throughout this note we shall restrict ourselves to such a G-structure.

Let

$$\Omega=\begin{pmatrix}\Omega_1 & & & 0\\ & \Omega_2 & & \\ & & \ddots & \\ 0 & & & \Omega_n\end{pmatrix}$$

be a decomposition of Ω into primary components Ω_p belonging to distinct eigenvalues λ_p. Morever, each Ω_p has a decomposition

$$\Omega_p=\begin{pmatrix}\Omega_{p1} & & & 0\\ & \Omega_{p2} & & \\ & & \ddots & \\ 0 & & & \Omega_{pn_p}\end{pmatrix}\qquad (p=1, 2, \cdots, n) \tag{1.2}$$

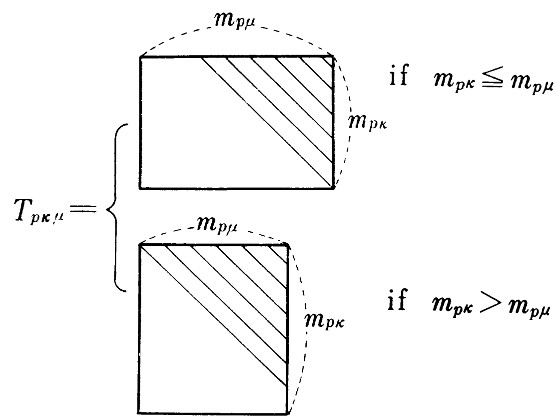
where each $\Omega_{p\nu}$ is the matrix of order $m_{p\nu}$,

$$\Omega_{p\nu}=\begin{cases}[\lambda_p] & \text{if}\quad m_{p\nu}=1\\ \begin{pmatrix}\lambda_p & 1 & & & 0\\ & \lambda_p & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ 0 & & & & \lambda_p\end{pmatrix} & \text{if}\quad m_{p\nu}>1\end{cases}\qquad (\nu=1, 2, \cdots, n_p).$$

In this case, we may assume that $m_{p1} \geqq m_{p2} \geqq \cdots \geqq m_{pn_p} \geqq 1$ for each p.

Then, any element $T \in G$ must be of the form

$$T = \begin{pmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_n \end{pmatrix}, \qquad T_p = \begin{pmatrix} T_{p11} & \cdots & T_{p1n_p} \\ \vdots & & \vdots \\ T_{pn_p1} & \cdots & T_{pn_pn_p} \end{pmatrix} \qquad (p=1, 2, \cdots, n),$$



where T_p is the matrix of order m_p, and $T_{p\kappa\mu}$ is the matrix having $m_{p\kappa}$ rows and $m_{p\mu}$ columns, all elements of which are equal and arbitrary on the obliques, and the other elements of which are zero. Consequently we can assume that a basis A_ρ is the matrix of order m, all elements of which are equal to 1 on one of the obliques, and the other elements of which are zero.

3) *The Nijenhuis tensor.* It is well known that, given any vector 1-form k on M, the Nijenhuis tensor associated with k is a contravariant vector 2-form N defined by

$$N(u, v) = k^2[u, v] + [ku, kv] - k[ku, v] - k[u, kv],$$

where $[u, v]$ denotes the Poisson bracket of vector fields u and v over M. Suppose that θ^* is an adapted coframe for the G-structure defined by the vector 1-form h and that

$$d\theta^i = \frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k. \tag{1.3}$$

Then the corresponding components of N associated with h are given by

$$N^i_{jk} = \Omega^i_r (\Omega^s_j C^r_{sk} + \Omega^s_k C^r_{js}) - \Omega^r_j \Omega^s_k C^i_{rs} - \Omega^i_r \Omega^r_s C^s_{jk}. \tag{1.4}$$

4) *Theorem.* We are now on the point of stating the following theorem.

THEOREM. *Let h be any special vector 1-form whose eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real and let the multiplicity of each λ_p be m_p respectively. Then the Chern invariant for the G-structure defined by h vanishes if and only if the following two conditions are satisfied:*

i) *The Nijenhus tensor associated with h vanishes;*

ii) *The kernel of every $(h-\lambda_p I)^q$ is involutive, where $p=1, 2, \cdots, n$ and $q=1, 2, \cdots, m_p$.*

This theorem generalizes both the result ([3], p. 135) obtained by R. S. Clark and M. Bruckheimer for almost product and almost tangent structures, and the result ([4], THÉORÈME) obtained by J. L. Lejeune for the generalized almost tangent structure. The proof goes as follows. Let

$$R_1(A^i_{jk})=0, \quad R_2(A^i_{jk})=0, \;\cdots\cdots,\; R_\chi(A^i_{jk})=0 \tag{1.5}$$

be a complete system of linearly independent relations among the A's, which will be called the fundamental relations. Then, on account of (1.1) the vanishing of the Chern invariant is characterized by a system of equations

$$R_1(C^i_{jk})=0, \quad R_2(C^i_{jk})=0, \;\cdots\cdots,\; R_\chi(C^i_{jk})=0. \qquad (1.5)_C{}^{(2)}$$

Using this fact, we prove in §2 that the theorem is reduced to the case where h is nilpotent, and then we accomplish in §3 the proof of the theorem for this simple case.

Combining the theorem ([5], p. 967) obtained by E. T. Kobayashi and the above theorem, and attending to Remark in §3, we have

COROLLARY. *Let h be a special vector 1-form whose eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real and whose each λ_p-component Ω_p is equal to*

$$\begin{pmatrix} \lambda_p & & 0 \\ & \lambda_p & \\ & & \ddots & \\ 0 & & & \lambda_p \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} \lambda_p & 1 & & 0 \\ & \lambda_p & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ 0 & & & & \lambda_p \end{pmatrix}.$$

Then the following are equivalent:

a) *The Chern invariant of the G-structure defined by h vanishes;*

b) *The Nijenhuis tensor associated with h vanishes;*

c) *The G-structure defined by h is integrable*(3).

(2) By the subscript C, we mean the substitution of C for A in (1.5). Such a convention will be used later.

(3) The G-structure defined by a special vector 1-form h is integrable if and only if, for each point $x \in M$, we can find a local coordinate system around x, in which the coordinate expression of h is its Jordan canonical form Ω.

It is an immediate consequence of (1.1) that the vanishing of the Chern invariant is necessary for the G-structure to be integrable. However, it should be remarked that this is not always sufficient. Our corollary gives an example of the G-structures in which the vanishing of the Chern invariant is a sufficient condition for its integrability.

§ 2. Reduction of the theorem.

In this section we show that our theorem is reduced to the simple case where h is nilpotent.

1) *Preliminaries.* Let I_p be n sets of integers i such as

$$\sum_{k=1}^{p-1} m_k < i \leqq \sum_{k=1}^{p} m_k \qquad (p=1, 2, \cdots, n),$$

and let $I_{p\nu}$ be n_p sets of integers i such as

$$\sum_{k=1}^{p-1} m_k + \sum_{\kappa=1}^{\nu-1} m_{p\kappa} < i \leqq \sum_{k=1}^{p-1} m_k + \sum_{\kappa=1}^{\nu} m_{p\kappa} \qquad (\nu=1, 2, \cdots, n_p).$$

Let an index u take a range I_p, then we assume that u_ν takes all integers belonging to $I_{p\nu}$ for each ν. For the convenience' sake, the symbols $\tilde{u}$, $\tilde{u}_\nu$ are defined to be

$$\tilde{u} = u - \sum_{k=1}^{p-1} m_k, \qquad \tilde{u}_\nu = u_\nu - \left(\sum_{k=1}^{p-1} m_k + \sum_{\kappa=1}^{\nu-1} m_{p\kappa}\right),$$

and called the normalized indices of I_p, $I_{p\kappa}$ respectively. By contraries, we define

$$i^*(p, \nu) = i + \sum_{k=1}^{p-1} m_k + \sum_{\kappa=1}^{\nu-1} m_{p\kappa} \qquad (i=1, 2, \cdots, m_{p\nu})$$

and write merely i^* for $i^*(p, \nu)$ unless an ambiguity should be introduced.

Now we classify the components Γ^i_{jk} into the following three parts:

$$\begin{array}{ll} [\mathrm{I}] & \Gamma^A_{a\mu} \ \text{for} \ r \neq s, t; \\ [\mathrm{II}] & \Gamma^A_{a\mu} \ \text{for} \ r=s \ \text{and} \ r \neq t, \ \text{or} \ r \neq s \ \text{and} \ r=t; \\ [\mathrm{III}] & \Gamma^A_{a\mu} \ \text{for} \ r=s=t, \end{array}$$

where and throughout this section Γ denotes any one of N, C, A, and $A \in I_r$, $a \in I_s$, $\mu \in I_t$. Then the component $\Gamma^A_{a\mu}$ is called type I, II or III as it belongs to the part [I], [II] or [III], respectively.

If we transcribe each λ_p-component Ω_p in the form

$$\Omega_p=\begin{pmatrix} \lambda_p & \varepsilon_1^* & & & 0 \\ & \lambda_p & \varepsilon_2^* & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varepsilon^*_{(m_p-1)} \\ 0 & & & & \lambda_p \end{pmatrix} \qquad (p=1, 2, \cdots, n),$$

where the ε's are zero or 1 and $*j=j+\sum_{k=1}^{p-1} m_k$ for $j=1, 2, \cdots, m_p-1$, then we have from (1.4)

$$N^A_{a\mu}=(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)C^A_{a\mu}-(2\lambda_r-\lambda_s-\lambda_t)\varepsilon_A C^{A+1}_{a\mu}+\varepsilon_{a-1}(\lambda_r-\lambda_t)C^A_{a-1\mu}$$

$$+\varepsilon_{\mu-1}(\lambda_r-\lambda_s)C^A_{a\mu-1}+\varepsilon_A(\varepsilon_{a-1}C^{A+1}_{a-1\mu}-\varepsilon_{A+1}C^{A+2}_{a\mu})-\varepsilon_{\mu-1}(\varepsilon_{a-1}C^A_{a-1\mu-1}-\varepsilon_A C^{A+1}_{a\mu-1}). \quad (2.1)$$

Hence, any component N^i_{jk} of the Nijenhuis tensor is expressed as a linear combination of some components C^i_{jk} of the same type as that of the given N^i_{jk}. On the other hand, owing to the choice of the basis A_ρ, we may assume that all the components A's which appear in each relation $R_\omega(A^i_{jk})=0$ of (1.5) are of the same type Π, where Π indicates any one of I, II, III. Then, such a relation is called type Π, and the set of the relations of type Π will be denoted by (1.5_Π).

In the following 2) (resp. 3)), we prove that all the components of type I (resp. II) of the Nijenhuis tensor N vanish if and only if the equations $(1.5_{\mathrm{I}})_C$ (resp. $(1.5_{\mathrm{II}})_C$) hold.

2) *Type I.* Evidently, (1.5_{I}) are reduced to

$$A^A_{a\mu}=0 \qquad (r\neq s;\ r\neq t). \quad (2.2)$$

Then we have

PROPOSITION 2.1. *In order that the equations $(2.2)_C$ hold, it is necessary and sufficient that every component of type* I *of the Nijenhuis tensor vanishes.*

PROOF. On account of (2.1), necessity is evident. For the proof of sufficiency, it is enough to consider the case where $A=A_\alpha$ and $\mu=\mu_\beta$.

First, we show that sufficiency holds for the case $\mu_\beta=1^*$. If we put $\mu_\beta=1^*$ in (2.1), we have

$$N^{A_\alpha}_{a1^*}=(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)C^{A_\alpha}_{a1^*}-\varepsilon_{A_\alpha}(2\lambda_r-\lambda_s-\lambda_t)C^{A_\alpha+1}_{a1^*}$$

$$+\varepsilon_{a-1}(\lambda_r-\lambda_t)C^{A_\alpha}_{a-11^*}+\varepsilon_{A_\alpha}(\varepsilon_{a-1}C^{A_\alpha+1}_{a-11^*}-\varepsilon_{A_\alpha+1}C^{A_\alpha+2}_{a1^*}). \quad (2.3)$$

Furthermore, from (2.3) by putting $A_\alpha=\phi^*$, where $\phi=m_{r\alpha}$, we obtain

$$N^{\phi^*}_{a1^*}=(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)C^{\phi^*}_{a1^*}+\varepsilon_{a-1}(\lambda_r-\lambda_t)C^{\phi^*}_{a-11^*}.$$

Remarking that $(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)\neq 0$, we find from the above equations that $N^{\phi^*}_{a1^*}=0$ imply $C^{\phi^*}_{b1^*}=0$ $(a, b \in I_s)$. If we put $A_\alpha=\phi^*-1$ in (2.3) and use the results just obtained, then we have

$$N^{\phi^*-1}_{a1^*}=(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)C^{\phi^*-1}_{a1^*}+\varepsilon_{a-1}(\lambda_r-\lambda_t)C^{\phi^*-1}_{a-11^*},$$

from which we see that $N^{\phi^*-1}_{a1^*}=0$ imply $C^{\phi^*-1}_{b1^*}=0$ $(a, b \in I_s)$. If we repeat in order the similar discussions as above in the cases $A_\alpha=\phi^*-2, \dots, 2^*, 1^*$, then we have the following:

$$N^{A\alpha}_{a1^*}=0 \quad \text{imply} \quad C^{B\alpha}_{b1^*}=0 \qquad (A_\alpha, B_\alpha \in I_{r\alpha};\, a, b \in I_s). \tag{2.4}$$

Hence, sufficiency holds for the case $\mu_\beta=1^*$, as desired.

Secondly, putting $\mu_\beta=2^*$ in (2.1) and then using (2.4), we get

$$\begin{aligned} N^{A\alpha}_{a2^*}=&(\lambda_r-\lambda_s)(\lambda_t-\lambda_r)C^{A\alpha}_{a2^*}-\varepsilon_{A\alpha}(2\lambda_r-\lambda_s-\lambda_t)C^{A\alpha+1}_{a2^*}\\ &+\varepsilon_{a-1}(\lambda_r-\lambda_t)C^{A\alpha}_{a2^*}+\varepsilon_{A\alpha}(\varepsilon_{a-1}C^{A\alpha+1}_{a-12^*}-\varepsilon_{A\alpha+1}C^{A\alpha+2}_{a2^*}). \end{aligned}$$

If we proceed in the similar manner as in the case $\mu_\beta=1^*$ with the above equations, we know that the equations $N^{A\alpha}_{a2^*}=0$ imply $C^{B\alpha}_{b2^*}=0$ $(A_\alpha, B_\alpha \in I_{r\alpha};$ $a, b \in I_s)$.

After we continued this process until the case $\mu_\beta=m^*_{t\beta}$, we can conclude that the equations $N^{A\alpha}_{a\mu_\beta}=0$ imply $C^{B\alpha}_{a\nu_\beta}=0$ $(A_\alpha, B_\alpha \in I_{r\alpha};\, a, b \in I_s;\, \mu_\beta, \nu_\beta \in I_{t\beta})$.

Thus, the proof is completed.

3) *Type II.* Let [II$_1$] (resp. [II$_2$]) be the first (resp. the second) part of [II]. In the following considerations we will be concerned with the components $\Gamma^a_{b\mu}$ of type II$_1$, because, as for the components $\Gamma^\mu_{a\nu}$ of type II$_2$, we have the analogous results in the similar way.

It is easily verified that (1.5_{II_1}) are reduced to the equations

$$A^{a\alpha}_{b\beta\mu}=0 \qquad \text{for} \quad \tilde{b}_\beta-\tilde{a}_\alpha<f(s;\, \alpha, \beta), \tag{2.5}$$

$$A^{a\alpha}_{b\beta\mu}=A^{c\alpha}_{d\beta\mu} \qquad \text{for} \quad \tilde{b}_\beta-\tilde{a}_\alpha=\tilde{d}_\beta-\tilde{c}_\alpha\geqq f(s;\, \alpha, \beta), \tag{2.6}$$

where $a_\alpha, c_\alpha \in I_{s\alpha};\, b_\beta, d_\beta \in I_{s\beta}$, and also

$$f(s;\, \alpha, \beta)=\begin{cases} m_{s\beta}-m_{s\alpha} & \text{if} \quad m_{s\alpha}<m_{s\beta} \\ 0 & \text{if} \quad m_{s\alpha}\geqq m_{s\beta}. \end{cases}$$

On the other hand, if we put

$$P^a_{b\mu}=\varepsilon_{b-1}C^a_{b-1\mu}-\varepsilon_a C^{a+1}_{b\mu}, \tag{2.7}$$

then the equations (2.1) turn out to be

$$N^{a}_{b\mu}=(\lambda_s-\lambda_t)P^{a}_{b\mu}+\varepsilon_a P^{a+1}_{b\mu}-\varepsilon_{\mu-1}P^{a}_{b\mu-1}. \tag{2.8}$$

Similarly as in Proposition 2.1, we have

LEMMA 2.1. $N^{a}_{b\mu}=0$ *if and only if* $P^{c}_{d\nu}=0$ $(a, b, c, d \in I_s; \mu, \nu \in I_t)$.

We now prove

PROPOSITION 2.2. *In order that the equations* $(2.5)_C$ *and* $(2.6)_C$ *hold, it is necessary and sufficient that every component of type* II_1 *of the Nijenhuis tensor vanishes.*

PROOF. For the proof, owing to Lemma 2.1, it is enough to show that $(2.5)_C$ and $(2.6)_C$ hold if and only if $P^{a}_{b\mu}=0$ $(a, b \in I_s; \mu \in I_t)$.

Necessity. By applying respectively $(2.5)_C$ or $(2.6)_C$ to (2.7), according as $\tilde{b}_\beta-\tilde{a}_\alpha-1<f(s;\alpha,\beta)$ or $\geqq f(s;\alpha,\beta)$, it will be easily verified that we have $P^{a_\alpha}_{b_\beta\mu}=0$ $(a_\alpha \in I_{s\alpha}, b_\beta \in I_{s\beta}, \mu \in I_t)$.

Sufficiency. In the first place, let us assume $f(s;\alpha,\beta)=0$. Then it is sufficient to show that from the equations $P^{a_\alpha}_{b_\beta\mu}=0$ we have

$$C^{a_\alpha}_{b_\beta\mu}=0 \qquad \text{for} \quad \tilde{a}_\alpha>\tilde{b}_\beta, \tag{2.5$'_C$}$$

$$C^{a_\alpha}_{b_\beta\mu}=C^{c_\alpha}_{d_\beta\mu} \qquad \text{for} \quad \tilde{b}_\beta-\tilde{a}_\alpha=\tilde{d}_\beta-\tilde{c}_\alpha\geqq 0, \tag{2.6$'_C$}$$

where $a_\alpha, c_\alpha \in I_{s\alpha}$; $b_\beta, d_\beta \in I_{s\beta}$; $\mu \in I_t$. First, suppose $b_\beta=1^*$ then $P^{a_\alpha}_{1^*\mu}=-\varepsilon_{a_\alpha}C^{a_\alpha+1}_{1^*\mu}=0$ $(1^*\leqq a_\alpha\leqq m^*_{s\alpha}, \mu \in I_t)$, from which one gets

$$C^{b_\alpha}_{1^*\nu}=0 \qquad (2^*\leqq b_\alpha\leqq m^*_{s\alpha}, \nu \in I_t), \tag{2.9}$$

that is to say, $(2.5)'_C$ with $b_\beta=1^*$. Secondly, suppose $b_\beta=2^*$ $(\leqq m^*_{s\beta})$ then $P^{a_\alpha}_{2^*\mu}=C^{a_\alpha}_{1^*\mu}-\varepsilon_{a_\alpha}C^{a_\alpha+1}_{2^*\mu}=0$ $(1^*\leqq a_\alpha\leqq m^*_{s\alpha}, \mu \in I_t)$. Here each equation $P^{1^*}_{2^*\mu}=0$ gives

$$C^{1^*}_{1^*\mu}=C^{2^*}_{2^*\mu} \qquad (\mu \in I_t), \tag{2.10}$$

and, when $2^*\leqq a_\alpha\leqq m^*_{s\alpha}$, on account of (2.9) we have $P^{a_\alpha}_{2^*\mu}=-\varepsilon_{a_\alpha}C^{a_\alpha+1}_{2\ \mu}=0$, from which we have

$$C^{b_\alpha}_{2^*\nu}=0 \qquad (3^*\leqq b_\alpha\leqq m^*_{s\alpha}, \nu \in I_t), \tag{2.11}$$

that is to say, $(2.5)'_C$ with $b_\beta=2^*$. Thirdly, suppose $b_\beta=3^*$ $(\leqq m^*_{s\beta})$ then $P^{a_\alpha}_{3^*\mu}=C^{a_\alpha}_{2^*\mu}-\varepsilon_{a_\alpha}C^{a_\alpha+1}_{3^*\mu}=0$ $(1^*\leqq a_\alpha\leqq m^*_{s\alpha}, \mu \in I_t)$. Here, $P^{1^*}_{3^*\mu}=0$ gives

$$C^{1^*}_{2^*\mu}=C^{2^*}_{3^*\mu} \qquad (\mu \in I_t) \tag{2.12}$$

and $P^{2^*}_{3^*\mu}=0$ gives

$$C^{2^*}_{2^*\mu}=C^{3^*}_{3^*\mu} \qquad (\mu \in I_t), \tag{2.13}$$

and also, from $P^{a\alpha}_{3^*\mu}=0$ $(3^* \leqq a_\alpha \leqq m^*_{s\alpha},\ \mu \in I_t)$ and (2.11), we have

$$C^{b\alpha}_{3^*\nu}=0 \qquad (4^* \leqq b_\alpha \leqq m^*_{s\alpha},\ \nu \in I_t), \tag{2.14}$$

that is to say, $(2.5)'_C$ with $b_\beta=3^*$. After repeating successively this process in the cases $b_\beta=4^*, \cdots, m^*_{s\alpha}$, taking (2.9), (2.11), (2.14), etc. together, we have $(2.5)'_C$, and also taking (2.10), (2.12), (2.13), etc. together, we have $(2.6)'_C$.

In the second place, let us assume $f(s;\alpha,\beta)>0$. Then it is sufficient to show that from the equations $P^{a\alpha}_{b\beta\mu}=0$ we have

$$C^{a\alpha}_{b\beta\mu}=0 \qquad \text{for} \quad \tilde{b}_\beta-\tilde{a}_\alpha<f(s;\alpha,\beta), \tag*{$(2.5)''_C$}$$

$$C^{a\alpha}_{b\beta\mu}=C^{c\alpha}_{d\beta\mu} \qquad \text{for} \quad \tilde{b}_\beta-\tilde{a}_\alpha=\tilde{d}_\beta-\tilde{c}_\alpha \geqq f(s;\alpha,\beta), \tag*{$(2.6)''_C$}$$

where $a_\alpha, c_\alpha \in I_{s\alpha}$; $b_\beta, d_\beta \in I_{s\beta}$; $\mu \in I_t$. The proof goes in the similar way as in the case $f(s;\alpha,\beta)=0$, but we start with $a_\alpha=m^*_{s\alpha}$. Then the equations $P^{m^*_{s\alpha}}_{b\beta\mu}=\varepsilon_{b_\beta-1}C^{m^*_{s\alpha}}_{b_\beta-1\mu}=0$ $(b_\beta \in I_{s\beta},\ \mu \in I_t)$ give

$$C^{m^*_{s\alpha}}_{c\beta\nu}=0 \qquad (1^* \leqq c_\beta \leqq m^*_{s\beta}-1,\ \nu \in I_t). \tag{2.15}$$

In the next place, suppose $a_\alpha=m^*_{s\alpha}-1$, then we have $P^{m^*_{s\alpha}-1}_{b\beta\mu}=\varepsilon_{b_\beta-1}C^{m^*_{s\alpha}-1}_{b_\beta-1\mu}-C^{m^*_{s\alpha}}_{b\beta\mu}=0$ $(b_\beta \in I_{s\beta},\ \mu \in I_t)$. Here, when $1^* \leqq b_\beta \leqq m^*_{s\beta}-1$, we get $P^{m^*_{s\alpha}}_{b\beta\mu}=\varepsilon_{b_\beta-1}C^{m^*_{s\alpha}-1}_{b_\beta-1\mu}=0$ in consequence of (2.15), from which we obtain

$$C^{m^*_{s\alpha}-1}_{c\beta\nu}=0 \qquad (1^* \leqq c_\beta \leqq m^*_{s\beta}-2,\ \nu \in I_t). \tag{2.16}$$

And also, when $b_\beta=m^*_{s\beta}$, we get

$$C^{m^*_{s\alpha}-1}_{m^*_{s\alpha}-1\mu}=C^{m^*_{s\alpha}}_{m^*_{s\alpha}\mu} \qquad (\mu \in I_t). \tag{2.17}$$

Repeating this process in the cases $a_\alpha=m^*_{s\alpha}-2, \cdots, 1^*$, we have $(2.5)''_C$ from (2.15), (2.16), etc., and we have $(2.6)''_C$ from (2.17), etc., as desired.

Thus the proof is completed.

4) *Reduction of the theorem.* Finally we consider the condition ii) in our theorem, which is now written explicitly as follows:

$$\begin{aligned} &C^A_{ab}=0 \qquad (r \neq s) && (2.18)\\ &C^{a\alpha}_{a\beta a\gamma}=0 \qquad \text{for} \quad \tilde{a}_\alpha>\max(\tilde{a}_\beta,\tilde{a}_\gamma) && (2.19) \end{aligned} \qquad \begin{pmatrix} r, s=1, 2, \cdots, n; \\ \alpha, \beta, \gamma=1, 2, \cdots, n_s \end{pmatrix}.$$

Since (2.18) is automatically satisfied if every component of type I of the Chern invariant C vanishes and the equations (2.19), so to speak, are of type III, we find from Propositions 2.1 and 2.2 that the theorem has been reduced to the case in which Ω consists of a single component Ω_s. In this case, $\Omega=\Omega_s$, the vector 1-form h may be changed into $h-\lambda_s I$ without altering the condition ii), namely, the equations (2.19). Moreover, any special vector 1-form k may be changed into $k-cI$ (c=const.) without altering both the Nijenhuis tensor N and the structure group G associated with it. Consequently, we can assume that h is nilpotent. Thus, the theorem has been reduced to the case where h is nilpotent.

§ 3. Proof of the theorem when h is nilpotent.

The purpose of this section is to prove the theorem in the case where h is any special nilpotent vector 1-form.

1) *Preliminaries.* Now, let Φ be the Jordan canonical form of h and let

$$\Phi=\begin{pmatrix} \Phi_1 & & & 0 \\ & \Phi_2 & & \\ & & \ddots & \\ 0 & & & \Phi_\pi \end{pmatrix}$$

be its decomposition corresponding to (1.2), where each Φ_τ is of order l_τ and of the form

$$\Phi_\tau=\begin{cases} \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix} & \text{if} \quad l_\tau>1 \\ [0] & \text{if} \quad l_\tau=1 \end{cases} \qquad (\tau=1, 2, \cdots, \pi).$$

Here, we may assume that $l_1 \geqq l_2 \geqq \cdots \geqq l_\pi$. Let us transcribe Φ in the form

$$\Phi=\begin{pmatrix} 0 & \varepsilon_1 & & & 0 \\ & 0 & \varepsilon_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varepsilon_{m-1} \\ 0 & & & & 0 \end{pmatrix},$$

then the equations (2.1) turn out to be

$$N^i_{jk}=-\varepsilon_{j-1}\varepsilon_{k-1}C^i_{j-1\,k-1}-\varepsilon_i\varepsilon_{i+1}C^{i+2}_{jk}+\varepsilon_i(\varepsilon_{j-1}C^{i+1}_{j-1\,k}+\varepsilon_{k-1}C^{i+1}_{j\,k-1}). \tag{3.1}$$

Let the index ranges J_τ be the sets of integers p such as $\sigma_{\tau-1}<p\leqq\sigma_\tau$

where $\sigma_0=0$ and $\sigma_\tau=\sum_{u=1}^{\tau} l_u$ $(\tau=1, 2, \cdots, \pi)$. A notation "—" relative to a fixed range J_α means to operate on each index $\iota \in J_\tau$ as follows:

$$\bar{\iota}=\iota-\sigma_{\tau-1}-f(\alpha, \tau) \tag{3.2}$$

where

$$f(\alpha, \tau)=\begin{cases} l_\tau-l_\alpha & \text{if } l_\tau>l_\alpha \\ 0 & \text{if } l_\tau \leqq l_\alpha. \end{cases}$$

The $\bar{\iota}$ is called the normalized index of J_τ and has the range

$$-f(\alpha, \tau)<\bar{\iota} \leqq l'_\tau,$$

where $l'_\tau=\min(l_\alpha, l_\tau)$. In the sequel, we assume that the letters a, b and c denote the normalized idices of the fixed ranges J_α, J_β and J_γ, respectively, and we are concerned only with the normalized components Γ^a_{bc}, which have 1-1 correspondence with the ordinary components Γ^i_{jk} by means of (3.2), where $i \in J_\alpha$, $j \in J_\beta$, $k \in J_\gamma$. Without loss of generality, we may assume $\beta \leqq \gamma$, so that $l_\beta \geqq l_\tau$, because Γ^i_{jk} is skew-symmetric with respect to the lower indices j, k.

It is desirable now to introduce a convention which will be used throughout this section, namely, that when any one of the indices a, b and c appearing in a component Γ^a_{bc} in question lies outside of its index range, it is understood that the component Γ^a_{bc} is zero. Using this convention, we can rewrite (3.1) in the following form:

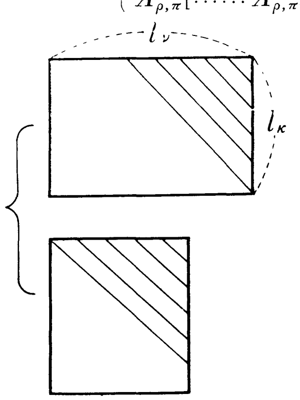
$$N^a_{bc}=-C^a_{b-1\ c-1}-C^{a+2}_{bc}+C^{a+1}_{b-1\,c}+C^{a+1}_{b\,c-1}. \tag{3.1$'$}$$

2) *The fundamental relations.* Here, we intend to seek an explicit form of the fundamental relations (1.5) which are nothing but the relations (1.5_{III}) in the sense of §2.

The basis A_ρ is now of the form

$$A_\rho=\begin{pmatrix} A_{\rho,11} & \cdots\cdots & A_{\rho,1\pi} \\ \vdots & & \vdots \\ A_{\rho,\pi 1} & \cdots\cdots & A_{\rho,\pi\pi} \end{pmatrix},$$

$A_{\rho,\kappa\nu}=$



if $l_\kappa \leqq l_\nu$

if $l_\kappa > l_\nu$,

all elements of which are equal to 1 on one of the obliques, and the other elements of which are zero. Let us give a number to the obliques from the left at each block, and let us write $\bar{\rho}(\kappa, \nu)=i$ if the non-zero oblique of A_ρ is the i-th one at the (κ, ν)-block $A_{\rho,\kappa\nu}$. Then we have

$$A^a_{\rho b}=\begin{cases}1; & \text{if } b-a+1=\bar{\rho}(\alpha, \beta)\\ 0; & \text{if otherwise.}\end{cases} \tag{3.3}$$

As an immediate consequence of (3.3) we have

$$\text{(I)}\quad A^a_{bc}=0 \qquad \text{for} \quad a>\max(b, c).$$

The $A^a_{bc}(\xi)$ is said to be of the first kind, or of the second kind according as the ξ's occur exactly once, or twice in its definition, $A^a_{bc}(\xi)=A^a_{\rho c}\xi^\rho_b-A^a_{\rho b}\xi^\rho_c$. We have then useful characterizations for components to be a certain kind, namely, that $A^a_{bc}(\xi)$ is of the first kind if and only if $\max(b, c)\geqq a>\min(b, c)$, and it is of the second kind if and only if $a\leqq\min(b, c)$.

It is easy to verify the following relations which consist of the components of the first kind:

$$\text{(II)}\quad \begin{cases}(\mathrm{II}_1)\quad A^{c+l}_{c+p+l\ c}-A^{c+1}_{c+p+1\ c}=0\\ (\mathrm{II}_2)\quad A^{b+l}_{b\ b+p+l}-A^{b+1}_{b\ b+p+1}=0\end{cases} \qquad (p\geqq 0,\ l\geqq 2),$$

where $1\leqq b+1,\ b+l,\ c+1,\ c+l\leqq l'_\alpha;\ -f(\alpha, \beta)<c+p+1,\ c+p+l\leqq l'_\beta;$ $-f(\alpha, \gamma)<b+p+1,\ b+p+l\leqq l'_\gamma$.

Now, let us define as follows:

$$R(a; r, s)=A^a_{a+r\ a+s}-A^{a+s+1}_{a+r+s+1\ a+s}-A^{a+r+1}_{a+r\ a+r+s+1}$$
$$(r, s\geqq 0;\ a+r+s<l'_\gamma),$$
$$S(a; u, s)=A^a_{a+u\ a+s}-A^{a+1}_{a+u\ a+s+1}-A^{a+1}_{a+u+1\ a+s}+A^{a+2}_{a+u+1\ a+s+1}$$
$$(s\geqq 0,\ u\geqq 1;\ a+u<l'_\beta,\ a+s<l'_\gamma).$$

Then, a direct calculation gives $R(a; r, s)=0$ and $S(a; u, s)=0$; in particular it follows

$$\text{(III)}\quad R(a; r, s)=0$$

and

$$\text{(IV)}\quad S(a; u, s)=0 \qquad \text{for} \quad a+u+s\geqq l'_\gamma.$$

In what follows, we shall show that the fundamental relations (1.5) are reduced to the equations (I)$\sim$(IV). For this purpose, it is enough to prove that any relation among the A's can be expressed in terms of the relations (I)$\sim$(IV), because they are evidently independent of each other. The types of the relations to be considered are as follows:

(1) The relations in which only the components of the first kind appear;
(2) The relations in which only the components of the second kind appear;
(3) The relations in which the components of both kinds appear.

First of all, by using the characterization for A^a_{bc} to be of the first kind, it is verified that the relations (II) construct a complete system of the relations for the case (1).

In the second place, let us define as follows:

$$V(a;r,s;u,v)=A^{a}_{a+r+u\ a+s+v}-A^{a+u}_{a+r+u\ a+s+u+v}-A^{a+v}_{a+r+u+v\ a+s+v}+A^{a+u+v}_{a+r+u+v\ a+s+u+v}$$
$$(r, s\geqq 0,\ u, v\geqq 1;\ a+r+u+v\leqq l'_\beta,\ a+s+u+v\leqq l'_\gamma).$$

Then, it is seen that the relations $V(a;r,s;u,v)=0$ form a complete system of the relations for the case (2). And that they are derived from (III) and (IV). In fact, on account of the identity

$$V(a;r,s;u,v)=\sum_{k=1}^{u}\sum_{h=1}^{v}S(a+u+v-h-k;\,r+k,\,s+h),$$

the relations $V(a;r,s;u,v)=0$ are obtained from $S(a;u,s)=0$ $(s\geqq 1)$. And yet, as an immediate consequence of the following Lemma 3.1, the latter is derived from (III) and (IV).

Lemma 3.1. *The relations* (III) *and* (IV) *are equivalent to*

$$(\mathrm{III})'\quad\begin{cases}(\mathrm{III}_1)' & R(a;r,0)=0\\ (\mathrm{III}_2)' & R(a;0,s)=0\end{cases}$$

and

$$(\mathrm{IV})'\quad\begin{cases}(\mathrm{IV}_1)' & S(a;u,s)=0 \qquad (s\geqq 1)\\ (\mathrm{IV}_2)' & S(a;u,0)=0 \qquad (a+u\geqq l'_\gamma).\end{cases}$$

Proof. By making use of the identity

$$S(a;u,s)=R(a;u,s)-R(a+1;u-1,s)-R(a+1;u,s-1)+R(a+2;u-1,s-1)$$
$$(s\geqq 1;\ a+u+s<l'_\gamma),$$

we find that if we assume the relations (III)$'$ then the relations $S(a;u,s)=0$ $(s\geqq 1; a+u+s<l'_\gamma)$ are equivalent to (III).

Thus the proof is completed.

Finally, all types of the relations coming under the case (3) are as follows;

(i) The relations, in which a component of the second kind is expressed as the sum of two components of the first kind;

(ii) The relations, in which a sum of two components of the second kind is expressed as the sum of two components of the first kind.

It is easily found that the relations (III) give a complete system for the relations, which are of type (i) and independent of (II). On the other hand, a complete system for the relations, which are of type (ii) and independent of (II), is constructed by the following:

$$A^{a}_{a+r\ a+s+u}-A^{a+r+1}_{a+r\ a+r+s+u+1}-A^{a+u}_{a+r+u\ a+s+u}+A^{a+r+u+1}_{a+r+u\ a+r+s+u+1}=0 \tag{3.4}$$

$$(r,\ s\geqq 0,\ u\geqq 1;\ a+r+s+u<l'_\gamma),$$

$$W(a;u;r,s)\equiv A^{a}_{a+r+u\ a+s}-A^{a+u}_{a+r+u\ a+s+u}+A^{a+s+u+1}_{a+r+s+u+1\ a+s+u}-A^{a+s+1}_{a+r+s+u+1\ a+s}=0$$

$$(r,\ s\geqq 0,\ u\geqq 1;\ a+r+s+u<l'_\beta,\ a+s+u\leqq l'_\gamma).$$

But, these relations are derived from (III) and (IV). Indeed, the equations (3.4) are nothing but the equations

$$R(a;r,s+u)-R(a+u;r,s)=0.$$

Next, we consider the relations $W(a;u;r,s)=0$. Since $S(a;u,0)=R(a;u,0)-R(a+1;u-1,0)$, we have $S(a;u,0)=0$ $(a+u<l'_\gamma)$ from (III) with $s=0$. Taking these relations and (IV) with $s=0$ together, we have $S(a;u,0)=0$, or equivalently, $W(a;1;r,0)=0$. Futhermore, we find that the relations $W(a;1;r,0)=0$ imply $W(a;u;r,0)=0$, because we have

$$W(a;u;r,0)=\sum_{h=1}^{u}W(a+u-h;1;r+h-1,0).$$

And that we have the identity

$$W(a;u;r,w)=W(a+w;u;r,0)+V(a;r,0;u,w) \qquad (w\geqq 1),$$

so the relations $W(a;u;r,0)=0$ imply $W(a;u;r,s)=0$ because (III) and (IV) give $V(a;r,s;u,v)=0$ as it has been shown. Hence the relations $W(a;u;r,w)=0$ are derived from (III) and (IV).

Thus, any relation among the A's is derived from the relations (I)$\sim$(IV). Therefore, we have

PROPOSITION 3.1. *The relations* (I)$\sim$(IV) *are the fundamental relations.*

3) *Proof of the theorem.* We are now on the point of proving the theorem where h is nilpotent. For this aim, it is enough to verify that the equations $(\mathrm{I})_C\sim(\mathrm{IV})_C$ are equivalent to the following:

i)′ $N^a_{bc}=0$;
ii)′ $C^a_{bc}=0$ for $a>\max(b+f(\alpha, \beta), c+f(\alpha, \gamma))$.

In the first place, we have

LEMMA 3.2. *When* $l_\alpha\geqq l_\beta\geqq l_\gamma$, *the equations* $(\mathrm{I})_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad a\geqq\max(b, c) \tag{3.5}$$

and the equations ii)′. *When* $l_\beta\geqq l_\gamma>l_\alpha$, *the equations* $(\mathrm{I})_C$ *are equivalent to* (3.5).

PROOF. *Necessity.* Evidently, the equations $(\mathrm{I})_C$ imply ii)′, and also they give (3.5) because of (3.1)′. *Sufficiency.* When $l_\alpha\geqq l_\beta\geqq l_\gamma$, $(\mathrm{I})_C$ and ii)′ are identical; hence the sufficiency is trivial. When $l_\beta\geqq l_\gamma>l_\alpha$, we can obtain the equations $(\mathrm{I})_C$ from (3.5) in the similar way as in Proposition 2.1.

EXAMPLE (cf. [5], p. 976). Let M be the 4-dimensional euclidean space and suppose x, y, z, t are the coordinates. Let

$$X_1=\frac{\partial}{\partial x}, \quad X_2=\frac{\partial}{\partial y}, \quad X_3=\frac{\partial}{\partial z}, \quad X_4=\frac{\partial}{\partial t}+(1+z)\frac{\partial}{\partial y},$$

and define h by $hX_1=hX_3=hX_4=0$, and $hX_2=X_1$. Then the Nijenhuis tensor of h vanishes but we have $C^2_{34}=-1$, relative to the adapted frame X_i ($i=1, 2, 3, 4$). Therefore, the condition ii)′ is essential for the completeness of Lemma 3.2.

LEMMA 3.3. *When* $l_\alpha\geqq l_\beta\geqq l_\gamma$ *or* $l_\beta\geqq l_\gamma>l_\alpha$, *if we assume* $(\mathrm{I})_C$ *then the equations* $(\mathrm{II})_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad \max(b, c)>a\geqq\min(b, c). \tag{3.6}$$

PROOF. If $\max(b, c)=b$, it follows $b>a\geqq c$. So we set $a=c+s$ ($s\geqq 0$) and $b=c+s+u$ ($u\geqq 1$). Then we have

$$N^a_{bc}=N^{c+s}_{c+s+u\ c}=(C^{c+s+1}_{c+s+u\ c-1}-C^{c+s}_{c+s+u-1\ c-1})-(C^{c+s+2}_{c+s+u\ c}-C^{c+s+1}_{c+s+u-1\ c}). \tag{3.7}$$

On the other hand, if $\max(b, c)=c$, it follows $c>a\geqq b$. So we set $a=b+s$ $(s\geqq 0)$ and $c=b+s+u$ $(u\geqq 1)$. Then we have

$$N_{b\,c}^{a}=N_{b\,b+s+u}^{b+s}=(C_{b-1\,b+s+u}^{b+s+1}-C_{b-1\,b+s+u-1}^{b+s})-(C_{b\,b+s+u}^{b+s+2}-C_{b\,b+s+u-1}^{b+s+1}). \quad (3.8)$$

Necessity. (3.6) follows because the four brackets in (3.7) and (3.8) vanish respectively in consequence of $(\mathrm{I})_C$ or $(\mathrm{II})_C$.

Sufficiency. Let us show that the equations $N_{b\,c}^{a}=0$ $(b>a\geqq c)$ give $(\mathrm{II}_1)_C$. When $u=1$, the second bracket of (3.7) vanishes because of $(\mathrm{I})_C$; hence we get $C_{c+s+1\,c-1}^{c+s+1}-C_{c+s\,c-1}^{c+s}=0$, which give $(\mathrm{II}_1)_C$ with $p=0$. When $u=2$, the second bracket of (3.7) vanishes because of $(\mathrm{II}_1)_C$ with $p=0$; hence we have $C_{c+s+2\,c-1}^{c+s+1}-C_{c+s+1\,c-1}^{c+s}=0$, which give $(\mathrm{II}_1)_C$ with $p=1$. Putting in order $u=3, 4, \cdots$, in (3.7) and repeating this process, we have succesively $(\mathrm{II}_1)_C$ with $p=2, 3, \ldots$. Thus the equations $N_{b\,c}^{a}=0$ $(b>a\geqq c)$ give $(\mathrm{II}_1)_C$. On the other hand, the similar arguments with (3.8) instead of (3.7) can be applied to establish that the equations $N_{b\,c}^{a}=0$ $(c>a\geqq b)$ give $(\mathrm{II}_2)_C$.

Thus the proof is completed.

LEMMA 3.4. *When $l_\beta>l_\alpha\geqq l_\gamma$, the equations* $(\mathrm{I})_C$ *and* $(\mathrm{II})_C$ *are equivalent to*

$$N_{b\,c}^{a}=0 \qquad \text{for} \quad a\geqq\min(b, c). \tag{3.9}$$

PROOF. *Necessity.* Similarly as in Lemmas 3.2 and 3.3, $(\mathrm{I})_C$ or $(\mathrm{II})_C$ implies (3.5) or (3.6) respectively. Hence we have (3.9) from $(\mathrm{I})_C$ and $(\mathrm{II})_C$.

Sufficiency. The proof of sufficiency is based on the following three steps. First, in consequence of (3.9), where $b>a\geqq c$, we have $(\mathrm{II}_1)_C$ and

$$C_{c+s\,c}^{c+s+1}=0 \qquad (s\geqq 0;\ c+s<l_\alpha). \tag{3.10}$$

In fact, by putting in order $c=1, 2, \cdots, \min(l_\alpha-1, l_\gamma)$ in (3.6), we obtain $(\mathrm{II}_1)_C$ and (3.10) in the similar way as in Proposition 2.1. Secondly, we get $(\mathrm{I})_C$ from (3.10) and the equations

$$N_{b\,c}^{a}=0 \qquad \text{for} \quad a\geqq\max(b, c). \tag{3.11}$$

In fact, when $c=1$ we find that (3.10) and (3.11) give $(\mathrm{I})_C$ with $c=1$. When $c=2$, in consequence of the result just obtained we have $(\mathrm{I})_C$ with $c=2$ from (3.10) and (3.11). Repeating this process in the cases $c=3, \cdots, l_\gamma$, we gain $(\mathrm{I})_C$. Finally, using (3.8) as in Lemma 3.3, we have $(\mathrm{II}_2)_C$ from (3.9), where $c>a\geqq b$.

Thus the proof is completed.

Combining Lemmas 3.2, 3.3 and 3.4, we have

PROPOSITION 3.2. *The equations* $(\mathrm{I})_C$ *and* $(\mathrm{III})_C$ *are equivalent to* (3.9) *and* ii)$'$.

LEMMA 3.5. *On the assumption* $(\mathrm{I})_C$, *the equations* $(\mathrm{III})'_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad a+1=\min(b, c) \quad \textit{and} \quad \max(b, c)\leqq l'_\gamma.$$

PROOF. When $b=c$, it follows $b=c=a+1$. Then by making use of $(\mathrm{I})_C$ we have

$$N^a_{bc}=N^a_{a+1a-1}=-R(a;0,0)_C. \tag{3.12}$$

When $b<c$, it follows $b=a+1$ and $c=a+w+1$ $(w\geqq 1)$, and we have

$$N^a_{bc}=N^a_{a+1\ a+w+1}=R(a+1;0,w-1)_C-R(a;0,w)_C. \tag{3.13}$$

Also when $b>c$, it follows $c=a+1$ and $b=a+w+1$ $(w\geqq 1)$, and we have

$$N^a_{bc}=N^a_{a+w+1\ a+1}=R(a+1;w-1,0)_C-R(a;w,0)_C. \tag{3.14}$$

Necessity. From (3.12), (3.13) and (3.14), it is evident.

Sufficiency. Now we assume that the right sides of (3.12), (3.13) and (3.14) are respectively zero. In the first place, from (3.12) we have $R(a;0,0)_C=0$. From (3.13) we have $R(a+1;0,w-1)_C=R(a;0,w)_C$, so that $R(a;0,w)_C=R(a+w;0,0)_C$. And these give $(\mathrm{III}_2)'_C$ because of $R(a;0,0)_C=0$. Similarly, from (3.14) we have $R(a+1;w-1,0)_C=R(a;w,0)_C$, so that $R(a;w,0)_C=R(a+w;0,0)_C$. And these give $(\mathrm{III}_1)'_C$ because of $R(a;0,0)_C=0$.

Thus the proof is completed.

LEMMA 3.6. *The equations* $(\mathrm{IV}_2)'_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad a+1=\min(b, c) \quad \textit{and} \quad \max(b, c)>l'_\gamma.$$

PROOF. Since $\max(b, c)=b$, we may set $c=a+1$ and $b=a+w+1$ $(w\geqq 1)$. Then we have

$$N^a_{bc}=N^a_{a+w+1\ a+1}=-S(a;w,0)_C \qquad (a+w\geqq l_\gamma).$$

Therefore, the proof is evident.

LEMMA 3.7. *The equations* $(\mathrm{IV}_1)'_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad a+1<\min(b, c).$$

Proof. When $c \geqq b$, we may set $b=a+u+1$ $(u \geqq 1)$ and $c=a+u+s+1$ $(s \geqq 0)$. Then we have

$$N^a_{bc}=N^a_{a+u+v+1\ a+u+1}=-S(a;\ u+v,\ u)_C.$$

On the other hand, when $b>c$, we may set $c=a+u+1$ $(u \geqq 1)$ and $b=a+u+v+1$ $(v \geqq 1)$. Then we have

$$N^a_{bc}=N^a_{a+u+v+1\ a+u+1}=-S(a;\ u+v,\ u)_C.$$

As an immediate consequence of these relations, we have Lemma 3.7.

Taking Lemmas 3.1, 3.5, 3.6 and 3.7 together, we have

Proposition 3.3. *On the assumption* $(\mathrm{I})_C$, *the equations* $(\mathrm{III})_C$ *and* $(\mathrm{IV})_C$ *are equivalent to*

$$N^a_{bc}=0 \qquad \textit{for} \quad a<\min(b,\ c).$$

Combining Propositions 3.1, 3.2 and 3.3, we find that the proof of our theorem where h is nilpotent has been accomplished.

Remark. We can remove the condition ii)$'$ from the first half of Lemma 3.2, when $l_\alpha=l_\beta=l_\gamma$. In fact, we can obtain $(\mathrm{I})_C$ from (3.5) in the similar way as in Proposition 2.1. Hence, when $l_1=l_2=\cdots=l_\pi$; in particular

$$\Phi=0 \quad \text{or} \quad \Phi=\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix},$$

the Chern invariant vanishes if and only if the Nijenhuis tensor vanishes.

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