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On a One-step Method of Order 4

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1. Introduction

Given a differential equation

$$y' = f(x, y) \tag{1.1}$$

and the initial condition

$$y(x_0) = y_0, \tag{1.2}$$

where $f(x, y)$ is assumed to be a sufficiently smooth function. We are concerned with the case where the equation (1.1) is integrated numerically by a one-step method of order 4.

It is well known that the one-step method of order 4 requires at least four evaluations of the derivative per step and that, in the course of numerical integration, if the same step-size is used twice in succession, the approximate value of the truncation error is obtained by integrating again with the double step-size. This method of approximating the truncation error requires at least three additional evaluations of the derivative per two steps of integration. Hence we raise the question whether there is or not a one-step formula of order 4 such that only one additional evaluation of the derivative makes it possible to approximate the truncation error for two steps of integration with the same step-size.

In this paper, it is shown that such a formula really exists and the method is illustrated by numerical examples.

2. Preliminaries

Put

$$x_j = x_0 + jh \qquad (j=1, 2, \cdots) \tag{2.1}$$

$$z_1 = y_0 + h\sum_{i=1}^{4} p_i k_i, \tag{2.2}$$

and

(2.3) $$z_1 - y(x_1) = T(x_0, y_0; h),$$

where h is an increment of x,

(2.4) $$\begin{cases} k_1 = f(x_0, y_0) \\ k_2 = f(x_0 + ah, y_0 + ahk_1), \\ k_3 = f(x_0 + bh, y_0 + (b-c)hk_1 + chk_2), \\ k_4 = f(x_0 + lh, y_0 + (l-d-e)hk_1 + dhk_2 + ehk_3), \end{cases}$$

and a, b, c, d, e, l and p_i's ($i=1, 2, 3, 4$) are constants to be determined so that

(2.5) $$T(x_0, y_0; h) = O(h^5).$$

Then, under the condition (2.5), it is known that l must be equal to 1 [1][1].

Let D be a differential operator defined by

(2.6) $$D = \frac{\partial}{\partial x} + k_1 \frac{\partial}{\partial y}$$

and put, for simplicity,

(2.7) $$D^j f(x_0, y_0) = T^j, \quad D^j f_y(x_0, y_0) = S^j \qquad (j=1, 2, \cdots, 5),$$

(2.8) $$(Df)^2(x_0, y_0) = P, \quad (Df_y)^2(x_0, y_0) = Q, \quad Df_{yy}(x_0, y_0) = R,$$

(2.9) $$f_y(x_0, y_0) = f_y, \quad f_{yy}(x_0, y_0) = f_{yy},$$

then $y(x_0+h)$ is expanded into power series in h as follows [2]:

(2.10) $$\begin{aligned} y(x_0+h) = {} & y_0 + hk_1 + \frac{1}{2!}h^2 T + \frac{1}{3!}h^3(T^2 + f_y T) + \frac{1}{4!}h^4(T^3 + 3TS + \\ & + f_y T^2 + f_y^2 T) + \frac{1}{5!}h^5(T^4 + 6TS^2 + 4T^2 S + 3f_{yy}P + f_y T^3 + f_y^2 T^2 + \\ & + f_y^3 T + 7f_y TS) + \frac{1}{6!}h^6(T^5 + 10TS^3 + 10T^2S^2 + 5T^3 S + 15PR + \\ & + 15TQ + 13f_y f_{yy} P + 16f_y TS^2 + 9f_y T^2 S + 12f_y^2 TS + 10f_{yy} TT^2 + \\ & + f_y T^4 + f_y^2 T^3 + f_y^3 T^2 + f_y^4 T) + O(h^7). \end{aligned}$$

Also the expansions of k_i's ($i=2, 3, 4$) are obtained as follows:

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1) Numbers in square brackets refer to the references listed at the end of this paper.

$$k_2 = k_1 + ahT + \frac{1}{2!}h^2a^2T^2 + \frac{1}{3!}h^3a^3T^3 + \frac{1}{4!}h^4a^4T^4 + \cdots, \tag{2.11}$$

$$k_3 = k_1 + hbT + \frac{1}{2!}h^2(b^2T^2 + 2acf_yT) + \frac{1}{3!}h^3(b^3T^3 + 6abcTS + \\ + 3a^2cf_yT^2) + \frac{1}{4!}h^4(b^4T^4 + 12ab^2cTS^2 + 12a^2bcT^2S + 12a^2c^2f_{yy}P + \\ + 4a^2cf_yT^3) + \cdots, \tag{2.12}$$

$$k_4 = k_1 + hT + \frac{1}{2!}h^2(T^2 + 2Xf_yT) + \frac{1}{3!}h^3(T^3 + 6XTS + 3Zf_yT^2 + \\ + 6acef_y^2T) + \frac{1}{4!}h^4(T^4 + 12XTS^2 + 12ZT^2S + 12X^2f_{yy}P + \\ + 4(aZ + beY)f_yT^3 + 12a^2cef_y^2T^2 + 24(1+b)acef_yTS) + \cdots, \tag{2.13}$$

where

$$X = ad + be, \quad Y = b(b-a), \quad Z = aX + eY. \tag{2.14}$$

Substituting these into (2.3), we obtain the expansion of $T(x_0, y_0; h)$. We write this expansion as follows:

$$T(x_0, y_0; h) = hC_1k_1 + \frac{1}{2!}h^2C_2T + \frac{1}{3!}h^3(C_3T^2 + C_4f_yT) + \\ + \frac{1}{4!}h^4(B_1T^3 + B_2TS + B_3f_yT^2 + B_4f_y^2T)) + \frac{1}{5!}h^5(A_1T^4 + \\ + A_2TS^2 + A_3T^2S + A_4f_{yy}P + A_5f_yT^3 + A_6f_y^2T^2 + A_7f_y^3T + \\ A_8f_yTS) + \cdots. \tag{2.15}$$

The condition

$$C_i = B_i = 0 \qquad (i=1, 2, 3, 4) \tag{2.16}$$

leads to the equations

$$\begin{cases} p_1 + p_2 + p_3 + p_4 = 1, \\ ap_2 + bp_3 + p_4 = \dfrac{1}{2}, \\ acp_3 + Xp_4 = \dfrac{1}{6}, \\ acep_4 = \dfrac{1}{24}, \end{cases} \tag{2.17}$$

and

$$(2.18)\qquad \begin{cases} Yp_3+(1-a)p_4=\dfrac{1}{6}(2-3a) \\ (1-a)(1-b)p_4=\dfrac{1}{12}J, \\ (1-b)Xp_4=\dfrac{1}{24}(3-4b), \\ eYp_4=\dfrac{1}{12}(1-2a) \end{cases}$$

where

$$(2.19)\qquad J=3-4(a+b)+6ab.$$

Evidently it must hold that

$$(2.20)\qquad acep_4\neq 0.$$

Then p_i's are determined from (2.17). With these the equations in (2.18) must be compatible. The compatibility conditions are as follows:

$$(2.21)\qquad \begin{cases} Y=2(1-2a)ac, \\ (1-b)X=(3-4b)ace, \\ (1-a)(1-b)=2Jace. \end{cases}$$

From these it follows that

$$(2.22)\qquad b\neq 1,$$

and A_j' ($j=1, 2, \cdots, 8$) become as follows:

$$(2.23)\qquad \begin{cases} A_1=\dfrac{1}{12}(3-5a-5b+10ab), \\ A_2=\dfrac{1}{2}(3-5b),\quad A_3=\dfrac{1}{2}(2-5a), \\ A_4=\dfrac{5}{2(1-b)}[(3-4b)X+ae]-3, \\ A_5=-4A_1,\quad A_6=-A_3,\quad A_7=-1,\quad A_8=5b-2. \end{cases}$$

In the sequel we impose the condition (2.21).

Put

$$(2.24)\qquad \begin{cases} k_5 = f(x_1, z_1), \\ k_6 = f(x_1+ah, z_1+ahk_5), \\ k_7 = f\big(x_1+bh, z_1+(b-c)hk_5+chk_6\big), \\ k_8 = f\big(x_1+h, z_1+(1-d-e)hk_5+dhk_6+ehk_7\big), \end{cases}$$

$$(2.25)\qquad z_2 = z_1 + h\sum_{i=1}^{4} p_i k_{4+i}$$

and

$$(2.26)\qquad T^*(x_0, y_0; h) = z_2 - y(x_2).$$

Then the expansions of k_i's ($i=5, 6, 7, 8$) are as follows:

$$(2.27)\qquad k_5 = k_1 + hT + \frac{1}{2!}h^2(T^2+f_yT) + \frac{1}{3!}h^3(T^3+3TS+f_yT^2+$$
$$+f_y^2T) + \frac{1}{4!}h^4(T^4+6TS^2+4T^2S+3f_{yy}P+f_yT^3+f_y^2T^2+$$
$$+f_y^3T+7f_yTS)+\cdots,$$

$$(2.28)\qquad k_6 = k_1+(1+a)hT+\frac{1}{2!}h^2[(1+a)^2T^2+(1+2a)f_yT]+$$
$$+\frac{1}{3!}h^3[(1+a)^3T^3+3(1+a)(1+2a)TS+(1+3a)f_yT^2+$$
$$+(1+3a)f_y^2T]+\frac{1}{4!}h^4[(1+a)^4T^4+6(1+a)^2(1+2a)TS^2+$$
$$+4(1+a)(1+3a)T^2S+3(1+2a)^2f_{yy}P+(1+4a)f_yT^3+$$
$$+(1+4a)f_y^2T^2+(1+4a)f_y^3T+(7+28a+12a^2)f_yTS]+\cdots,$$

$$(2.29)\qquad k_7 = k_1+(1+b)hT+\frac{1}{2!}h^2[(1+b)^2T^2+(1+2b+2ac)f_yT]+$$
$$+\frac{1}{3!}h^3[(1+b)^3T^3+3(1+b)(1+2b+2ac)TS+\big(1+3b+$$
$$+3(2+a)ac\big)f_yT^2++(1+3b+6ac)f_y^2T]+\frac{1}{4!}h^4[(1+b)^4T^4+$$

$$+6(1+b)^2(1+2b+2ac)TS^2+4(1+b)(1+3b+3(2+a)ac)T^2S+$$
$$+3(1+2b+2ac)^2f_{yy}P+(1+4b+4(3+3a+a^2)ac)f_yT^3+$$
$$+(1+4b+12ac)f_y^2T^2+(1+4b+12ac)f_y^3T+(7+28b+12b^2+$$
$$+4(4+2a+3b)ac)f_yTS]+\cdots,$$

(2.30)
$$k_8=k_1+2hT+\frac{1}{2!}h^2[4T^2+(3+2X)f_yT]+\frac{1}{3!}h^3[8T^3+$$
$$+6(3+2X)TS+(4+6X+3Z)f_yT^2+(4+6X+6ace)f_y^2T]+$$
$$+\frac{1}{4!}h^4[16T^4+24(3+2X)TS^2+8(4+6X+3Z)T^2S+$$
$$+3(3+2X)^2f_{yy}P+(5+12X+12Z+4aZ+2beY)f_yT^3+$$
$$+(5+12X+12(2+a)ace)f_y^2T^2+(5+12X+24ace)f_y^3T+$$
$$+(47+84X+24Z+24(3+b)ace)f_yTS]+\cdots.$$

$T^*(x_0, y_0; h)$ is expanded as follows:

(2.31)
$$T^*(x_0, y_0; h)=\frac{2}{5!}h^5[A_1T^4+A_2TS^2+A_3T^2S+A_4f_{yy}P+A_5f_yT^3+$$
$$+A_6f_y^2T^2+A_7f_y^3T+A_8f_yTS]+\cdots.$$

Now we evaluate the determinant of the matrix

$$M=\begin{pmatrix} a & b & 1 & 1 & (1+a) & (1+b) & 2 \\ a^2 & b^2 & 1 & 1 & (1+a)^2 & (1+b)^2 & 4 \\ 0 & 2ac & 2X & 1 & 1+2a & 1+2b+2ac & 3+2X \\ a^3 & b^3 & 1 & 1 & (1+a)^3 & (1+b)^3 & 8 \\ 0 & 6abc & 6X & 3 & 3(1+a)(1+2a) & 3(1+b)(1+2b+2ac) & 6(3+2X) \\ 0 & 3a^2c & 3Z & 1 & 1+3a & 1+3b+3(2+a)ac & 4+6X+3Z \\ 0 & 0 & 0 & 1 & 1+3a & 1+3b+6ac & 4+6X+6ace \end{pmatrix}$$

and find that

$$\det M = -108a^4(2ac)^2(2ace)^2 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} = 0$$

and

$$\text{rank } M = 6.$$

Hence we can determine the values of the coefficients q_i's ($i=1, 2, \cdots, 8$) so that the expansion of

$$w = h\sum_{i=1}^{8} q_i k_i \tag{2.32}$$

may begin with the terms of h^5. In fact, if we put

$$\begin{aligned} &m_2 = \frac{1}{a}(k_2 - k_1), \quad m_3 = \frac{1}{2ac}(k_3 - k_1 - bm_2) \\ &m_4 = \frac{1}{2ace}(k_4 - k_1 - m_2) - \frac{(3-4b)m_3}{1-b}, \quad m_5 = k_5 - k_1 - m_2, \\ &m_6 = \frac{1}{a}(k_6 - k_5 - k_2 + k_1), \quad m_7 = \frac{1}{2ac}(k_7 - k_5 - k_3 + k_1 - bm_6), \\ &m_8 = \frac{1}{4ace}(k_8 - k_5 - k_4 + k_1 - m_6) - \frac{(3-4b)}{2(1-b)}m_7, \end{aligned} \tag{2.33}$$

and

$$w = \frac{1}{5}h(m_8 - m_6 + 6m_5 - m_4 - 4m_3), \tag{2.34}$$

then we have

$$\begin{aligned} w = \frac{1}{5!}h^5[&A_1^* T^4 + A_2^* TS^2 + A_3^* T^2 S + A_4^* f_{yy} P + A_5^* f_y T^3 + \\ &+ A_6^* f_y^2 T^2 + A_7^* f_y^3 T + A_8^* f_y TS] + \cdots, \end{aligned} \tag{2.35}$$

where

$$
(2.36)\quad \begin{cases} A_1^* = 5-7a-7b+14ab, \quad A_2^*=6(5-4a-7b+8ab), \\ A_3^* = 20-6a-24b, \quad A_4^* = 4-4b-4ac-(3-4b)(X-ac)/(1-b), \\ A_5^* = 4(2-8a-b+2ab), \quad A_6^*=8-6a, \quad A_7^* = 8, \\ A_8^* = 4(11-6a-9b). \end{cases}
$$

3. Existence of the formula

In this paragraph, we show first that the values of a, b and k can be determined so that

$$
(3.1)\qquad 2A_i = kA_i^* \qquad (i=1, 2, 3, 4)
$$

may hold. These equations can be written as follows:

$$
(3.2)\qquad 3-5a-5b+10ab = k(30-42a-42b+84ab),
$$

$$
(3.3)\qquad 3-5b = k(30-24a+42b+48ab),
$$

$$
(3.4)\qquad 2-5a = k(20-6a-24b),
$$

$$
(3.5)\qquad (5+6k)(3-4b)(X-ac) = 2(1-b)[12k(1-b-ac)+3-10ac].
$$

Subtracting (3.3) from (3.2), we have

$$
(3.6)\qquad a(18k-5)(2b-1) = 0.
$$

Since $a \neq 0$, it follows that $k=\frac{5}{18}$ or $b=\frac{1}{2}$.

Substituting $k=\frac{5}{18}$ into (3.3) and (3.4) and eliminating a from these, we have

$$
(3.7)\qquad 60b^2-77b+28 = 0.
$$

This equation has no real roots.

For $b=\frac{1}{2}$, from (3.3) and (3.4) we have $k=\frac{1}{18}$ and $a=\frac{1}{3}$.

Then, from (2.21), it follows that

$$
(3.8)\qquad ac = \frac{1}{8}, \quad ace = \frac{1}{4}, \quad X = \frac{1}{2}
$$

Table 1.

		k_2	k_3	k_4	k_5	k_6	k_7	k_8
h	T	1/3	1/2	1	1	4/3	3/2	2
$\frac{h^2}{2!}$	T^2	1/9	1/4	1	1	16/9	9/4	4
	$f_y T$		1/4	1	1	5/3	9/4	4
$\frac{h^3}{3!}$	T^3	1/27	1/8	1	1	64/27	27/8	8
	TS		3/8	3	3	20/3	81/8	24
	$f_y T^2$		1/8	1	1	2	27/8	8
	$f^2_y T$			3/2	1	2	13/4	17/2
$\frac{h^4}{4!}$	T^4	1/81	1/16	1	1	256/81	81/16	16
	TS^2		3/8	6	6	160/9	243/8	96
	T^2S		1/4	4	4	32/3	81/4	64
	$f_{yy}P$		3/16	3	3	25/3	243/16	48
	$f_y T^3$		1/18	7/9	1	7/3	91/18	142/9
	$f^2_y T^2$			1	1	7/3	9/2	18
	$f^3_y T$				1	7/3	9/2	17
	$f_y TS$			9	7	53/3	34	118
$\frac{h^5}{5!}$	T^5	1/243	1/32	1	1	1024/243	243/32	32
	TS^3		5/16	10	10	3200/81	1215/16	320
	T^2S^2		5/16	10	10	320/9	1215/16	320
	T^3S		5/36	35/9	5	140/9	455/12	1420/9
	$f_{yy}TT^2$		5/16	10	10	100/3	1215/16	320
	PR		15/32	15	15	500/9	3645/32	480
	TQ			15	15	140/3	435/4	500
	$f_y f_{yy} P$			135/8	105/8	995/24	375/4	440
	$f_y TS^2$			75/4	65/4	1865/36	2795/24	1595/3
	$f_y T^2 S$			15/2	55/6	475/18	725/12	950/3
	$f^2_y TS$				25/2	625/18	315/4	805/2
	$f_y T^4$		5/216	115/216	25/24	65/24	205/27	1645/54
	$f^2_y T^3$			5/9	5/6	5/2	35/6	655/18
	$f^3_y T^2$				5/6	5/2	35/6	185/6
	$f^4_y T$					5/3	5	30

and (3.5) is satisfied. Thus (3.1) has a unique solution

$$a=\frac{1}{3},\quad b=\frac{1}{2},\quad k=\frac{1}{18}, \tag{3.9}$$

For these values, other constants are determined as follows:

$$c=\frac{3}{8},\quad d=-\frac{3}{2},\quad e=2, \tag{3.10}$$

$$p_1=p_4=\frac{1}{6},\quad p_2=0,\quad p_3=\frac{4}{6} \tag{3.11}$$

and the coefficients in the expansions of k_i's ($i=2, 3, \cdots, 8$) are listed in Table 1.

From Table 1 we have

$$v=\frac{w}{18}=\frac{1}{90}h(k_1-4k_3+6k_5-4k_7+k_8+3(k_5-k_4)), \tag{3.12}$$

$$\begin{aligned} T^*(x_0, y_0; h)=&\frac{1}{5!}h^5\Big[\frac{1}{12}T^4+\frac{1}{2}TS^2+\frac{1}{3}T^2S+\frac{1}{4}f_{yy}P- \\ &-\frac{1}{3}f_yT^3-\frac{1}{3}f_y^2T^2-2f_y^3T+f_yTS\Big]+\frac{1}{6!}h^6\Big[\frac{1}{2}T^5+5TS^3+ \\ &+5T^2S^2-\frac{10}{9}T^3S+5f_{yy}TT^2+5TQ+\frac{15}{2}PR+13f_yf_{yy}P+ \\ &+\frac{17}{2}f_yTS^2-f_yT^2S-38f_y^2TS-\frac{3}{2}f_yT^4-\frac{26}{9}f_y^2T^3-9f_y^3T^2- \\ &-14f_y^4T\Big]+\cdots, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} T^*(x_0, y_0; h)-v=&\frac{1}{5!}h^5\Big[-\frac{4}{9}f_yT^3-\frac{2}{3}f_y^2T^2-\frac{22}{9}f_y^3T\Big]+ \\ &+\frac{1}{6!}h^6\Big[-\frac{100}{27}T^3S-\frac{16}{3}TQ+\frac{25}{6}f_yf_{yy}P-\frac{17}{9}f_yTS^2-10f_yT^2S- \\ &-\frac{154}{3}f_y^2TS-\frac{109}{54}fT^4-\frac{112}{27}f_y^2T^3-10f_y^3T^2-\frac{44}{3}f_y^4T\Big]+\cdots. \end{aligned} \tag{3.14}$$

Now put

(3.15) $$p=h\sum_{i=1}^{8}r_ik_i=c_1hk_1+\frac{1}{2!}h^2c_2T+\frac{1}{3!}h^3(c_3T^2+c_4f_yT)+\frac{1}{4!}h^4(c_5T^3+ +c_6TS+c_7f_yT^2+c_8f_y^2T)+\frac{1}{5!}h^5(a_1T^4+a_2TS^2+a_3T^2S+ a_4f_{yy}P+a_5f_yT^3+a_6f_y^2T^2+a_7f_y^3T+a_8f_yTS)+\cdots.$$

We require that

(3.16) $$c_i=0\quad(i=1,2,3,5,6),\quad a_j=0\quad(j=1,2,3,4).$$

Then it is readily shown that

(3.17) $$r_i=0\qquad(i=1,2,3,6,7,8),\quad r_4+r_5=0,$$

and we have

(3.18) $$k_5-k_4=\frac{1}{3!}h^3\Big(-\frac{1}{2}f_y^2T\Big)+\frac{1}{4!}h^4\Big(\frac{2}{9}f_yT^3+f_y^3T-2f_yTS\Big)+ +\frac{1}{5!}h^5\Big(\frac{10}{9}T^3S-\frac{15}{4}f_yf_{yy}P-\frac{5}{2}f_yTS^2+\frac{5}{3}f_yT^2S+\frac{25}{2}f_y^2TS+ +\frac{55}{108}f_yT^4+\frac{5}{18}f_y^2T^3+\frac{5}{6}f_y^3T^2\Big)+\cdots.$$

We write (2.4) and (2.24) as

(3.19) $$k_i=f(x_0+q_ih,\ w_i)\qquad(i=1,2,\cdots,8)$$

and define k_i^*'s by

(3.20) $$k_i^*=f(x_0+q_ih,\ w_i+p).$$

Then, for the choice $c_1=c_2=c_3=0$, we have

(3.21) $$k_i^*-k_i=\frac{1}{3!}h^3c_4f_y^2T+\frac{1}{4!}h^4[c_5f_yT^3+c_7f_y^2T^2+c_8f_y^3T+ +(c_6+4q_ic_4)f_yTS]+\frac{1}{5!}h^5[5q_ic_5T^5S+5q_ic_6TQ+ +(a_4+10q_i^2c_4)f_yf_{yy}P+(a_2+10q_i^2c_4)f_yTS^2+(a_3+5q_ic_7)f_yT^2S+ +(a_8+5q_ic_8)f_y^2TS+a_1f_yT^4+a_5f_y^2T^3+a_6f_y^3T^2+a_7f_y^4T]+\cdots,$$

and it follows that

(3.22)
$$g_i = h[r(k_5-k_4)+s(k_i^*-k_i)]$$
$$=\frac{1}{4!}h^4\Big[4\Big(-\frac{r}{2}+sc_4\Big)f_y^2 T\Big]+\frac{5}{5!}h^5\Big[\Big(\frac{2}{9}r+sc_5\Big)f_y T^3+sc_7 f_y^2 T^2+$$
$$+(r+sc_8)f_y^3 T+(-2r+s(c_6+4q_ic_4))f_y TS\Big]+\cdots.$$

Taking into consideration (3.14) and (3.18), we require that

(3.23)
$$-r+2sc_4=0,\quad 5\Big(\frac{2}{9}r+sc_5\Big)=-\frac{4}{9},\quad 5sc_7=-\frac{2}{3},$$
$$5(r+sc_8)=-\frac{22}{9},\quad -2r+s(c_6+4q_ic_4)=0.$$

Then we have

(3.24)
$$r_1=\frac{14t}{3(q_i-1)}+3t+r_8,\quad r_2=-\frac{21t}{(q_i-1)}-3t,\quad r_3=\frac{56t}{3(q_i-1)}-4t-4r_8,$$
$$r_4=-18t-\frac{7t}{3(q_i-1)}-3r_8,\quad r_5=23t+9r_8,\quad r_6=3t,\quad r_7=-4t-4r_8,$$

(3.25)
$$r=\frac{7}{45(q_i-1)},\quad s=\frac{1}{90t},$$

(3.26)
$$c_4=\frac{7t}{(q_i-1)},\quad c_5=-8t-\frac{28t}{9(q_i-1)},\quad c_6=-28t,$$
$$c_7=-12t,\quad c_8=-44t-\frac{14t}{(q_i-1)},$$

(3.27)
$$a_1=5\Big(-\frac{77t}{54(q_i-1)}-\frac{109t}{18}+\frac{3}{2}r_8\Big),\quad a_2=5\Big(-\frac{7t}{(q_i-1)}-\frac{119t}{3}+9r_8\Big),$$
$$a_3=5\Big(-\frac{14t}{3(q_i-1)}-30t+6r_8\Big),\quad a_4=5\Big(-\frac{7t}{2(q_i-1)}-\frac{43t}{2}+\frac{9}{2}r_8\Big)$$
$$a_5=5\Big(-\frac{7t}{9(q_i-1)}-\frac{40t}{9}+2r_8\Big),\quad a_6=5\Big(-\frac{7t}{3(q_i-1)}-6t+6r_8\Big),$$
$$a_7=5(12t+8r_8),\quad a_8=5\Big(-\frac{21t}{(q_i-1)}-84t+18r_8\Big),$$

and

(3.28)
$$T^*(x_0,\ y_0;\ h)-m=\frac{1}{3}\frac{1}{6!}h^6\Big[8(q_i-1)T^3S+4(7q_i-4)TQ+$$
$$+\Big(2(10-7q_i)-\frac{9}{2}k\Big)f_yf_{yy}P+(2(10-7q_i)-9k)f_y TS^2+$$
$$+6(2q_i-3k)f_y T^2S+(4(11q_i-14)-18k)f_y^2 TS-$$
$$-\frac{3}{2}kf_y T^4-2(4+k)f_y^2 T^3-6(4+k)f_y^3 T^2-8(7+k)f_y^4 T\Big]+\cdots,$$

where

(3.29) $$m = v + g_i, \quad k = \frac{1}{t} r_8.$$

Referring to Table 2, we take $k=0$ and $i=6\left(q_i=\frac{4}{3}\right)$ and put $t=\frac{1}{45}$ so that the coefficient of $\frac{1}{7!}h^7 f_y^2 f_{yy} P$ in $T^*(x_0, y_0; h)-m$ may

Table 2.

q_i	2	3/2	4/3	1/2	1/3	0
$8(q_i-1)$	8	4	8/3	−4	−16/3	−8
$4(7q_i-4)$	40	26	64/3	−2	−20/3	−16
$2(10-7q_i)$	−8	−1	4/3	13	46/3	20
$12q_i$	24	18	16	6	4	0
$4(11q_i-14)$	32	10	8/3	−34	−124/3	−56

be small. Then we have the formulas as follows:

(3.30) $$\begin{cases} k_1 = f(x_0, y_0), \quad k_2 = f\left(x_0 + \frac{h}{3}, y_0 + \frac{h}{3}k_1\right), \\ k_3 = f\left(x_0 + \frac{h}{2}, \quad y_0 + \frac{h}{8}(k_1 + 3k_2)\right), \\ k_4 = f\left(x_0 + h, y_0 + h\left(\frac{1}{2}k_1 - \frac{3}{2}k_2 + 2k_3\right)\right) \end{cases}$$

(3.31) $$z_1 = y_0 + \frac{h}{6}(k_1 + 4k_3 + k_4),$$

(3.32) $$\begin{cases} k_5 = f(x_1, z_1), \quad k_6 = f\left(x_1 + \frac{h}{3}, z_1 + \frac{h}{3}k_5\right), \\ k_7 = f\left(x_1 + \frac{h}{2}, z_1 + \frac{h}{8}(k_5 + 3k_6)\right), \\ k_8 = f\left(x_1 + h, z_1 + h\left(\frac{1}{2}k_5 - \frac{3}{2}k_6 + 2k_7\right)\right) \end{cases}$$

(3.33) $$z_2 = z_1 + \frac{h}{6}(k_5 + 4k_7 + k_8),$$

(3.34) $$p = \frac{h}{45}(17k_1 - 66k_2 + 52k_3 - 25k_4 + 23k_5 + 3k_6 - 4k_7),$$

(3.35) $$k_6^*=f\left(x_1+\frac{h}{3},\ z_1+\frac{h}{3}k_5+p\right),$$

(3.36) $$m=h\left[\frac{1}{90}(k_1-4k_3+6k_5-4k_7+k_8)+\frac{1}{2}(k_5-k_4+(k_6^*-k_6))\right]$$

(3.37) $$z_2-m-y(x_2)=\frac{1}{3}\frac{1}{6!}h^6\left[\frac{8}{3}T^3S+\frac{64}{3}TQ+\frac{4}{3}f_yf_{yy}P+\right.$$
$$+\frac{4}{3}f_yTS^2+16f_yT^2S+\frac{8}{3}f_y^2TS-8f_y^2T^3-24f_y^3T^2-$$
$$\left.-56f_y^4T\right]+O(h^7).$$

To compare our method with the usual one, let

(3.38) $$\begin{cases}\hat{k}_2=f\left(x_0+\frac{2}{3}h,\ y_0+\frac{2}{3}hk_1\right),\\ \hat{k}_3=f\left(x_0+h,\ y_0+\frac{h}{4}(k_1+3\hat{k}_2)\right),\\ \hat{k}_4=f(x_0+2h,\ y_0+h(k_1-3\hat{k}_2+4\hat{k}_3)),\end{cases}$$

and

(3.39) $$u=\frac{h}{90}(k_1+8\hat{k}_3+2\hat{k}_4-4k_3-k_4-k_5-4k_7-k_8),$$

then we have

(3.40) $$z_2-u-y(x_2)=\frac{1}{3}\frac{1}{6!}h^6\left[\frac{32}{9}T^3S+16TQ-8f_yf_{yy}P-8f_yTS^2+\right.$$
$$\left.16f_yTS^2+32f_y^2TS-\frac{32}{9}f_y^2T^3-16f_y^3T^2-32f_y^4T\right]+O(h^7).$$

From (3.37) and (3.40) it is seen that our method may compare favorably with the usual one.

4. Numerical examples

In the following examples, two methods are used for numerical integration.

In *Method 1*, the step-size is halved until the inequality

(4.1) $$|m|\leq\varepsilon|z_2|\qquad(\varepsilon=0.5\times10^{-7})$$

is satisfied; the value

$$v = e + m + 2h[f(x_1, z_1 + e) - k_5] \tag{4.2}$$

is computed and e, y_0 and x_0 are replaced by v, z_2 and x_2 respectively. Initially e is set equal to 0.

In *Method 2*, the step-size is halved until the inequality

$$|m| \leqq \varepsilon |z_2 - m| \qquad (\varepsilon = 0.5 \times 10^{-7}) \tag{4.3}$$

is satisfied and y_0 and x_0 are replaced by $z_2 - m$ and x_2 respectively.

Example 1.

$$y' = 2xy, \quad y(0) = 1. \tag{4.4}$$

Example 2.

$$y' = -5y, \quad y(0) = 1. \tag{4.5}$$

In both examples, at the start of integration h is set equal to 0.05, and the error E, the truncation error T and u are computed for comparison. The computation is carried out in the floating-point arithmetic with 39 bits mantissa and rounding is done by chopping.

Table 3. $y' = 2xy$, Method 1

X	m	u	T	e	E
0. 1	−3. 736-10	−8. 387-11	8. 736-10	−3. 736-10	8. 367-10
0. 2	2. 786-09	2. 680-09	4. 053-09	2. 401-09	4. 919-09
0. 3	5. 215-09	5. 207-09	6. 574-09	7. 736-09	1. 174-08
0. 4	6. 156-09	5. 723-09	7. 563-09	1. 443-08	2. 023-08
0. 5	3. 991-09	2. 675-09	5. 042-09	1. 972-08	2. 715-08
0. 6	−4. 501-09	−7. 609-09	−4. 904-09	1. 739-08	2. 548-08
0. 7	−2. 544-08	−3. 210-08	−2. 974-08	−5. 788-09	−7. 167-10
0. 8	−7. 016-08	−8. 371-08	−8. 374-08	−7. 682-08	−8. 457-08
0. 9	−6. 406-09	−6. 924-09	−7. 116-09	−1. 015-07	−1. 126-07
1. 0	−1. 327-08	−1. 427-08	−1. 480-08	−1. 452-07	−1. 621-07
1. 1	−2. 618-08	−2. 811-08	−2. 931-08	−2. 242-07	−2. 523-07
1. 2	−5. 019-08	−5. 386-08	−5. 651-08	−3. 696-07	−4. 197-07
1. 3	−9. 450-08	−1. 014-07	−1. 070-07	−6. 400-07	−7. 339-07
1. 4	−1. 760-07	−1. 891-07	−2. 004-07	−1. 147-06	−1. 330-06
1. 5	−3. 260-07	−3. 507-07	−3. 736-07	−2. 105-06	−2. 468-06
1. 6	−6. 027-07	−6. 495-07	−6. 954-07	−3. 924-06	−4. 656-06
1. 7	−4. 025-08	−4. 183-08	−4. 395-08	−5. 533-06	−6. 633-06
1. 8	−7. 501-08	−7. 803-08	−8. 201-08	−8. 010-06	−9. 703-06
1. 9	−1. 405-07	−1. 464-07	−1. 537-07	−1. 192-05	−1. 460-05
2. 0	−2. 651-07	−2. 764-07	−2. 905-07	−1. 826-05	−2. 262-05

Table 4. y′=2xy, Method 2

X	m	u	T	E
0. 1	−3. 736-10	−3. 387-11	8. 367-10	1. 208-09
0. 2	2. 786-09	2. 980-09	4. 056-09	2. 517-09
0. 3	5. 215-09	5. 207-09	6. 577-09	3. 998-09
0. 4	6. 156-09	5. 723-09	7. 563-09	5. 770-09
0. 5	3. 991-09	2. 675-09	5. 046-09	7. 345-09
0. 6	−4. 501-09	−7. 609-09	−4. 904-09	7. 869-09
0. 7	−2. 544-08	−3. 210-08	−2. 974-08	4. 668-09
0. 8	−7. 016-08	−8. 371-08	−8. 375-08	−8. 153-09
0. 9	−6. 406-09	−6. 924-09	−7. 116-09	−1. 087-08
1. 0	−1. 327-08	−1. 427-08	−1. 479-08	−1. 563-08
1. 1	−2. 618-08	−2. 811-08	−2. 931-08	−2. 458-08
1. 2	−5. 019-08	−5. 386-08	−5. 649-08	−4. 211-08
1. 3	−9. 450-08	−1. 014-07	−1. 070-07	−7. 618-08
1. 4	−1. 760-07	−1. 891-07	−2. 004-07	−1. 438-07
1. 5	−3. 260-07	−3. 507-07	−3. 736-07	−2. 792-07
1. 6	−6. 027-07	−6. 495-07	−6. 955-07	−5. 509-07
1. 7	−4. 025-08	−4. 183-08	−4. 406-08	−7. 767-07
1. 8	−7. 501-08	−7. 803-08	−8. 201-08	−1. 122-06
1. 9	−1. 405-07	−1. 464-07	−1. 538-07	−1. 665-06
2. 0	−2. 651-07	−2. 764-07	−2. 905-07	−2. 541-06

Table 5. y′=−5y, Method 1

X	m	u	T	e	E
0. 1	1. 119-08	1. 074-08	1. 016-08	4. 420-08	4. 036-08
0. 2	6. 788-09	6. 512-09	6. 165-09	5. 272-08	4. 928-08
0. 3	4. 117-09	3. 950-09	3. 739-09	4. 716-08	4. 484-08
0. 4	2. 497-09	2. 396-09	2. 269-09	3. 751-08	3. 626-08
0. 5	1. 515-09	1. 453-09	1. 376-09	2. 797-08	2. 749-08
0. 6	9. 187-10	8. 813-10	8. 343-10	2. 002-08	2. 001-08
0. 7	5. 572-10	5. 346-10	5. 061-10	1. 394-08	1. 416-08
0. 8	3. 380-10	3. 242-10	3. 070-10	9. 505-09	9. 813-09
0. 9	2. 050-10	1. 967-10	1. 862-10	6. 381-09	6. 695-09
1. 0	1. 243-10	1. 193-10	1. 129-10	4. 232-09	4. 512-09
1. 1	7. 541-11	7. 234-11	6. 850-11	2. 778-09	3. 010-09
1. 2	4. 574-11	4. 388-11	4. 154-11	1. 809-09	1. 991-09
1. 3	2. 774-11	2. 661-11	2. 520-11	1. 170-09	1. 308-09
1. 4	1. 683-11	1. 614-11	1. 528-11	7. 523-10	8. 546-10
1. 5	1. 021-11	9. 791-12	9. 269-12	4. 813-10	5. 553-10
1. 6	6. 190-12	5. 938-12	5. 622-12	3. 066-10	3. 592-10
1. 7	3. 755-12	3. 602-12	3. 410-12	1. 945-10	2. 315-10
1. 8	2. 277-12	2. 185-12	2. 069-12	1. 230-10	1. 487-10
1. 9	1. 381-12	1. 325-12	1. 255-12	7. 757-11	9. 518-11
2. 0	8. 378-13	8. 037-13	7. 609-13	4. 878-11	6. 077-11

Table 6. $y' = -5y$, Method 2

X	m	u	T	E
0. 1	1. 119-08	1. 074-08	1. 016-08	−4. 140-09
0. 2	6. 788-09	6. 512-09	6. 165-09	−5. 028-09
0. 3	4. 117-09	3. 950-09	3. 739-09	−4. 570-09
0. 4	2. 497-09	2. 396-09	2. 268-09	−3. 701-09
0. 5	1. 515-09	1. 453-09	1. 376-09	−2. 806-09
0. 6	9. 187-10	8. 813-10	8. 343-10	−2. 041-09
0. 7	5. 572-10	5. 346-10	5. 061-10	−1. 446-09
0. 8	3. 380-10	3. 242-10	3. 070-10	−1. 003-09
0. 9	2. 050-10	1. 967-10	1. 862-10	−6. 847-10
1. 0	1. 243-10	1. 193-10	1. 129-10	−4. 614-10
1. 1	7. 541-11	7. 234-11	6. 850-11	−3. 084-10
1. 2	4. 574-11	4. 388-11	4. 154-11	−2. 043-10
1. 3	2. 774-11	2. 661-11	2. 520-11	−1. 343-10
1. 4	1. 683-11	1. 614-11	1. 528-11	−8. 777-11
1. 5	1. 021-11	9. 791-12	9. 271-12	−5. 709-11
1. 6	6. 190-12	5. 938-12	5. 622-12	−3. 695-11
1. 7	3. 755-12	3. 602-12	3. 410-12	−2. 382-11
1. 8	2. 277-12	2. 185-12	2. 069-12	−1. 531-11
1. 9	1. 381-12	1. 325-12	1. 255-12	−9. 801-12
2. 0	8. 377-13	8. 037-13	7. 608-13	−6. 260-12

References

[1] Ceschino, F. et J. Kuntzmann: *Problèmes différentiels de conditions initiales*, Paris, 1963.

[2] Ralston, A. and H. S. Wilf: *Mathematical methods for digital computers*, New York, 1960.

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