J. Sci. Hiroshima Univ. Ser. A-I
30 (1966), 75-90

Direct Methods for the Numerical Solution of Partial Difference Equations for a Rectangle

Hisayoshi Shintani
(Received March 15, 1966)

1. Introduction

The problem of solving approximately elliptic partial differential equations over a rectangle with Dirichlet boundary conditions is often reduced to the problem of solving the system of linear equations of the following form

$$\text{(1.1)}\qquad \begin{cases} A_1\boldsymbol{x}_1 - B_1\boldsymbol{x}_2 = \boldsymbol{\Gamma}_1, \\ -C_k\boldsymbol{x}_{k-1} + A_k\boldsymbol{x}_k - B_k\boldsymbol{x}_{k+1} = \boldsymbol{\Gamma}_k \qquad (k=2, 3, \cdots, m-1) \\ -C_m\boldsymbol{x}_{m-1} + A_m\boldsymbol{x}_m = \boldsymbol{\Gamma}_m, \end{cases}$$

where $\boldsymbol{x}_i$ and $\boldsymbol{\Gamma}_i$ $(i=1, 2, \cdots, m)$ are n-vectors and A_i, B_i and C_i are $n\times n$ diagonal or tridiagonal matrices [4][1].

The system (1.1) is usually solved by the iterative methods and the direct methods are rarely used because of the storage capacity [4]. Among the direct methods, however, there are the square root method [4], the hypermatrix method [2, 3, 13], the tensor product method [8, 9], and so on [1, 6, 7, 12, 14]. As G.E. Forsythe and W.R. Wasow [4] indicate, direct methods are of practical use when they need not so large storage space and the inverse matrices can be generated or the problem is reduced to the inversion of the matrices of the small order.

In this paper, direct methods are derived in an elementary manner for (1.1), for the periodic boundary problems of Poisson's equations and of one-dimensional heat equations, and for biharmonic equations. Stability of the numerical process is discussed in some cases.

2. Dirichlet problem

In this paragraph, we assume that the system (1.1) has a unique solution and that B_k $(k=1, 2, \cdots, m-1)$ are non-singular.

1) Numbers in square brackets refer to the references listed at the end of this paper.

We define the matrices P_k and the vectors $\boldsymbol{q}_k$ ($k=0, 1, \cdots, m$) as follows:

$$P_0 = I, \quad P_1 = B_1^{-1}A_1, \tag{2.1}$$

$$P_k = B_k^{-1}(A_kP_{k-1} - C_kP_{k-2}) \qquad (k=2, 3, \cdots, m), \tag{2.2}$$

$$\boldsymbol{q}_0 = \boldsymbol{0}, \quad \boldsymbol{q}_1 = B_1^{-1}\boldsymbol{\Gamma}_1, \tag{2.3}$$

$$\boldsymbol{q}_k = B_k^{-1}(A_k\boldsymbol{q}_{k-1} - C_k\boldsymbol{q}_{k-2} + \boldsymbol{\Gamma}_k) \qquad (k=2, 3, \cdots, m), \tag{2.4}$$

where B_m is the identity matrix I. Then the system (1.1) can be rewritten as follows:

$$\begin{cases} \boldsymbol{x}_{k+1} = P_k\boldsymbol{x}_1 - \boldsymbol{q}_k & (k=1, 2, \cdots, m) \\ \boldsymbol{x}_{m+1} = \boldsymbol{0}. \end{cases} \tag{2.5}$$

Hence we have

$$P_m\boldsymbol{x}_1 = \boldsymbol{q}_m. \tag{2.6}$$

Since B_k ($k=1, 2, \cdots, m$) are non-singular diagonal or tridiagonal matrices, P_k and $\boldsymbol{q}_k$ can be obtained easily and, to obtain $\boldsymbol{x}_1$, we have only to solve the system (2.6) of linear equations with n unknowns. Once $\boldsymbol{x}_1$ is obtained, $\boldsymbol{x}_2$ can be obtained similarly from the system

$$\begin{cases} B_2\boldsymbol{x}_3 = A_2\boldsymbol{x}_2 - (C_2\boldsymbol{x}_1 + \boldsymbol{\Gamma}_2), \\ B_k\boldsymbol{x}_{k+1} = A_k\boldsymbol{x}_k - C_k\boldsymbol{x}_{k-1} - \boldsymbol{\Gamma}_k & (k=2, 3, \cdots, m), \\ \boldsymbol{x}_{m+1} = \boldsymbol{0}. \end{cases} \tag{2.7}$$

In the case where C_k ($k=1, 2, \cdots, m-1$) are non-singular, to reduce the propagation of round-off errors, it seems to be better to solve first $\boldsymbol{x}_1$ and $\boldsymbol{x}_m$ and then $\boldsymbol{x}_2$ and $\boldsymbol{x}_{m-1}$ and so on.

The process becomes simpler when A_k, B_k and C_k are of the form

$$A_k = a_kI + \alpha_kJ, \quad B_k = b_kI + \beta_kJ, \quad C_k = c_kI + \gamma_kJ \qquad (k=1, 2, \cdots, m), \tag{2.8}$$

where a_k, b_k, c_k, α_k, β_k, and γ_k are scalars and

$$J = \begin{pmatrix} 0, & 1, & & \mathbf{0} \\ 1, & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ \mathbf{0} & & 1, & 0 \end{pmatrix}. \tag{2.9}$$

In this case, if we put

$$T=(t_{ij}), \quad t_{ij}=\sin ij\theta, \tag{2.10}$$

$$G=\text{Diag}\,(\cos\theta, \cos 2\theta, \cdots, \cos n\theta) \tag{2.11}$$

$$\theta=\pi/N, \quad N=n+1, \tag{2.12}$$

then, since we have

$$A_k=T^{-1}(a_k I+2\alpha_k G)T, \tag{2.13}$$

$$B_k=T^{-1}(b_k I+2\beta_k G)T, \tag{2.14}$$

$$C_k=T^{-1}(c_k I+2\gamma_k G)T, \tag{2.15}$$

the system (1.1) can be rewritten as follows:

$$\begin{cases} \boldsymbol{y}_2=D_1\boldsymbol{y}_1-\boldsymbol{g}_1, \\ \boldsymbol{y}_{k+1}=D_k\boldsymbol{y}_k-E_k\boldsymbol{y}_{k-1}-\boldsymbol{g}_k \qquad (k=1, 2, \cdots, m), \\ \boldsymbol{y}_{m+1}=0, \end{cases} \tag{2.16}$$

where

$$F_k=(b_k I+2\beta_k G)^{-1}, \tag{2.17}$$

$$D_k=F_k(a_k I+2\alpha_k G), \quad E_k=F_k(c_k I+2\gamma_k G), \tag{2.18}$$

$$\boldsymbol{g}_k=F_k T\Gamma_k, \quad \boldsymbol{y}_k=T\boldsymbol{x}_k. \tag{2.19}$$

Put

$$P_0=I, \quad P_1=D_1, \tag{2.20}$$

$$P_k=D_kP_{k-1}-E_kP_{k-2} \qquad (k=2, 3, \cdots, m), \tag{2.21}$$

$$\boldsymbol{q}_0=0, \quad \boldsymbol{q}_1=\boldsymbol{g}_1, \tag{2.22}$$

$$\boldsymbol{q}_k=D_k\boldsymbol{q}_{k-1}-E_k\boldsymbol{q}_{k-2}+\boldsymbol{g}_k \qquad (k=2, 3, \cdots, m), \tag{2.23}$$

then $\boldsymbol{x}_1$ is given by the formula

$$\boldsymbol{x}_1=\frac{2}{N}TP_m^{-1}\boldsymbol{q}_m, \tag{2.24}$$

because

$$T^{-1}=\frac{2}{N}T. \tag{2.25}$$

Since F_k, E_k, and D_k are diagonal matrices, P_m is also a diagonal matrix and its inverse matrix is easily obtained. We note that the elements of the matrix T need not be stored, because

$$r_i=\sum_{j=1}^{n}t_{ij}f_j \tag{2.26}$$

can be computed by the following recurrence formula

$$p_n=p_{n+1}=0, \tag{2.27}$$

$$p_k=(2\cos i\theta)p_{k+1}-p_{k+2}+f_{k+1} \qquad (k=n-1,\ n-2,\ \cdots,\ 0), \tag{2.28}$$

$$r_i=p_0\sin i\theta. \tag{2.29}$$

In the case where

$$a_k=a,\quad \alpha_k=\alpha,\quad b_k=c_k=b,\quad \beta_k=\gamma_k=\beta \qquad (k=1,\ 2,\ \cdots,\ m), \tag{2.30}$$

the method becomes much simpler. We define the polynomials $P_i(x)$ $(i=-1,\ 0,\ \cdots,\ m)$ by the formula

$$P_{-1}(x)=0,\quad P_0(x)=1, \tag{2.31}$$

$$P_{k+1}(x)=xP_k(x)-P_{k-1}(x) \qquad (k=0,\ 1\ \cdots,\ m-1), \tag{2.32}$$

then it follows that

$$P_k=P_k(D),\quad \boldsymbol{q}_k=\sum_{i=1}^{k}P_{k-i}\boldsymbol{g}_i \qquad (k=0,\ 1,\ \cdots,\ m), \tag{2.33}$$

where

$$D=D_k,\quad P_{-1}=0. \tag{2.34}$$

From the equations

$$(2.35)\qquad \begin{cases} y_{k+1} = Dy_k - (g_k + y_{k-1}) & (k=0, 1, \cdots, m), \\ y_{l+1} = Dy_l - y_{l-1} - g_l & (l=k+1, \cdots, m) \\ y_{m+1} = y_0 = 0, \end{cases}$$

we have

$$(2.36)\qquad y_k = P_{m+1-k}^{-1}\left[P_{m-k}y_{k-1} + \sum_{i=k}^{m} P_{m-i}g_i\right] \qquad (k=1, 2, \cdots, m)$$

and, in the same way, we have

$$(2.37)\qquad y_{m-k} = P_{m-k-1}^{-1}\left[P_{m-k-2}y_{m-k+1} + \sum_{i=0}^{m-k-2} P_i g_{i+k+1}\right] \qquad (k=0, 1, \cdots, m-1).$$

From among the numerical processes that apply these formulas, the following two may be mentioned.

1°. One-sided process that utilizes the formula (2.36) to compute x_1, x_2, $\cdots$, x_m successively.

2°. Two-sided process that uses (2.36) to compute x_1, x_2, $\cdots$, and applies (2.37) to compute x_m, x_{m-1}, $\cdots$.

For the numerical process (2.36) (or (2.37)) to be stable, the eigenvalues of $P_i^{-1}P_j$ $(j=i-1, \cdots, 0;\ i=1, 2, \cdots, m)$ must be less than one in modulus. As is well-known, $P_k(x)$ can be expressed as follows:

$$(2.38)\qquad P_k(x) = \begin{cases} \sinh(k+1)\mu/\sinh\mu, & 2\cosh\mu = x \quad (x>2), \\ \left(\frac{x}{2}\right)^k(1+k) & (|x|=2), \\ \sin(k+1)\mu/\sin\mu, & 2\cos\mu = x \quad (|x|<2), \\ (-1)^k\sinh(k+1)\mu/\sinh\mu, & 2\cosh\mu = |x| \quad (x<-2). \end{cases}$$

Hence, let

$$(2.39)\qquad D = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

then, since

$$(2.40)\qquad P_i^{-1}P_j = \mathrm{Diag}\left(P_j(\lambda_1)/P_i(\lambda_1), \cdots, P_j(\lambda_n)/P_i(\lambda_n)\right),$$

the process (2.36) (or (2.37)) is stable, provided

$$(2.41)\qquad |\lambda_i| \geqq 2 \qquad (i=1, 2, \cdots, n).$$

For instance, according as the Laplace's operator is discretized by the five-point formula or by the nine-point formula, we have

(2.42) $$\lambda_i = 4 - 2\cos i\theta = 2\left(1 + 2\sin^2\frac{i\theta}{2}\right)$$

or

(2.43) $$\lambda_i = \frac{20 - 8\cos i\theta}{4 + 2\cos i\theta} = 2 + \frac{6(1 - \cos i\theta)}{2 + \cos i\theta} \qquad (i = 1, 2, \cdots, n)$$

In both cases, the process (2.36) (or (2.37)) is stable and the scale factor to be multiplied with $P_k(\lambda_i)$ to guard against the overflow is easily determined from (2.38).

Substituting

(2.44) $$\boldsymbol{y}_1 = P_m^{-1}\sum_{i=1}^{m} P_{m-i}\boldsymbol{g}_i$$

into

(2.45) $$\boldsymbol{y}_k = P_{k-1}\boldsymbol{y}_1 - \sum_{i=k-1}^{m} P_{k-1-i}\boldsymbol{g}_i \qquad (k = 2, 3, \cdots, m),$$

and making use of the relation

(2.46) $$P_r(x)P_{m-i}(x) - P_m(x)P_{r-i}(x) = P_{m-r-1}(x)P_{i-1}(x)$$

we can write $\boldsymbol{x}_k$ explicitly in the following form

(2.47) $$\boldsymbol{x}_k = \frac{2}{N}\, TP_m^{-1}[P_{m-k}\sum_{i=1}^{k-1} P_{i-1}\boldsymbol{g}_i + P_{k-1}\sum_{i=k}^{m} P_{m-i}\boldsymbol{g}_i]. \qquad (k = 1, 2, \cdots, m).$$

This result coincides with that of E. Egérvary [3].

Numerical example

The problem is to find the function u that satisfies the equation

(2.48) $$\Delta u(x, y) = 0$$

in the domain

(2.49) $$R: \quad -1 < x < 1, \quad -1 < y < 1,$$

and the boundary condition

(4.50) $$u(1, y) = u(-1, y) = 0 \qquad (|y| \leqq 1),$$

(4.51) $$u(x, -1) = 0, \quad u(x, 1) = 100\sin\pi x \qquad (|x| \leqq 1).$$

The Laplacian is discretized by the five-point formula with the meshsize 1/20, so that $m=n=38$. The computation is carried out in the floating-point arithmetic with 39 binary bits mantissa and rounding is done by chopping. $P_k(\lambda_i)$'s range from 1 to 5.7×10^{21}, so that $2^{-32}P_k(\lambda_i)$'s are computed.

Since the problem is symmetric with respect to the y-axis, unknowns are arranged so that the vector $\boldsymbol{x}_i$ may coincide with the vector $\boldsymbol{x}_{m+1-i}$. To check on the stability of our numerical process, the one-sided process is used and the computed vectors $\boldsymbol{x}_i$ and $\boldsymbol{x}_{m+1-i}$ are compared. The maximum discrepancy between the corresponding elements of the computed vectors $\boldsymbol{x}_i$ and $\boldsymbol{x}_{m+1-i}$ was one unit in the tenth significant digit.

3. Periodic boundary problems

3.1 Poisson's equation

R. W. Hockney [5] treated the problem of finding the approximate solution of Poisson's equation over a rectangle with the boundary condition that the solution be periodically repeated in both x- and y-directions. This problem is reduced to that of solving the following system of linear equations

$$(3.1)\qquad \begin{cases} A\boldsymbol{x}_1-\boldsymbol{x}_2-\boldsymbol{x}_m=\boldsymbol{\Gamma}_1, \\ -\boldsymbol{x}_{k-1}+A\boldsymbol{x}_k-\boldsymbol{x}_{k+1}=\boldsymbol{\Gamma}_k \qquad (k=2,3,\cdots,m-1), \\ -\boldsymbol{x}_1-\boldsymbol{x}_{m-1}+A\boldsymbol{x}_m=\boldsymbol{\Gamma}_m, \end{cases}$$

where

$$(3.2)\qquad A=\begin{pmatrix} 4, & -1, & 0, & \dots, & & 0, & -1 \\ -1, & 4, & -1, & 0, & & & 0 \\ 0, & -1, & 4, & & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & 0 \\ 0 & & & -1, & 4, & & -1 \\ -1, & 0, & & & 0, & -1, & 4 \end{pmatrix}=4I-K$$

Hockney solved (3.1) using Fourier analysis, but we shall show that it can be solved in the same way as in the preceding paragraph.

Put

$$(3.3)\qquad H=\mathrm{Diag}\,(1,\cos\theta,\cos 2\theta,\cdots,\cos(n-1)\theta),$$

(3.4)
$$R = \begin{pmatrix} \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}}, & \cdots, & \frac{1}{\sqrt{2}} \\ 1, & \cos\theta, & \cdots, & \cos(n-1)\theta \\ \vdots & & & \\ 1, & \cos(l-1)\theta, & \cdots, & \cos(n-1)(l-1)\theta \\ \delta, & \delta\cos l\theta, & \cdots, & \delta\cos(n-1)l\theta \\ 0, & \sin(l+1)\theta, & \cdots, & \sin(n-1)(l+1)\theta \\ \vdots & & & \\ 0, & \sin(n-1)\theta, & \cdots, & \sin(n-1)(n-1)\theta \end{pmatrix}$$

where

(3.5)
$$\theta = 2\pi/n, \quad \delta = \begin{cases} 1 & (n: \text{ odd}) \\ \frac{1}{\sqrt{2}} & (n: \text{ even}) \end{cases}$$

and l is the greatest integer not exceeding $n/2$. Then since K is a circulant matrix [10], it follows that

(3.6)
$$A = R^{-1}DR, \quad D = 4I - 2H, \quad R^{-1} = \frac{2}{n}R^T.$$

Making use of this result, we can rewrite (3.1) as follows:

(3.7)
$$\begin{cases} D\boldsymbol{y}_1 - \boldsymbol{y}_2 - \boldsymbol{y}_m = \boldsymbol{g}_1 \\ -\boldsymbol{y}_{k-1} + D\boldsymbol{y}_k - \boldsymbol{y}_{k+1} = \boldsymbol{g}_k \qquad (k=2, 3, \cdots, m), \\ \boldsymbol{y}_{m+1} = \boldsymbol{y}_1, \end{cases}$$

where

(3.8)
$$\boldsymbol{y}_j = R\boldsymbol{x}_j, \quad \boldsymbol{g}_j = R\boldsymbol{\Gamma}_j \qquad (j=1, 2, \cdots, m).$$

Then, as in the preceding paragraph, we have

(3.9)
$$\boldsymbol{y}_{k+1} = P_k\boldsymbol{y}_1 - P_{k-1}\boldsymbol{y}_m - \sum_{i=1}^{k} P_{k-i}\boldsymbol{g}_i \qquad (k=1, 2, \cdots, m).$$

From this and the last equation in (3.7), it follows that

(3.10)
$$\begin{cases} P_{m-1}\boldsymbol{y}_m = (P_m - I)\boldsymbol{y}_1 - \sum_{i=1}^{m} P_{m-i}\boldsymbol{g}_i, \\ (I + P_{m-2})\boldsymbol{y}_m = P_{m-1}\boldsymbol{y}_1 - \sum_{i=1}^{m-1} P_{m-1-i}\boldsymbol{g}_i, \end{cases}$$

and hence

(3.11) $$\Delta \boldsymbol{y}_1 = (I + P_{m-2})\sum_{i=1}^{m} P_{m-i}\boldsymbol{g}_i - P_{m-1}\sum_{i=1}^{m-1} P_{m-1-i}\boldsymbol{g}_i,$$

(3.12) $$\Delta \boldsymbol{y}_m = P_{m-1}\sum_{i=1}^{m} P_{m-i}\boldsymbol{g}_i - (P_m - I)\sum_{i=1}^{m-1} P_{m-1-i}\boldsymbol{g}_i,$$

where

(3.13) $$\begin{aligned}\Delta &= (I + P_{m-2})(P_m - I) - P_{m-1}^2 \\ &= P_{m-2}P_m - P_{m-1}^2 + P_m' - P_{m-2} - I.\end{aligned}$$

As is readily seen from the definition of $P_k(x)$, $P_k(x)$ is a polynomial in x of degree k with the leading coefficient 1. Hence the direct application of the formulas (3.11), (3.12) and (3.13) will result in the loss of significant figures and some protection must be done.

As is easily checked, there hold the relations

(3.14) $$P_{m-1}(x)P_{m-i}(x) - P_m(x)P_{m-1-i}(x) = P_{i-1}(x) \qquad (i=1, 2, \cdots, m-1)$$

(3.15) $$P_{m-2}(x)P_{m-i}(x) - P_{m-1}(x)P_{m-1-i}(x) = P_{i-2}(x). \qquad (i=2, 3, \cdots, m-1)$$

Substituting these into (3.11), (3.12) and (3.13), we have

(3.16) $$\boldsymbol{y}_1 = (P_m - P_{m-2} - 2I)^{-1}\sum_{i=1}^{m}(P_{m-i} + P_{i-2})\boldsymbol{g}_i,$$

(3.17) $$\boldsymbol{y}_m = (P_m - P_{m-2} - 2I)^{-1}\sum_{i=1}^{m}(P_{m-1-i} + P_{i-1})\boldsymbol{g}_i.$$

From the equations

(3.18) $$\begin{cases} D\boldsymbol{y}_k - \boldsymbol{y}_{k+1} = \boldsymbol{g}_k + \boldsymbol{y}_{k-1} & (k=1, 2, \cdots, m-1) \\ -\boldsymbol{y}_{j-1} + D\boldsymbol{y}_j - \boldsymbol{y}_{j+1} = \boldsymbol{g}_i & (j=k+1, \cdots, m-1), \end{cases}$$

we have

(3.19) $$\boldsymbol{y}_k = P_{m-k}^{-1}[\boldsymbol{y}_m + P_{m-1-k}\boldsymbol{y}_{k-1} + \sum_{i=k}^{m-1} P_{m-1-i}\boldsymbol{g}_i] \qquad (k=2, \cdots, m-1)$$

In the same way, we have also

(3.20) $$\boldsymbol{y}_{m-k} = P_{m-k-1}^{-1}[\boldsymbol{y}_1 + P_{m-k-2}\boldsymbol{y}_{m-k+1} + \sum_{i=0}^{m-k-2} P_i\boldsymbol{g}_{k+1+i}] \qquad (k=1, \cdots, m-2),$$

and the solution of (3.1) is given by the formula

(3.21) $$\boldsymbol{x}_k = \frac{2}{n} R^T \boldsymbol{y}_k \qquad (k=1, 2, \cdots, m).$$

The numerical process (3.19) (or (3.20)) is stable, because the eigenvalues of D are all not less than 2 in modulus.

Finally, substituting (3.16) and (3.17) into (3.9), and making use of (2.38), we can write the solution of (3.1) explicitly in the following form

$$(3.22)\qquad \boldsymbol{x}_{k+1}=\frac{2}{n}R^T(P_m-P_{m-2}-2I)^{-1}\Big[\sum_{i=1}^{k}(P_{m-k-2+i}+P_{k-i})\boldsymbol{g}_i+ + \sum_{i=k+1}^{m}(P_{m+k-i}+P_{i-k-2})\boldsymbol{g}_i\Big] \qquad (k=0,\ 1,\ \cdots,\ m-1).$$

3.2 Heat equation

G.J. Tee [15] considered the following periodic parabolic problem

$$(3.23)\qquad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (0<x<1)$$

with the boundary condition

$$(3.24)\qquad u(0,\ t)=f(t),\quad u(1,\ t)=g(t),\quad u(x,\ 0)=u(x,\ T),$$

where

$$(3.25)\qquad f(t+T)=f(t),\quad g(t+T)=g(t) \qquad (t\geqq 0).$$

Put

$$(3.26)\qquad l=T/m,\quad h=1/(n+1),\quad \sigma=l/h^2,$$

then, according as $\frac{\partial^2 u}{\partial x^2}$ is discretized by the explicit formula or by the implicit formula, the problem is reduced to the solution of the system of linear e-quations

$$(3.27)\qquad \begin{cases} \boldsymbol{x}_1-M\boldsymbol{x}_m=\boldsymbol{\Gamma}_1, \\ \boldsymbol{x}_k-M\boldsymbol{x}_{k-1}=\boldsymbol{\Gamma}_k \qquad (k=2,\ 3,\ \cdots,\ m), \end{cases}$$

or

$$(3.28)\qquad \begin{cases} N\boldsymbol{x}_1-\boldsymbol{x}_m=\boldsymbol{\Gamma}_1 \\ -\boldsymbol{x}_{k-1}+N\boldsymbol{x}_k=\boldsymbol{\Gamma}_k \qquad (k=2,\ 3,\ \cdots,\ m), \end{cases}$$

where

$$(3.29)\qquad M=(1-2\sigma)I+\sigma J,\quad N=(1+2\sigma)I-\sigma J.$$

Tee solved these systems by the iterative method using the theory of p-cyclic matrix, but we shall show that they can be solved directly as in the preceding paragraph.

Since

$$M = T^{-1}DT, \quad D = (1-2\sigma)I + 2\sigma G, \tag{3.30}$$

(3.27) can be rewritten as follows:

$$\begin{cases} \boldsymbol{y}_1 = D\boldsymbol{y}_m + \boldsymbol{g}_1, \\ \boldsymbol{y}_k = D\boldsymbol{y}_{k-1} + \boldsymbol{g}_k \qquad (k=2, 3, \cdots, m), \end{cases} \tag{3.31}$$

where

$$\boldsymbol{y}_j = T\boldsymbol{x}_i, \quad \boldsymbol{g}_j = T\boldsymbol{\Gamma}_j \qquad (j=1, 2, \cdots, m). \tag{3.32}$$

Therefore we have

$$\boldsymbol{y}_k = D^k\boldsymbol{y}_m + \sum_{i=1}^{k} D^{k-i}\boldsymbol{g}_i \qquad (k=1, 2, \cdots, m). \tag{3.33}$$

From this it follows that

$$\boldsymbol{y}_m = (I - D^m)^{-1}\sum_{i=1}^{m} D^{m-i}\boldsymbol{g}_i, \tag{3.34}$$

and $\boldsymbol{y}_k$ $(k=1, 2, \cdots, m-1)$ are obtained from (3.31).

For this process to be stable, the eigenvalues of D must be less than one in modulus. For this it is sufficient that

$$\sigma \leqq 1/2, \tag{3.35}$$

because

$$\lambda_i = 1 - 2\sigma + 2\sigma \cos\frac{i\theta}{N} = 1 - 4\sigma \sin^2\frac{i\theta}{2N}. \tag{3.36}$$

Next we are concerned with the system (3.28). Put

$$E = ((1+2\sigma)I - 2\sigma G)^{-1}, \tag{3.37}$$

then we have

$$N^{-1} = T^{-1}ET \tag{3.38}$$

and (3.28) can be rewritten as follows:

(3.39)
$$\begin{cases} \boldsymbol{y}_1 = E\boldsymbol{y}_m + E\boldsymbol{g}_1 \\ \boldsymbol{y}_k = E\boldsymbol{y}_{k-1} + E\boldsymbol{g}_k \qquad (k=2, 3, \cdots, m). \end{cases}$$

From this, as before, we have

(3.40)
$$\boldsymbol{y}_k = E^k \boldsymbol{y}_m + \sum_{i=1}^{k} E^{k+1-i} \boldsymbol{g}_i \qquad (k=1, 2, \cdots, m)$$

so that

(3.41)
$$\boldsymbol{y}_m = (I-E^m)^{-1} \sum_{i=1}^{m} E^{m+1-i} \boldsymbol{g}_i,$$

and $\boldsymbol{y}_k$ $(k=1, 2, \cdots, m-1)$ are obtained from (3.39).

Since the eigenvalues of E are all less than one in modulus, this process is always stable.

Substituting (3.34) and (3.41) into (3.33) and (3.40) respectively, we can write explicitly the solution of (3.27) and (3.28) respectively as follows:

(3.42)
$$\boldsymbol{x}_k = \frac{2}{N} T(I-D^m)^{-1} [\sum_{i=1}^{k} D^{k-i} \boldsymbol{g}_i + \sum_{i=k+1}^{m} D^{k+m-i} \boldsymbol{g}_i],$$

(3.43)
$$\boldsymbol{x}_k = \frac{2}{N} T(I-E^m)^{-1} E [\sum_{i=1}^{k} E^{k-i} \boldsymbol{g}_i + \sum_{i=k+1}^{m} E^{k+m-i} \boldsymbol{g}_i]$$

$$(k=1, 2, \cdots, m).$$

4. Biharmonic equation

Consider the following biharmonic equation

(4.1)
$$\Delta\Delta u(x, y) = f(x, y)$$

in the domain

(4.2)
$$R: \quad 0 < x < L, \quad 0 < y < M.$$

In the case where the function u is given on the entire boundary, the first normal derivative u_y is given on the horizontal sides and the second normal derivative u_{xx} is given on the vertical sides, the problem of finding the approximate solution of (4.1) satisfying the boundary conditions is reduced to that of solving the following system of equations [12]

$$(4.3)\qquad \begin{cases} B\boldsymbol{x}_1+C\boldsymbol{x}_2+\alpha\boldsymbol{x}_3=\boldsymbol{\Gamma}_1, \\ C\boldsymbol{x}_1+B\boldsymbol{x}_2+C\boldsymbol{x}_3+\alpha\boldsymbol{x}_4=\boldsymbol{\Gamma}_2, \\ \alpha\boldsymbol{x}_{k-4}+C\boldsymbol{x}_{k-3}+B\boldsymbol{x}_{k-2}+C\boldsymbol{x}_{k-1}+\alpha\boldsymbol{x}_k=\boldsymbol{\Gamma}_{k-2} \qquad (k=5,\ \cdots,\ m), \\ \alpha\boldsymbol{x}_{m-3}+C\boldsymbol{x}_{m-2}+B\boldsymbol{x}_{m-1}+C\boldsymbol{x}_m=\boldsymbol{\Gamma}_{m-1}, \\ \alpha\boldsymbol{x}_{m-2}+C\boldsymbol{x}_{m-1}+B\boldsymbol{x}_m=\boldsymbol{\Gamma}_m, \end{cases}$$

where

$$(4.4)\qquad \Delta x=L/N,\quad \Delta y=M/(m+1),\quad N=n+1,\quad m=2p,$$

$$(4.5)\qquad \theta_x=\Delta x^2/2(\Delta x^2+\Delta y^2),\quad \theta_y=\Delta y^2/2(\Delta x^2+\Delta y^2),$$

$$(4.6)\qquad R=1+2\theta_x{}^2+2\theta_y{}^2,\quad \theta_x+\theta_y=1/2,\quad \alpha=\theta_y{}^2/R,$$

$$(4.7)\qquad A=(I-\theta_x J),\quad B=(A^2+\theta_y{}^2 I)/R,\quad C=-2\theta_y A/R.$$

If we put

$$(4.8)\qquad \boldsymbol{g}_i=T\boldsymbol{\Gamma}_i/\alpha,\quad T\boldsymbol{x}_i=\boldsymbol{y}_i \qquad (i=1,\ 2,\ \cdots,\ m),$$

$$(4.9)\qquad E=(1/\theta_y)(I-2\theta_x G),\quad D=I+E^2,$$

$$(4.10)\qquad \boldsymbol{g}_{-1}=\boldsymbol{g}_0=0,$$

then (4.3) can be rewritten as follows:

$$(4.11)\qquad \begin{cases} \boldsymbol{y}_{-1}=\boldsymbol{y}_0=0, \\ \boldsymbol{y}_k=2E\boldsymbol{y}_{k-1}-D\boldsymbol{y}_{k-2}+2E\boldsymbol{y}_{k-3}-\boldsymbol{y}_{k-4}+\boldsymbol{g}_{k-2} \qquad (k=3,\ 4,\ \cdots,\ m+2), \\ \boldsymbol{y}_{m+1}=\boldsymbol{y}_{m+2}=0. \end{cases}$$

Let the matrices P_k, Q_k and the vectors $\boldsymbol{q}_k$ $(k=-1,\ 0,\ \cdots,\ m+2)$ be defined by the formulas

$$(4.12)\qquad \begin{cases} P_{-1}=P_0=0,\quad P_1=I,\quad P_2=0, \\ P_k=2EP_{k-1}-DP_{k-2}+2EP_{k-3}-P_{k-4} \qquad (k=3,\ 4,\ \cdots,\ m+2), \end{cases}$$

$$(4.13)\qquad \begin{cases} Q_{-1}=Q_0=Q_1=0,\quad Q_2=I, \\ Q_k=2EQ_{k-1}-DQ_{k-2}+2EQ_{k-3}-Q_{k-4} \qquad (k=3,\ 4,\ \cdots,\ m+2), \end{cases}$$

$$(4.14)\qquad \begin{cases} \boldsymbol{q}_{-1}=\boldsymbol{q}_0=\boldsymbol{q}_1=\boldsymbol{q}_2=0, \\ \boldsymbol{q}_k=\boldsymbol{g}_{k-2}+2E\boldsymbol{q}_{k-1}-D\boldsymbol{q}_{k-2}+2E\boldsymbol{q}_{k-3}-\boldsymbol{q}_{k-4} \qquad (k=3,\ 4,\ \cdots,\ m+2), \end{cases}$$

then (4.11) can be written as follows:

(4.15) $$\begin{cases} \boldsymbol{y}_k = P_k \boldsymbol{y}_1 + Q_k \boldsymbol{y}_2 + \boldsymbol{q}_k & (k=-1,\ 0,\ \cdots,\ m+2), \\ \boldsymbol{y}_{m+1} = \boldsymbol{y}_{m+2} = 0. \end{cases}$$

It is readily seen that

(4.16) $$\boldsymbol{q}_k = \sum_{l=1}^{k-2} Q_{k-l} \boldsymbol{g}_l \qquad (k=3,\ 4,\ \cdots,\ m+2)$$

and that P_k and Q_k are polynomials in E of degree $k-1$ and $k-2$ respectively such that

(4.17) $$P_k = -(k-2)E^{k-1} + \text{terms of degree less than } k-2,$$

(4.18) $$Q_k = (k-1)E^{k-2} + \text{terms of degree less than } k-3.$$

From (4.15) it follows that

(4.19) $$\begin{cases} P_{m+1}\boldsymbol{y}_1 + Q_{m+1}\boldsymbol{y}_2 = -\boldsymbol{q}_{m+1} \\ P_{m+2}\boldsymbol{y}_1 + Q_{m+2}\boldsymbol{y}_2 = -\boldsymbol{q}_{m+2} \end{cases}$$

and we have

(4.20) $$\begin{cases} \Delta \boldsymbol{y}_1 = Q_{m+1}\boldsymbol{q}_{m+2} - Q_{m+2}\boldsymbol{q}_{m+1} = \sum_{k=1}^{m} U_k \boldsymbol{g}_{m+1-k}, \\ \Delta \boldsymbol{y}_2 = P_{m+2}\boldsymbol{q}_{m+1} - P_{m+1}\boldsymbol{q}_{m+2} = \sum_{k=1}^{m} V_k \boldsymbol{g}_{m+1-k}, \end{cases}$$

where

(4.22) $$\begin{cases} \Delta = Q_{m+2}P_{m+1} - Q_{m+1}P_{m+2}, \\ U_k = Q_{m+1}Q_{k+1} - Q_{m+2}Q_k, \\ V_k = P_{m+2}Q_k - P_{m+1}Q_{k+1}. \end{cases}$$

From (4.17) and (4.18) it is readily seen that the leading terms of Δ, U_k and V_k are as follows:

(4.23) $$\begin{cases} \Delta = [m^2 - (m+1)(m-1)]E^{2m} + \cdots = E^{2m} + \cdots, \\ U_k = [mk - (m+1)(k-1)]E^{m+k-2} + \cdots = (m+1-k)E^{m+k-2} + \cdots, \\ V_k = [(m-1)k - (k-1)m]E^{m+k-1} + \cdots = (m-k)E^{m+k-1} + \cdots. \end{cases}$$

Hence the direct application of (4.22) will result in the loss of significant figures, and so we introduce the recurrence formulas as follows:

$$(4.24)\qquad \begin{cases} R_k = D(R_{k-1} - R_{k-3}) - 2E(S_{k-2} - S_{k-3}) + R_{k+4}, \\ U_k = 2EU_{k-1} - DU_{k-2} + 2EU_{k-3} - U_{k-4}, \\ V_k = 2EV_{k-1} - DV_{k-2} + 2EV_{k-3} - V_{k-4} \qquad (k=3,\ 4,\ \cdots,\ m), \end{cases}$$

where

$$(4.25)\qquad \begin{cases} R_{-1} = R_0 = 0, \quad R_1 = I, \quad R_2 = D, \\ S_{-1} = S_0 = 0, \quad S_1 = 2E, \\ U_{-1} = U_0 = 0, \quad U_1 = Q_{m+1}, \quad U_2 = DQ_m - 2EQ_{m-1} + Q_{m-2}, \\ V_{-1} = V_0 = 0, \quad V_1 = -P_{m+1}, \quad V_2 = -DP_m + 2EP_{m-1} - P_{m-2}. \end{cases}$$

Then it is readily shown that

$$(4.26)\qquad R_k = P_kQ_{k+1} - Q_kP_{k+1}, \quad S_k = P_kQ_{k+2} - Q_kP_{k+2}, \quad \Delta = R_{m+1}.$$

Thus we have

$$(4.27)\qquad \boldsymbol{x}_1 = \frac{2}{N}\, TR_{m+1}^{-1} \sum_{k=1}^{m} U_k \boldsymbol{g}_{m+1-k}, \quad \boldsymbol{x}_2 = \frac{2}{N}\, TR_{m+1}^{-1} \sum_{k=1}^{m} V_k \boldsymbol{g}_{m+1-k},$$

and

$$(4.28)\qquad \boldsymbol{x}_m = \frac{2}{N}\, TR_{m+1}^{-1} \sum_{k=1}^{m} U_k \boldsymbol{g}_k, \quad \boldsymbol{x}_{m-1} = \frac{2}{N}\, TR_{m+1}^{-1} \sum_{k=1}^{m} V_k \boldsymbol{g}_k.$$

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Department of Mathematics
Faculty of Science
Hiroshima University