J. Sci. Hiroshima Univ. Ser. A-I
29 (1965), 225-241

On the Multiplicative Products of x_+^α and x_+^β

Mitsuyuki Itano

(Received September 24, 1965)

In my previous paper [3] we examined the relationship between the different approaches of defining multiplication between distributions. We consider only distributions defined on the real line R. The definition of multiplicative product due to Y. Hirata and H. Ogata [2] is equivalent to the one given by J. Mikusiński [4]. In the sequel the multiplicative product in this sense of two distributions S, T, if it exists, will be denoted by ST. We have shown in [6] that ST exists if and only if $(\phi S)*\check{T}$, $\phi \in \mathcal{D}$, when restricting it to a neighbourhood of 0, is a bounded function continuous at 0. Another approach suggested by H. G. Tillmann runs as follows: let $\hat{S}(z)$ and $\hat{T}(z)$ be locally analytic functions corresponding to S and T respectively ([7], p. 122). Putting $\hat{S}_\varepsilon(x)=\hat{S}(x+i\varepsilon)-\hat{S}(x-i\varepsilon)$ and $\hat{T}_\varepsilon(x)=\hat{T}(x+i\varepsilon)-\hat{T}(x-i\varepsilon)$, $\varepsilon>0$, he defined the product $S\cdot T$ to be $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon\hat{T}_\varepsilon$ if it exists, or more generally the finite part of $\hat{S}_\varepsilon\hat{T}_\varepsilon$ (in Hadamard's sense) if it exists. As in my previous paper [3], we understand by $S\bigcirc T$ the distributional limit $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon\hat{T}_\varepsilon$ if it exists. We have shown in [3] that if ST exists, then $S\bigcirc T$ exists and coincides with ST, but not conversely.

The main purpose of this paper is to make a comparison between the various multiplications indicated above when S and T are x_+^α and x_+^β respectively.

1. Preliminaries

It is shown in [6] that if $\dfrac{dS}{dx}T$ exists, then $S\dfrac{dT}{dx}$, ST exist and $\dfrac{d}{dx}(ST)$ $=\dfrac{dS}{dx}T+S\dfrac{dT}{dx}$. Let $\mathcal{D}'_+$ be the set of all distributions with supports in the positive real axis.

Proposition 1. *Let Y be the Heaviside function. Let T be $\dfrac{dS}{dx}$. Then YT exists if and only if there exists a neighbourhood U of 0 in R such that S is a bounded function in U and is continuous at 0. When YT exists, $YT=$ $\dfrac{d}{dx}(YS)-S(0)\delta$ and especially $YT=T$ for $T\in\mathcal{D}'$.*

Proof. Suppose YT exists. Then YS exists. In view of the relation:

$$(\phi T)*\check{Y}=-\phi S-\left(\frac{d\phi}{dx}S\right)*\check{Y}, \qquad \phi \in \mathcal{D},$$

by taking $\phi=1$ in a neighbourhood of 0 in R, we see that S is a bounded function in a neighbourhood of 0 in R and is continuous at 0 since $(\phi T)*\check{Y}$ and $\left(\frac{d\phi}{dx}S\right)*\check{Y}$ have these properties.

Conversely if S is a bounded function in a neighbourhood of 0 in R and is continuous at 0, then YS exists, therefore we see that ϕS and $\left(\frac{d\phi}{dx}S\right)*\check{Y}$ are bounded functions in a neighbourhood of 0 in R and are continuous at 0. Hence $(\phi T)*\check{Y}$ has these properties also. Therefore YT exists, and

$$YT=YS'=(YS)'-S(0)\delta.$$

In addition, if $T\in\mathcal{D}'_+$, then we can take $S(0)=0$ and so $YT=T$.

Thus the proof is complete.

REMARK. We have in [6] (p. 229) that for $S, T\in\mathcal{D}'$, $(\tau_h S)T$ exists for every $h\in R$, if and only if $(\phi S)*\check{T}$ is a continuous function in R for every $\phi\in\mathcal{D}$. Therefore $(\tau_h Y)T$ exists for every $h\in R$ if and only if T is a distributional derivative of a continuous function in R. It is also easily shown that $(\tau_h\delta)S$ exists for every $h\in R$ if and only if S is continuous in R. From Proposition 1 it is easy to construct an example such that $S\bigcirc T$ exists, but not ST. For example, let $S=Y$, $T=\delta$. Then $T=\frac{dY}{dx}$, and Y is not continuous at 0, therefore $ST=Y\delta$ does not exist. On the other hand, $S\bigcirc T=Y\bigcirc\delta=\frac{1}{2}\delta$ ([3], p. 69).

COROLLARY. *Let $T=S^{(n+1)}$, n being a non-negative integer. Then $x_+^n T$ exists if and only if there exists a neighbourhood U of 0 in R such that the restriction of S to U is a bounded function continuous at 0. When $x_+^n T$ exists, $x_+^n T=x^n(YS')^{(n)}$ and especially $x_+^n T=x^n T$ for $T\in\mathcal{D}'_+$.*

PROOF. As an immediate consequence of Proposition 1, the first part of Corollary follows by the mathematical induction. If $x_+^n T$ exists,

$$x_+^n T=\sum_{k=0}^{n}(-1)^{n-k}(n-k)!\binom{n}{k}^2(x_+^k S')^{(k)},$$

and

$$(x_+^k S')^{(k)} = \begin{cases} ((x_+^k S)' - k x_+^{k-1} S)^{(k)} = (((Yx^k)S)' - k(Yx^{k-1})S)^{(k)} \\ \quad = ((Yx^k)S' + k(Yx^{k-1})S - k(Yx^{k-1})S)^{(k)} = (x^k(YS'))^{(k)} \quad \text{for } k \neq 0, \\ YS' \quad \text{for} \quad k=0. \end{cases}$$

Hence we have $x_+^n T = x^n (YS')^{(n)}$. In addition, if $T \in \mathcal{D}'_+$, $YS' = S'$ by Proposition 1 and so $x_+^n T = x^n T$.

We note that $x_+^\alpha x^m = x_+^{\alpha+m}$ for any non-negative integer m. In fact, $x_+^\alpha x^m$ exists since $x^m \in \mathcal{E}$. We have for any $\phi \in \mathcal{D}$

$$< x_+^\alpha x^m, \phi > = < x_+^\alpha, x^m \phi > = \mathrm{Pf} \int_0^\infty x^\alpha x^m \phi\, dx = \mathrm{Pf} \int_0^\infty x^{\alpha+m} \phi\, dx = < x_+^{\alpha+m}, \phi >,$$

where Pf denotes the finite part of the integral. Thus $x_+^\alpha x^m = x_+^{\alpha+m}$.

Let $Y_\alpha = \dfrac{x_+^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha \neq 0, -1, -2, \cdots$ and $= \delta^{(n)}$ for a non-positive integer $-n$ ([1], p. 64).

Proposition 2. *If ST exists for $S, T \in \mathcal{D}'_+$, then $(Y_\alpha * S)T$ exists also for* $\mathrm{Re}(\alpha) > 0$.

Proof. $S*(\phi T)^\vee$, $\phi \in \mathcal{D}$, is a bounded function in a neighbourhood of 0 in R and is continuous at 0. On the other hand $x_+^{\alpha-1}$ is locally summable and is a C^∞-function in $R \setminus \{0\}$. Therefore $(Y_\alpha * S)*(\phi T)^\vee$ is a continuous function near 0. This implies that $(Y_\alpha * S)T$ exists.

Remark. If $\mathrm{Re}(\alpha) = 0$ and $\alpha \neq 0$, $(Y_\alpha * S)T$ does not exist in general even if ST exists. In fact, let X be the set of all the continuous functions S with support in [0, 1] and let $T = \delta$. Then $ST = S(0)\delta$ exists. Suppose $(Y_\alpha * S)\delta$ exists for every $S \in X$. Then there exists a neighbourhood $U = \{t; |t| < A\}$ such that $Y_\alpha * S$ is a bounded function $f_S(t)$ in U. Since X is a Banach space, we can take the same U for every $S \in X$ and ess. sup $|f_S(t)| \leq K \|S\|$, where K is a positive constant and $\|S\|$ denotes $\sup |S(t)|$. This may be shown as in the proof of Proposition 2 in [3] (p. 53), so the proof is omitted. Using Banach-Steinhauss theorem we can find a point x_0, $0 < x_0 < A$, such that

$$(*) \qquad \lim_{h \to 0} \frac{1}{h} \int_0^h f_S(x_0 + t)\, dt$$

exists for every $S \in X$. (*) is the value of the distribution $f_S(t)$ at x_0 which we shall also denote by $f_S(x_0)$. Then

$$|f_S(x_0)| \leq K \|S\|.$$

Therefore we can find a function $F(t)$ of bounded variation such that

$$f_S(x_0)=\int_0^1 S(t)\,dF(t).$$

Let S be any $\phi\in\mathcal{D}$ with support in $(0, x_0)$. Then

$$\int_0^{x_0}\frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)}\phi(t)\,dt=\int_0^{x_0}\phi(t)\,dF(t).$$

Hence we have

$$-\frac{(x_0-t)^{\alpha}}{\Gamma(\alpha+1)}=F(t)+\text{const.}$$

in $(0, x_0)$. However this is a contradiction since $(x_0-t)^{\alpha}$ is not of bounded variation in $[0, x_0]$.

REMARK 2. Proposition 2 does not hold in general for $S\bigcirc T$. This will follow from Theorem 2 below.

L. Schwartz has noticed in [5] (p. 39) that the value $\mathrm{Pf}\int_0^{\infty}x^{\alpha}\phi(x)\,dx$, $\phi\in\mathcal{D}$, is invariant by change of the variable, but when α is a negative integer the statement does not hold in general. Here we note that if $h(x)$ is a C^{∞}-function on $[0, a]$ and n is a positive integer, then

$$\mathrm{Pf}\int_0^{a}\frac{h(x)}{x^n}\,dx=\frac{h^{(n-1)}(0)}{(n-1)!}\log t+\mathrm{Pf}\int_0^{\frac{a}{t}}\frac{h(tx)}{t^{n-1}x^n}\,dx,\qquad t>0.$$

2. The product $x_+^{\alpha}x_+^{\beta}$

It follows from Proposition 5 in [6] (p. 229) that the product $x_+^{\alpha}x_+^{\beta}$ exists if and only if, for any $\phi\in\mathcal{D}$, there exists a zero neighbourhood in which $\phi x_+^{\alpha}*(x_+^{\beta})^{\vee}$ is a bounded function continuous at 0. In this case $\langle x_+^{\alpha}x_+^{\beta}, \phi\rangle=$ $=(\phi x_+^{\alpha}*(x_+^{\beta})^{\vee})(0)$. We note that

$$(\phi x_+^{\alpha}*(x_+^{\beta})^{\vee})(t)=\begin{cases}\mathrm{Pf}\int_t^{\infty}\phi(x)x^{\alpha}(x-t)^{\beta}dx=\mathrm{Pf}\int_0^{\infty}\phi(x+t)\,(x+t)^{\alpha}x^{\beta}dx & \\ & \text{for}\quad t>0,\\ \mathrm{Pf}\int_0^{\infty}\phi(x)\,x^{\alpha}(x-t)^{\beta}dx & \text{for}\quad t<0.\end{cases}$$

PROPOSITION 3. *If* $\mathrm{Re}(\alpha+\beta)>-1$, *then* $x_+^{\alpha}x_+^{\beta}$ *exists and equals* $x_+^{\alpha+\beta}$.

PROOF. Let ϕ be any element of $\mathcal{D}$. We may assume that $\mathrm{supp}\,\phi\subset[a, b]$ with $b>0$.

Consider first the case $\mathrm{Re}(\beta)>0$. If $t>0$, we have $(\phi x_+^{\alpha}*(x_+^{\beta})^{\vee})(t)=$

$\int_t^b \phi(x)x^{\alpha+\beta}\left(1-\frac{t}{x}\right)^\beta dx$. Since $\mathrm{Re}(\alpha+\beta)>-1$, we have $\left|x^{\alpha+\beta}\left(1-\frac{t}{x}\right)^\beta\right| \leqq x^{\mathrm{Re}(\alpha+\beta)}$ for $x \geqq t$, where $\int_0^b x^{\mathrm{Re}(\alpha+\beta)}|\phi(x)|\,dx<\infty$. Therefore $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ tends to $\int_0^b \phi(x)\,x^{\alpha+\beta}dx=<x_+^{\alpha+\beta}, \phi>$ as $t\to 0$. If $t<0$, we have $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ $=\mathrm{Pf}\int_0^{t'}\phi(x)\,x^\alpha(t+t')^\beta dx+\int_{t'}^b \phi(x)x^\alpha(x+t')^\beta dx$ with $t'=-t$. Since $|\int_{t'}^b \phi(x)x^\alpha(x+t')^\beta dx|\leqq 2^{\mathrm{Re}(\beta)}\int_{t'}^b x^{\mathrm{Re}(\alpha+\beta)}|\phi(x)|\,dx<\infty$ for sufficiently small t', it follows that $\lim_{t'\to 0}\int_{t'}^b \phi(x)x^\alpha(x+t')^\beta dx=\int_0^b \phi(x)\,x^{\alpha+\beta}dx=<x_+^{\alpha+\beta}, \phi(x)>$. On the other hand, after a change of variable $x\to xt'$, we have for $\alpha\neq$ a negative integer

$$\lim_{t'\to 0}\mathrm{Pf}\int_0^{t'}\phi(x)\,x^\alpha(x+t')^\beta dx=\lim_{t'\to 0}\mathrm{Pf}\int_0^1 \phi(xt')\,(xt')^\alpha(xt'+t')^\beta t'\,dx$$

$$=\lim_{t'\to 0} t'^{\alpha+\beta+1}\,\mathrm{Pf}\int_0^1 \phi(xt')\,x^\alpha(x+1)^\beta=0,$$

and for $\alpha=$ a negative integer

$$\lim_{t'\to 0}\mathrm{Pf}\int_0^{t'}\phi(x)\,x^\alpha(x+t')^\beta dx=0.$$

Therefore $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ tends to $<x_+^{\alpha+\beta}, \phi>$ as $t\to 0$. Consequently $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ is continuous at 0 and has the limit $<x_+^{\alpha+\beta}, \phi>$.

Similarly in the case $\mathrm{Re}(\alpha)>0$.

Next consider the case $-1<\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)\leqq 0$. If $t>0$, $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)=$ $\int_t^b \phi(x)\,x^\alpha(x-t)^\beta dx=\int_0^{b-t}\phi(x+t)\,(x+t)^\alpha x^\beta dx$ and $|(x+t)^\alpha x^\beta|\leqq x^{\mathrm{Re}(\alpha+\beta)}$ for $x\geqq 0$. Since $\mathrm{Re}(\alpha+\beta)>-1$, $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ tends to $\int_0^b \phi(x)\,x^{\alpha+\beta}dx=$ $<x_+^{\alpha+\beta}, \phi>$ as $t\to 0$. If $t<0$, then $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)=\int_0^b \phi(x)\,x^\alpha(x-t)^\beta dx$ and $|x^\alpha(x-t)^\beta|\leqq x^{\mathrm{Re}(\alpha+\beta)}$. Since $\mathrm{Re}(\alpha+\beta)>-1$, $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ tends to $\int_0^b \phi(x)x^{\alpha+\beta}dx=<x_+^{\alpha+\beta}, \phi>$ as $t\to 0$. Consequently $(\phi x_+^\alpha*(x_+^\beta)^\vee)(t)$ is continuous at 0 and has the limit $<x_+^{\alpha+\beta}, \phi>$. Thus, $x_+^\alpha x_+^\beta$ exists and equals $x_+^{\alpha+\beta}$, which was to be proved.

PROPOSITION 4. *If* $\mathrm{Re}(\alpha+\beta)\leqq -1$, *then* $x_+^\alpha x_+^\beta$ *does not exist.*

PROOF. We put

$$g(x)=\begin{cases} x^\alpha & \text{for} \quad x\geqq 1,\\ 0 & \text{for} \quad x<1,\end{cases}$$

and $h(x)=x_+^\alpha-g(x)$, where $g(x)x_+^\beta$ always existsts.

(a) Consider the case where $\mathrm{Re}(\alpha+\beta)=-1$ and $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)\geqq-1$ but $\alpha, \beta\neq-1$. In the case $t>0$, by the substitution $xs=t$, we have

$$\left(h(x)*(x_+^\beta)^\vee\right)(t)=t^{\alpha+\beta+1}\,\mathrm{Pf}\int_t^1 s^{-(\alpha+\beta+2)}(1-s)^\beta\,ds.$$

Here if $\mathrm{Im}(\alpha+\beta)=0$,

$$\left|\left(h(x)*(x_+^\beta)^\vee\right)(t)\right|=\left|\int_t^1\frac{(1-s)^{\beta+1}}{s}\,ds+\frac{(1-t)^{\beta+1}}{\beta+1}\right|\to\infty$$

as $t\to0$. Thus $x_+^\alpha x_+^\beta$ does not exist. If $\mathrm{Im}(\alpha+\beta)\neq0$,

$$\left(h(x)*(x_+^\beta)^\vee\right)(t)=t^{\alpha+\beta+1}\left[\int_t^{\frac{1}{2}} s^{-(\alpha+\beta+2)}\left((1-s)^\beta-1\right)ds-\frac{2^{\alpha+\beta+1}}{\alpha+\beta+1}\right.$$

$$\left.+\,\mathrm{Pf}\int_{\frac{1}{2}}^1 s^{-(\alpha+\beta+2)}(1-s)^\beta\,ds\right]+\frac{1}{\alpha+\beta+1},$$

where for $\alpha\neq0$ the expression in the brackets tends to $B(-\alpha-\beta-1, \beta+1)$ as $t\to0$, hence $\left(h(x)*(x_+^\beta)^\vee\right)(t)$ is not continuous at 0. Thus $x_+^\alpha x_+^\beta$ does not exist. For $\alpha=0$, $x_+^0x_+^\beta=\dfrac{1}{\beta+1}Y(x_+^{\beta+1})'$, where $x_+^{\beta+1}$ is not continuous at 0. By Proposition 1, $x_+^\alpha x_+^\beta$ does not exist.

(b) Consider the case where $\mathrm{Re}(\alpha+\beta)=-1$ and α, β are not negative integers. Since $\mathrm{Re}(\alpha+\beta+1)=0$, we may assume $\mathrm{Re}(\alpha)<0$. Hence there is a positive integer n such that $-n\leqq\mathrm{Re}(\alpha)<-n+1$, that is, $-1\leqq\mathrm{Re}(\alpha+n-1)<0$ and $-1<\mathrm{Re}(\beta-n+1)\leqq0$. Suppose $x_+^\alpha x_+^\beta$ exists, then $x_+^{\alpha+n-1}x_+^{\beta-n+1}$ also exists by Remark 1 to Proposition 5 in [6] (p. 229), which contradicts the consequence of (a). Therefore $x_+^\alpha x_+^\beta$ does not exist in this case.

(c) Consider the case where $\mathrm{Re}(\alpha+\beta)=-1$ and α is a negative integer $-n$. We put $\beta=n-1+\tau i$. Let $\tau=0$. Suppose $x_+^{-n}x_+^{n-1}$ exists, then $x_+^{-n}(x^{n-1}Y)=(x_+^{-n}x^{n-1})Y=x_+^{-1}Y$ also exists, contradicting Proposition 1. If $\tau\neq0$, by the substitution $xs=t$, we have

$$\left(h(x)*(x_+^{n-1+\tau i})^\vee\right)(t)=t^{\tau i}\int_t^1 s^{-1-\tau i}(1-s)^{n-1+\tau i}ds$$

$$=t^{\tau i}\left[\int_t^{\frac{1}{2}} s^{-1-\tau i}\left((1-s)^{n-1+\tau i}-1\right)ds-\frac{2^{\tau i}}{\tau i}+\int_{\frac{1}{2}}^1 s^{-1-\tau i}(1-s)^{n-1+\tau i}ds\right]+\frac{1}{\tau i}.$$

Since the expression in the brackets tends to $B(-\tau i, n+\tau i)$ as $t\to0$, $\left(h(x)*(x_+^{n-1+\tau i})^\vee\right)(t)$ is not continuous at 0. Thus $x_+^\alpha x_+^\beta$ does not exist.

(d) Finally, consider the case $\mathrm{Re}(\alpha+\beta)<-1$. Let α be not a negative integer. There exists a complex number γ such that $\mathrm{Re}(\alpha+\beta+\gamma)=-1$ and $\mathrm{Im}(\alpha+\gamma)\neq 0$. According to Proposition 2, if $x_+^\alpha x_+^\beta$ exists, then $x_+^{\alpha+\gamma}x_+^\beta$ exists from the equation:

$$\frac{x_+^\alpha}{\Gamma(\alpha+1)} * \frac{x_+^{\gamma-1}}{\Gamma(\gamma)} = \frac{x_+^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}.$$

Similarly for the case where β is not a negative integer. If α, β are negative integers $-m$, $-n$ respectively, we have the equations $x_+^{-m}x^{m-1}=x_+^{-1}$, $x_+^{-n}x^n=Y$. Assuming $x_+^{-m}x_+^{-n}$ exists, then $x_+^{-1}Y$ would exist, which contradicts Proposition 1.

Thus the proof is complete.

As a consequence of Propositions 3 and 4, we have

THEOREM 1. *If and only if* $\mathrm{Re}(\alpha+\beta)>-1$, $x_+^\alpha x_+^\beta$ *exists and equals* $x_+^{\alpha+\beta}$.

3. Conditions for the existence of $x_+^\alpha \bigcirc x_+^\beta$

From Theorem 1 in [3] and Theorem 1, $x_+^\alpha \bigcirc x_+^\beta$ exists in the case $\mathrm{Re}(\alpha+\beta)>-1$ and $x_+^\alpha \bigcirc x_+^\beta = x_+^{\alpha+\beta}$. In the sequel we consider the multiplicative product $x_+^\alpha \bigcirc x_+^\beta$ in the case $\mathrm{Re}(\alpha+\beta)\leqq -1$. There exists a positive integer p such that $-p-1<\mathrm{Re}(\alpha+\beta)\leqq -p$. Let $S=x_+^\alpha$ and $T=x_+^\beta$. Then we have for any $\phi\in\mathcal{D}$

$$(1)\qquad <\hat{S}_\varepsilon\hat{T}_\varepsilon, \phi> = \left(\int_{-\infty}^{-1}+\int_1^\infty\right)\hat{S}_\varepsilon\hat{T}_\varepsilon\phi(x)\,dx + \int_{-1}^1 \hat{S}_\varepsilon\hat{T}_\varepsilon\left(\phi(x)-\sum_{k=0}^{p-1}\frac{\phi^{(k)}(0)}{k!}x^k\right)dx$$
$$+\sum_{k=0}^{p-1}\frac{\phi^{(k)}(0)}{k!}\int_{-1}^1 x^k\hat{S}_\varepsilon\hat{T}_\varepsilon\,dx.$$

Case A. α, β *are not integers.*

As $\widehat{x_+^\alpha}(z)$ we can take

$$(2)\qquad -\frac{1}{2i}\frac{1}{\sin\alpha\pi}(-z)^\alpha,$$

where $(-z)^\alpha = e^{\alpha(\log|z|+i(\arg z-\pi))}$, $0<\arg z<2\pi$. Then we have

$$\hat{S}_\varepsilon\hat{T}_\varepsilon = \frac{(x^2+\varepsilon^2)^{\frac{\alpha+\beta}{2}}}{\sin\alpha\pi\,\sin\beta\pi}\sin\alpha(\pi-\theta)\,\sin\beta(\pi-\theta),\qquad \theta=\tan^{-1}\frac{\varepsilon}{x}.$$

It is easily verified that for any integer k, $0\leqq k\leqq p-1$, we have

$$\int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon \, dx = \frac{1}{\sin \alpha\pi \sin \beta\pi} \int_0^1 x^k (x^2+\varepsilon^2)^{\frac{\alpha+\beta}{2}} f_k(\theta)\, dx \tag{3}$$

$$= \frac{\varepsilon^{\alpha+\beta+k+1}}{\sin \alpha\pi \sin \beta\pi} \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \sin^{-\alpha-\beta-k-2}\theta \cos^k\theta \, f_k(\theta)\, d\theta,$$

where $f_k(\theta)=\sin\alpha(\pi-\theta)\sin\beta(\pi-\theta)+(-1)^k \sin\alpha\theta \sin\beta\theta$. We also note that if k is any integer such that $0\leqq k\leqq p-1$ when $\alpha+\beta\neq -p$, or $0\leqq k\leqq p-2$ when $\alpha+\beta=-p$, then

$$\text{(the finite part of } \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx \text{ as } \varepsilon\to 0) = \frac{1}{\alpha+\beta+k+1}. \tag{4}$$

PROPOSITION 5. *When* $-2<\mathrm{Re}(\alpha+\beta)\leqq -1$ *and* α, β *are not integers, then* $x_+^\alpha \bigcirc x_+^\beta$ *exists if and only if* $\alpha-\beta$ *is an odd integer, and* $x_+^\alpha \bigcirc x_+^\beta = x_+^{\alpha+\beta}$.

PROOF. Let $\alpha+\beta\neq -1$. Let us consider the relation (1). Evidently we have

$$\left(\int_{-\infty}^{-1} + \int_1^\infty\right) \hat{S}_\varepsilon \hat{T}_\varepsilon \phi(x)\, dx = \int_1^\infty x^{\alpha+\beta}\phi(x)\, dx + o(1), \quad \text{as} \quad \varepsilon\to 0. \tag{5}$$

We may put $\phi(x)-\phi(0)=x\, g(x)$, $g(x)\in \mathcal{E}$. Since $x\hat{S}_\varepsilon\hat{T}_\varepsilon$ is bounded on $|x|\leqq 1$, we have

$$\int_{-1}^{1} \hat{S}_\varepsilon \hat{T}_\varepsilon (\phi(x)-\phi(0))\, dx \tag{6}$$

$$= \frac{1}{\sin \alpha\pi \sin \beta\pi} \int_{-1}^{1} x(x^2+\varepsilon^2)^{\frac{\alpha+\beta}{2}} \sin\alpha(\pi-\theta) \sin\beta(\pi-\theta) g(x)\, dx$$

$$= \int_0^1 x^{\alpha+\beta+1} g(x)\, dx + o(1)$$

$$= \int_0^1 x^{\alpha+\beta}(\phi(x)-\phi(0))\, dx + o(1), \qquad \text{as} \quad \varepsilon\to 0,$$

and

$$\int_{-1}^{1} \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx = \frac{\varepsilon^{\alpha+\beta+1}}{\sin \alpha\pi \sin \beta\pi} \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \sin^{-\alpha-\beta-2}\theta f_0(\theta)\, d\theta \tag{7}$$

$$= \frac{\varepsilon^{\alpha+\beta+1}}{\sin \alpha\pi \sin \beta\pi} \left(\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \sin^{-\alpha-\beta}\theta f_0(\theta)\, d\theta + \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \sin^{-\alpha-\beta-2}\theta \cos^2\theta f_0(\theta)\, d\theta\right)$$

$$= \frac{\varepsilon^{\alpha+\beta+1}}{\sin \alpha\pi \sin \beta\pi} \left(\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \sin^{-\alpha-\beta}\theta f_0(\theta)\, d\theta + \left[\frac{\sin^{-\alpha-\beta-1}\theta}{-\alpha-\beta-1} \cos\theta f_0(\theta)\right]_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\right.$$

$$+\frac{1}{\alpha+\beta+1}\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\sin^{-\alpha-\beta-1}\theta\big(\cos\theta f_0(\theta)\big)'d\theta\Big)$$

$$=\frac{1}{\alpha+\beta+1}+\frac{\varepsilon^{\alpha+\beta+1}}{\sin\alpha\pi\sin\beta\pi}\operatorname{Pf}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta-2}\theta f_0(\theta)\,d\theta+o(1),\qquad\text{as}\quad\varepsilon\to0.$$

By calculation we shall obtain

$$\text{(8)}\qquad \operatorname{Pf}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta-2}\theta f_0(\theta)\,d\theta=\frac{4\alpha\beta\cos(\alpha-\beta)\frac{\pi}{2}}{(\alpha+\beta+1)(\alpha+\beta)}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta,$$

where $g(\theta)=\cos(\alpha-\beta)\left(\frac{\pi}{2}-\theta\right)$.

Now we show that $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta\neq0$. To this end we assume $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta=0$ and we shall deduce a contradiction. By calculation in the same way as before

$$\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta=\frac{4(\alpha-1)(\beta-1)}{(\alpha+\beta-1)(\alpha+\beta-2)}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta+2}\theta\, g(\theta)\,d\theta.$$

If $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)d\theta=0$, then $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta+2n}\theta\, g(\theta)\,d\theta=0$ for every non-negative integer n, hence $\int_0^{\frac{\pi}{2}}P(\sin^2\theta)\sin^{-\alpha-\beta}\theta\, g(\theta)d\theta=0$ for any polynomial $P(x)$. Then, by the approximation theorem of Stone-Weierstrass we conclude that $\int_0^{\frac{\pi}{2}}\psi(\theta)\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta=0$ for any $\psi(\theta)\in C_{[0,\frac{\pi}{2}]}$. Therefore $\sin^{-\alpha-\beta}\theta\, g(\theta)\equiv0$, which is a contradiction.

Consequently, from the relations (1), (5), (6), (7) and (8) we obtain

$$<\hat{S}_\varepsilon\hat{T}_\varepsilon,\phi>=<x_+^{\alpha+\beta},\phi>+\frac{\varepsilon^{\alpha+\beta+1}}{\sin\alpha\pi\sin\beta\pi}\frac{4\alpha\beta\cos(\alpha-\beta)\frac{\pi}{2}}{(\alpha+\beta+1)(\alpha+\beta)}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\, g(\theta)\,d\theta$$
$$+o(1)\qquad\text{as}\quad\varepsilon\to0.$$

Thus $x_+^\alpha\bigcirc x_+^\beta$ exists only in the case where $\alpha-\beta$ is an odd integer.

Next, let $\alpha+\beta=-1$. As before, we have for any $\phi\in\mathcal{D}$

$$<\hat{S}_\varepsilon\hat{T}_\varepsilon,\phi>=\int_0^1 x^{-1}(\phi(x)-\phi(0))dx+\int_1^\infty x^{-1}\phi(x)dx+\phi(0)\int_{-1}^1\hat{S}_\varepsilon(x)\hat{T}_\varepsilon(x)\,dx+o(1)$$

$$=<x_+^{-1},\phi>+\phi(0)\int_{-1}^1\hat{S}_\varepsilon(x)\hat{T}_\varepsilon(x)\,dx+o(1),\qquad\text{as}\quad\varepsilon\to0.$$

By (3)

$$(9)\qquad \int_{-1}^{1} \hat{S}_\varepsilon \hat{T}_\varepsilon dx = \frac{1}{\sin \alpha\pi \sin \beta\pi} \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \frac{1}{\sin\theta} f_0(\theta)\, d\theta$$

$$= \frac{\cos(\alpha-\beta)\frac{\pi}{2}}{\sin \alpha\pi \sin \beta\pi} \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}} \frac{\cos(\alpha-\beta)\left(\frac{\pi}{2}-\theta\right)}{\sin\theta} d\theta$$

$$= \left(\log 2 + \frac{2}{\cos(\alpha-\beta)\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin(\alpha-\beta)\frac{\theta}{2} \sin(\alpha-\beta)\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}{\sin\theta} d\theta\right)$$

$$-\log \varepsilon + o(1), \qquad \text{as} \quad \varepsilon \to 0.$$

Hence it follows that $x_+^\alpha \bigcirc x_+^\beta$ does not exist.

Consequently, when $-2 < \mathrm{Re}(\alpha+\beta) \leqq -1$ and α, β are not integers, then $x_+^\alpha \bigcirc x_+^\beta$ exists if and only if $\alpha-\beta$ is an odd integer. From the foregoing proof we see that $x_+^\alpha \bigcirc x_+^\beta = x_+^{\alpha+\beta}$ if the left hand side exists. Thus the proof is complete.

PROPOSITION 6. *If* $\mathrm{Re}(\alpha+\beta) \leqq -2$ *and* α, β *are not integers,* $x_+^\alpha \bigcirc x_+^\beta$ *does not exist.*

PROOF. When $\alpha+\beta$ is not a negative integer, we can take a positive integer $p \geqq 2$ such that $-p-1 < \mathrm{Re}(\alpha+\beta) \leqq -p$. Then we have for any $\phi \in \mathcal{D}$

$$< \hat{S}_\varepsilon \hat{T}_\varepsilon, \phi > = \int_1^\infty x^{\alpha+\beta} \phi(x)\, dx + \int_0^1 x^{\alpha+\beta} \left(\phi(x) - \sum_{k=0}^{p-1} \frac{\phi^{(k)}(0)}{k!} x^k\right) dx$$

$$+ \sum_{k=0}^{p-1} \frac{\phi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx + o(1), \qquad \text{as} \quad \varepsilon \to 0,$$

where

$$(10)\qquad \int_{-1}^{1} \hat{S}_\varepsilon \hat{T}_\varepsilon dx = \frac{1}{\alpha+\beta+1} + \frac{\varepsilon^{\alpha+\beta+1}}{\sin\alpha\pi \sin\beta\pi} \frac{4\alpha\beta \cos(\alpha-\beta)\frac{\pi}{2}}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\times \int_0^{\frac{\pi}{2}} \sin^{-\alpha-\beta}\theta \cos(\alpha-\beta)\left(\frac{\pi}{2}-\theta\right) d\theta + o(1), \qquad \text{as} \quad \varepsilon \to 0,$$

and

$$(11)\quad \int_{-1}^{1} x\hat{S}_\varepsilon \hat{T}_\varepsilon\,dx=\frac{\varepsilon^{\alpha+\beta+2}}{\sin\alpha\pi\,\sin\beta\pi}\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\sin^{-\alpha-\beta-3}\theta\cos\theta f_1(\theta)\,d\theta$$

$$=\frac{1}{\alpha+\beta+2}+\frac{\varepsilon^{\alpha+\beta+2}}{\sin\alpha\pi\,\sin\beta\pi}\,\mathrm{Pf}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta-3}\theta\cos\theta f_1(\theta)\,d\theta+o(1)$$

$$=\frac{1}{\alpha+\beta+2}+\frac{\varepsilon^{\alpha+\beta+2}}{\sin\alpha\pi\,\sin\beta\pi}\,\frac{4\alpha\beta(\alpha-\beta)\sin(\alpha-\beta)\frac{\pi}{2}}{(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)}$$

$$\times\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\cos(\alpha-\beta)\Big(\frac{\pi}{2}-\theta\Big)\,d\theta+o(1),\qquad \text{as}\quad \varepsilon\to 0.$$

Since $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-\beta}\theta\cos(\alpha-\beta)\Big(\frac{\pi}{2}-\theta\Big)d\theta\neq 0$ (see the proof of Proposition 5), and $\cos(\alpha-\beta)\frac{\pi}{2}$, $(\alpha-\beta)\sin(\alpha-\beta)\frac{\pi}{2}$ do not vanish simultaneously, it follows from the equations (10), (11) that $x_+^\alpha \bigcirc x_+^\beta$ does not exist.

Next we suppose that $\alpha+\beta=-p$, p being a positive integer. Owing to the equation (3) we have

$$(12)\quad \int_{-1}^{1} x^{p-1}\hat{S}_\varepsilon \hat{T}_\varepsilon\,dx=\frac{1}{\sin\alpha\pi\,\sin\beta\pi}\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}f_{p-1}(\theta)\,d\theta$$

$$=\frac{1}{\sin\alpha\pi\,\sin\beta\pi}\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\big(f_{p-1}(\theta)-f_{p-1}(0)\big)\,d\theta$$

$$+\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\,d\theta$$

and

$$(13)\quad \int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\,d\theta=\begin{cases}-\log\varepsilon+o(1) & \text{for}\quad p=2,\\ -\log\varepsilon+(p-2)\displaystyle\int_0^{\frac{\pi}{2}}\cos^{p-3}\theta\sin\theta\log\sin\theta\;d\theta+o(1) & \\ & \text{for}\quad p\geqq 3,\quad \text{as}\quad \varepsilon\to 0.\end{cases}$$

Consequently, since ϕ is arbitrary, it follows that $x_+^\alpha \bigcirc x_+^\beta$ does not exist.

Thus the proof is complete.

Case B. *α, β are integers.*

When n is an integer, we can take as $\widehat{x_+^n}(z)$

$$(14)\quad -\frac{1}{2\pi i}z^n\,\mathrm{Log}(-z)=-\frac{1}{2\pi i}\big(\log|z|+i(\arg z-\pi)\big),$$

where $0<\arg z<2\pi$.

Let $S=x_+^{-n}$ and $T=x_+^{n-p}$, where n, p are integers such that $n\geqq 0, p\geqq 1$. Then we can write

$$\hat{S}_\varepsilon \hat{T}_\varepsilon=\frac{1}{\pi^2}|z_\varepsilon|^{-p}\big((\theta-\pi)\cos n\theta-\sin n\theta\log|z_\varepsilon|\big)$$
$$\times\big((\theta-\pi)\cos(n-p)\theta+\sin(n-p)\theta\log|z_\varepsilon|\big),$$

where $z_\varepsilon=x+i\varepsilon$ and $\theta=\tan^{-1}\dfrac{\varepsilon}{x}$.

We also note that for any integer k, $0\leqq k\leqq p-2$, we have

$$(15)\quad \int_{-1}^{1}x^k\hat{S}_\varepsilon\hat{T}_\varepsilon\,dx=\frac{\varepsilon^{k-p+1}}{\pi^2}\int_{\tan^{-1}\varepsilon}^{\pi-\tan^{-1}\varepsilon}\sin^{p-k-2}\theta\cos^k\theta\big((\theta-\pi)^2\cos n\theta\cos(n-p)\theta$$
$$-\sin n\theta\sin(n-p)\theta(\log|z_\varepsilon|)^2-(\theta-\pi)\sin p\theta\log|z_\varepsilon|\big)\,d\theta.$$

And it is easy to see that

$$(16)\qquad (\text{the finite part of }\int_{-1}^{1}x^k\hat{S}_\varepsilon\hat{T}_\varepsilon\,dx\quad\text{as}\quad\varepsilon\to 0)=\frac{1}{-p+k+1}.$$

With the aid of these relations we can show the following

PROPOSITION 7. *In the case* $\mathrm{Re}(\alpha+\beta)\leqq-1$, *where* $\alpha=-n$ *and* $\beta=n-p$ *are integers such that* $n\geqq 0$, $p\geqq 1$, $x_+^\alpha\circ x_+^\beta$ *does not exist.*

PROOF. For any $\phi\in\mathcal{D}$, we can write

$$<\hat{S}_\varepsilon\hat{T}_\varepsilon,\phi>=\int_1^\infty x^{-p}\phi(x)\,dx+\int_0^1 x^{-p}\Big(\phi(x)-\sum_{k=0}^{p-1}\frac{\phi^{(k)}(0)}{k!}x^k\Big)dx$$
$$+\sum_{k=0}^{p-1}\frac{\phi^{(k)}(0)}{k!}\int_{-1}^{1}x^k\hat{S}_\varepsilon\hat{T}_\varepsilon\,dx+o(1),\qquad\text{as}\quad\varepsilon\to 0.$$

Here from (15) we obtain

$$(17)\quad \int_{-1}^{1}x^{p-1}\hat{S}_\varepsilon\hat{T}_\varepsilon dx=\frac{1}{\pi^2}\int_{\tan^{-1}\varepsilon}^{\pi-\tan^{-1}\varepsilon}\frac{\cos^{p-1}\theta}{\sin\theta}\big((\theta-\pi)^2\cos n\theta\cos(n-p)\theta$$
$$-(\theta-\pi)\sin p\theta\log|z_\varepsilon|\big)d\theta$$
$$=\frac{2}{\pi}\int_{\tan^{-1}\varepsilon}^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\Big(\Big(\frac{\pi}{2}-\theta\Big)\cos n\theta\cos(n-p)\theta+\frac{1}{2}\sin p\theta\log|z_\varepsilon|\Big)d\theta$$
$$=\Big(\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\sin p\theta\,d\theta-1\Big)\log\varepsilon+\frac{2}{\pi}\,\mathrm{Pf}\int_0^{\frac{\pi}{2}}\frac{\cos^{p-1}\theta}{\sin\theta}\Big(\frac{\pi}{2}-\theta\Big)\cos n\theta$$

$$\times \cos(n-p)\theta\, d\theta - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^{p-1}\theta}{\sin\theta} \sin p\theta \log \sin\theta\, d\theta + o(1), \qquad \text{as} \quad \varepsilon \to 0.$$

Since the coefficient of $\log\varepsilon$ is $-\frac{1}{2}$ and ϕ is arbitrary, $x_+^\alpha \bigcirc x_+^\beta$ does not exist, which completes the proof.

Case C. *Either α or β is an integer.*

Let β be an integer n but α be not an integer. Let $S = x_+^\alpha$ and $T = x_+^n$, where $-p-1 < \mathrm{Re}(\alpha+n) \leqq -p$ for some integer $p \geqq 1$. From (2) and (14) we have, for any integer k such that $0 \leqq k \leqq p-1$,

$$(18) \qquad \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx = \frac{\varepsilon^{\alpha+n+k+1}}{\pi \sin \alpha\pi} \int_{\tan^{-1}\varepsilon}^{\pi - \tan^{-1}\varepsilon} \sin^{-\alpha-n-k-2}\theta \cos^k\theta \sin\alpha(\theta-\pi)$$

$$\times \big((\theta-\pi)\cos n\theta - \sin n\theta \log \sin\theta\big)\, d\theta$$

$$+ \frac{\varepsilon^{\alpha+n+k+1}\log\varepsilon}{\pi \sin \alpha\pi} \int_{\tan^{-1}\varepsilon}^{\pi - \tan^{-1}\varepsilon} \sin^{-\alpha-n-k-2}\theta \cos^k\theta \sin\alpha(\theta-\pi) \sin n\theta\, d\theta$$

and

$$(19) \qquad (\text{the finite part of } \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx \quad \text{as} \quad \varepsilon \to 0) = \frac{1}{\alpha+n+k+1}.$$

Proposition 8. *If* $\mathrm{Re}(\alpha+\beta) \leqq -1$, *where β is an integer n but α is not an integer, then $x_+^\alpha \bigcirc x_+^\beta$ does not exist.*

Proof. In the same way as in the proof of Proposition 5, we have

$$<\hat{S}_\varepsilon \hat{T}_\varepsilon, \phi> = \int_1^\infty x^{\alpha+n}\phi(x)\, dx + \int_0^1 x^{\alpha+n}\Big(\phi(x) - \sum_{k=0}^{p-1} \frac{\phi^{(k)}(0)}{k!} x^k\Big) dx$$

$$+ \sum_{k=0}^{p-1} \frac{\phi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon\, dx + o(1), \qquad \text{as} \quad \varepsilon \to 0.$$

Here we have by (18)

$$\int_{-1}^{1} \hat{S}_\varepsilon \hat{T}_\varepsilon dx = \frac{1}{\alpha+n+1} + \varepsilon^{\alpha+n+1}\Big(\frac{\alpha+n}{\alpha+n+1}\int_0^{\frac{\pi}{2}} \sin^{-\alpha-n}\theta\, d\theta$$

$$+ \frac{1}{\pi \sin\alpha\pi}\int_0^{\frac{\pi}{2}} \sin^{-\alpha-n-2}\theta\, h(\theta)\, d\theta\Big) + \frac{\varepsilon^{\alpha+n+1}\log\varepsilon}{\pi\sin\alpha\pi}\int_0^{\frac{\pi}{2}} \sin^{-\alpha-n-2}\theta\, g(\theta)\, d\theta + o(1),$$

$$\text{as} \quad \varepsilon \to 0,$$

where

$$g(\theta)=\sin\alpha(\theta-\pi)\sin n\theta-\sin\alpha\theta\sin n(\pi-\theta),$$

$$h(\theta)=\theta\big(\sin\alpha(\theta-\pi)\cos n\theta+\sin\alpha\theta\cos n(\pi-\theta)\big)$$
$$-\pi\big(\sin\alpha(\theta-\pi)\cos n\theta+\sin\alpha\pi\big)-g(\theta)\log\sin\theta.$$

Furthermore we have for any non-negative integer k

$$\frac{\varepsilon^{\alpha+n+1}\log\varepsilon}{\pi\sin\alpha\pi}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n-2}\theta\, g(\theta)\,d\theta=\frac{\varepsilon^{\alpha+n+1}\log\varepsilon}{\pi\sin\alpha\pi}\frac{4\alpha n}{(\alpha+n+1)(\alpha+n)}\cos\frac{\alpha+n}{2}\pi$$
$$\times\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)\,d\theta$$

$$=\frac{\varepsilon^{\alpha+n+1}\log\varepsilon}{\pi\sin\alpha\pi}\frac{4\alpha n}{(\alpha+n+1)(\alpha+n)}\frac{4(\alpha-1)(n-1)}{(\alpha+n-1)(\alpha+n-2)}\cdots\frac{4(\alpha-k)(n-k)}{(\alpha+n-2k+1)(\alpha+n-2k)}$$
$$\times\cos\frac{\alpha+n}{2}\pi\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n+2k}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)\,d\theta,$$

where $\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n-2}\theta\, g(\theta)d\theta\neq 0$ for $n<0$. Therefore $x_+^\alpha\bigcirc x_+^n$ does not exist for any negative integer n. Consequently,

$$\text{(20)}\qquad\begin{cases}\displaystyle\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n-2}\theta\, g(\theta)\,d\theta=0 & \text{for}\quad n\geqq 0,\\ \displaystyle\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)d\theta=0 & \text{for}\quad n\geqq 1.\end{cases}$$

Next we shall show that $x_+^\alpha\bigcirc x_+^n$ does not exist for $n\geqq 0$. In this case, with the aid of (20), we obtain

$$\int_{-1}^{1}\hat{S}_\varepsilon\hat{T}_\varepsilon dx-\frac{1}{\alpha+n+1}=\varepsilon^{\alpha+n+1}\Big(\frac{\alpha+n}{\alpha+n+1}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\,d\theta$$
$$+\frac{1}{\pi\sin\alpha\pi}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n-2}\theta\, h(\theta)\,d\theta\Big)+o(1)$$

$$=\varepsilon^{\alpha+n+1}\Big(\frac{\alpha+n}{\alpha+n+1}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\,d\theta$$
$$+\frac{1}{\pi\sin\alpha\pi}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\Big(\frac{\alpha+n}{\alpha+n+1}h(\theta)+\frac{1}{(\alpha+n+1)(\alpha+n)}h''(\theta)\Big)\,d\theta\Big)+o(1)$$

$$=\begin{cases}\displaystyle\frac{\varepsilon^{\alpha+1}}{\pi\sin\alpha\pi}\frac{2}{\alpha+1}\cos\frac{\alpha}{2}\pi\int_0^{\frac{\pi}{2}}\sin^{-\alpha}\theta\cos\Big(\theta-\frac{\pi}{2}\Big)\alpha\,d\theta+o(1) & \text{for}\quad n=0,\\ \displaystyle\frac{\varepsilon^{\alpha+n+1}}{\pi\sin\alpha\pi}\frac{4\alpha n}{(\alpha+n+1)(\alpha+n)}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta\, k(\theta)\,d\theta+o(1) & \text{for}\quad n\geqq 1,\ \text{as}\ \varepsilon\to 0,\end{cases}$$

where

$$k(\theta)=\cos\frac{\alpha+n}{2}\pi\cdot\theta\sin(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)-\frac{\pi}{2}\sin((\alpha-n)\theta-\pi\alpha)$$

$$-\cos\frac{\alpha+n}{2}\pi\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)\log\sin\theta.$$

Furthermore we have for $n\geqq 1$

$$\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n}\theta k(\theta)\,d\theta=\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n+2}\theta\Big(\frac{\alpha+n-2}{\alpha+n-1}k(\theta)+\frac{1}{(\alpha+n-1)(\alpha+n-2)}k''(\theta)\Big)\,d\theta$$

$$=\frac{4(\alpha-1)(n-1)}{(\alpha+n-1)(\alpha+n-2)}\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n+2}\theta\, k(\theta)\,d\theta$$

$$+\frac{2(\alpha-n)}{(\alpha+n-1)(\alpha+n-2)}\cos\frac{\alpha+n}{2}\pi\int_0^{\frac{\pi}{2}}\sin^{-\alpha-n+2}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)\,d\theta.$$

Consequently we obtain for $n\geqq 0$

$$\int_{-1}^{1}\hat{S}_\varepsilon\hat{T}_\varepsilon\,dx-\frac{1}{\alpha+n+1}=\frac{\varepsilon^{\alpha+n+1}}{\pi\sin\alpha\pi}\frac{2\cdot 4^n n!}{(\alpha+n+1)(\alpha+n)\cdots(\alpha+1)}\cos\frac{\alpha+n}{2}\pi$$

$$\times\int_0^{\frac{\pi}{2}}\sin^{-\alpha+n}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)\,d\theta+o(1),\qquad\text{as}\quad\varepsilon\to 0.$$

Suppose $\int_0^{\frac{\pi}{2}}\sin^{-\alpha+n}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)d\theta=0$, then we have $\int_0^{\frac{\pi}{2}}\sin^{-\alpha+n+2k}\theta\cos(\alpha-n)\Big(\theta-\frac{\pi}{2}\Big)d\theta=0$ for any non-negative integer k, which is a contradiction as shown in the same way as in the proof of Proposition 5. Therefore $x_+^\alpha\bigcirc x_+^n$ does not exist for any non-negative integer n.

Thus the proof is complete.

As a consequence of Propositions 5, 6, 7 and 8, we obtain

THEOREM 2. *$x_+^\alpha\bigcirc x_+^\beta$ exists if and only if $-1<\mathrm{Re}(\alpha+\beta)$, or $-2<\mathrm{Re}(\alpha+\beta)\leqq -1$ and $\alpha-\beta$ is an odd integer and $\alpha,\beta\neq\pm 1,\pm 2,\pm 3,\dots$. In these cases, $x_+^\alpha\bigcirc x_+^\beta=x_+^{\alpha+\beta}$ holds true.*

4. The product $x_+^\alpha\cdot x_+^\beta$

As noticed at the outset of Section 3, $x_+^\alpha\cdot x_+^\beta$ exists in the case where $\mathrm{Re}(\alpha+\beta)>-1$ and $x_+^\alpha\cdot x_+^\beta=x_+^\alpha\bigcirc x_+^\beta=x_+^\alpha x_+^\beta=x_+^{\alpha+\beta}$.

THEOREM 3. *$x_+^\alpha \cdot x_+^\beta$ exists for any α and β. $x_+^\alpha \cdot x_+^\beta = x_+^{\alpha+\beta}$ holds if $\alpha+\beta$ is not a negative integer, but it does not hold in general if $\alpha+\beta$ is a negative integer.*

PROOF. We can immediately see that $x_+^\alpha \cdot x_+^\beta$ exists always for any α, β from our discussions given in Section 3. Let $\mathrm{Re}(\alpha+\beta) \leqq -1$ and $\alpha+\beta$ be not a negative integer. We take an integer $p \geqq 1$ such that $-p-1 < \mathrm{Re}(\alpha+\beta) \leqq -p$. From the relations (4), (16), we have for any integer k such that $0 \leqq k \leqq p-1$,

$$(\text{the finite part of } \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon \, dx \quad \text{as} \quad \varepsilon \to 0) = \frac{1}{\alpha+\beta+k+1}.$$

Consequently if $\alpha+\beta$ is not a negative integer, $x_+^\alpha \cdot x_+^\beta = x_+^{\alpha+\beta}$ holds true.

It remains to show the last part of the theorem. Let $\alpha+\beta$ be a negative integer $-p$. In view of (4)

$$\begin{aligned} &\text{the finite part of } \int_{-1}^{1} x^k \hat{S}_\varepsilon \hat{T}_\varepsilon \, dx \qquad \text{as} \quad \varepsilon \to 0 \\ &\qquad = \frac{1}{\alpha+\beta+k+1} \qquad \text{for} \quad 0 \leqq k \leqq p-2. \end{aligned}$$

If α, β are not integers, then by (3)

$$\begin{aligned} &\text{the finite part of } \int_{-1}^{1} x^{p-1} \hat{S}_\varepsilon \hat{T}_\varepsilon \, dx \qquad \text{as} \quad \varepsilon \to 0 \\ &\quad = \frac{1}{\sin \alpha\pi \sin \beta\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^{p-1}\theta}{\sin\theta} \big(f_{p-1}(\theta) - f_{p-1}(0)\big) \, d\theta \\ &\qquad + \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta} (\cos^{p-1}\theta - 1) \, d\theta + \log 2, \end{aligned}$$

where $f_{p-1}(\theta) = \sin\alpha(\pi-\theta)\sin\beta(\pi-\theta) + (-1)^{p-1}\sin\alpha\theta\sin\beta\theta$. If $\alpha = -n$, $\beta = n-p$, then we have by (17)

$$\begin{aligned} &\text{the finite part of } \int_{-1}^{1} x^{p-1} \hat{S}_\varepsilon \hat{T}_\varepsilon \, dx \qquad \text{as} \quad \varepsilon \to 0 \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta} \left(\left(\frac{\pi}{2} - \theta \right) \cos^{p-1}\theta \cos n\theta \cos(n-p)\theta - \frac{\pi}{2} \right) d\theta \\ &\qquad - \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^{p-1}\theta}{\sin\theta} \sin p\theta \log \sin\theta \, d\theta + \log 2. \end{aligned}$$

Consequently we have

$$x_+^\alpha \cdot x_+^\beta = x_+^{\alpha+\beta} + (-1)^{p-1} \frac{\delta^{(n-1)}}{(n-1)!} \times \text{(the finite part of } \int_{-1}^1 \hat{S}_\varepsilon \hat{T}_\varepsilon dx),$$

where the last term does not vanish in general. Thus the proof is complete.

EXAMPLES. By actual calculation we can show the following formulas:

$$x_+^{-(n+1)} \cdot x_+^n = x_+^{-1} - \frac{1}{2}\left(\log 2 + 1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\delta,$$

$$x_+^{-(n+2)} \cdot x_+^n = x_+^{-2} + \frac{1}{4}\left(2 \log 2 + 2 + \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{n} + \frac{1}{n+1}\right)\delta',$$

for $n = 0, 1, 2, \cdots$.

References

[1] I. M. Gelfand und G. E. Schilow, *Verallgemeinerte Funktionen (Distributionen)*, I, Berlin (1960).

[2] Y. Hirata and H. Ogata, *On the exchange formula for distributions*, this Journal, **22** (1958), 147-152.

[3] M. Itano, *On the multiplicative products of distributions*, this Journal, **29** (1965), 51-74.

[4] J. Mikusiński, *Criteria of the existence of a distribution at a point*, Bull. Acad. Sci., Cl. VIII, **10** (1960), 681-683.

[5] L. Schwartz, *Théorie des distributions*, I, Paris, Hermann (1951).

[6] R. Shiraishi and M. Itano, *On the multiplicative products of distributions*, this Journal, **28** (1964), 223-235.

[7] H. G. Tillmann, *Darstellung der Schwartzschen Distributionen durch analytische Funktionen*, Math. Z., **77** (1961), 106-124.

Faculty of Education and Liberal Arts,
Kagawa University