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Generalized Capacity and Duality Theorem in Linear Programming

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Introduction

Recently certain results in the theory of games and linear programming have been applied to potential theory. We mention M. Nakai [2], B. Fuglede [1] and M. Ohtsuka [4]. Our paper is along this line.

More precisely, the minimax theorem in the theory of games was applied to the theory of capacity in [4]. For a compact Hausdorff space K and an extended real-valued lower semicontinuous function Φ on $K\times K$ which is bounded below, the author established

$$(1)\qquad \inf_{\mu\in\mathscr{U}}\sup_{x\in S_\mu}\int\Phi(x,\ y)d\mu(y)=\inf_{\nu\in\mathscr{U}}\sup_{y\in S_\nu}\int\Phi(x,\ y)d\nu(x)$$

and

$$(2)\qquad \sup_{\mu\in\mathscr{U}}\inf_{x\in S_\mu}\int\Phi(x,\ y)d\mu(y)=\sup_{\nu\in\mathscr{U}}\inf_{y\in S_\nu}\int\Phi(x,\ y)d\nu(x)\ ,$$

where $\mathscr{U}$ is the class of unit measures[1)] in K. See [3] for a simple proof of (1) in the case where K is discrete. We extend these results in the present paper. In §1 we consider a lower semicontinuous kernel, and generalize (1) by making use of a duality theorem in linear programming obtained in [5]. Next we are concerned with an upper semicontinuous kernel. A generalization of (2) is obtained there.

Let Φ be a function (called kernel) on $K\times K$ which is bounded above or below, and let g and f be upper or lower semicontinuous functions on K which are bounded above or below. We denote by $\mathscr{N}$($\mathscr{N}^+$ resp.) the class of measures (non-zero measures resp.) μ satisfying $\int\Phi(x,\ y)d\mu(y)\leqq g(x)$ on S_μ, and by $\check{\mathscr{N}}$($\check{\mathscr{N}}^+$ resp.) the class of measures (non-zero measures resp.) ν satisfying $\int\Phi(x,\ y)d\nu(x)\leqq f(y)$ on S_ν. We set

$$N=\sup_{\mu\in\mathscr{N}}\int fd\mu,\ N^+=\sup_{\mu\in\mathscr{N}^+}\int fd\mu,\ \check{N}=\sup_{\nu\in\check{\mathscr{N}}}\int gd\nu,\ \check{N}^+=\sup_{\nu\in\check{\mathscr{N}}^+}\int gd\nu$$

1) Here and throughout our paper a measure means a non-negative Radon measure.

in case each class is not empty. Each of these quantities may be regarded as a kind of capacity of K. We note that $0\in\mathcal{N}\cap\check{\mathcal{N}}$ and hence both N and $\check{N}$ are non-negative, but that $\mathcal{N}^+$ and $\check{\mathcal{N}}^+$ may be empty. Our interest lies in the equalities $N=\check{N}$ and $N^+=\check{N}^+$. If we want to specify the basic space K explicitly, we denote these classes and quantities by $\mathcal{N}(K)$, $\mathcal{N}^+(K)$, etc. and $N(K)$, $N^+(K)$, etc.

The primal problem in linear programming is to maximize $\int f d\mu$ with respect to μ belonging to $\mathcal{M}=\mathcal{M}(K)=\left\{\mu;\ \int\Phi(x,\ y)d\mu(y)\leqq g(x) \text{ on } K\right\}$. If this class is not empty, $\sup\int f d\mu$ will be denoted by M or $M(K)$. As the dual problem we consider $\mathcal{M}'=\mathcal{M}'(K)=\left\{\nu;\ \int\Phi(x,\ y)d\nu(x)\geqq f(y) \text{ on } K\right\}$ and $M'=M'(K)=\inf\int g d\nu$ for $\nu\in\mathcal{M}'$ in case $\mathcal{M}'\neq\emptyset$. For a lower semicontinuous kernel the author showed that $\mathcal{M}\neq\emptyset$ and $-\infty<M<\infty$ imply $\mathcal{M}'\neq\emptyset$ and $M=M'$ under some conditions in [5]. This duality theorem will play an important role in what follows.

§ 1. Lower semicontinuous kernel

Our first main theorem is

THEOREM 1. *Let Φ be a lower semicontinuous function on $K\times K$ which is bounded below, and g and f be upper semicontinuous functions on K which are bounded above. Assume one of the following conditions*:
(i) $\inf g>0$ *and* $\inf f>0$ *on* K,
(ii) $\inf\Phi>0$ *on* $K\times K$,[2)]
(iii) $\sup g<0$ *and* $\sup f<0$ *on* K,[2)]
(iv) $\sup\Phi<0$ *on* $K\times K$.
If $\mathcal{N}^+\neq\emptyset$ *and* $N^+\neq 0,\ \pm\infty$, *then* $\check{\mathcal{N}}^+\neq\emptyset$ *and* $N^+=\check{N}^+$.

We begin our proof with

LEMMA 1. *Let $\{\mu_n\}$ be a sequence of measures which converges vaguely to a non-zero measure μ_0. If $\int\Phi(x,\ y)d\mu_n(y)\leqq g(x)$ on S_{μ_n} for each n, then $\int\Phi(x,\ y)d\mu_0(y)\leqq g(x)$ on S_{μ_0}.*

PROOF. Let x_0 be any point of S_{μ_0}, and A be the directed set of neighborhoods of x_0. For every couple $(U,\ n)$ of $U\in A$ and n, we select any point $x(U,\ n)$ in $U\cap S_{\mu_{n'}}$ where n' is the smallest integer satisfying $n'\geqq n$ and $U\cap$

2) Since Φ is lower semicontinuous on K, the positivity of Φ is equivalent to $\inf\Phi>0$. However, we shall impose (ii) on an upper semicontinuous function in § 2 so that we write $\inf\Phi$ in (ii). A similar remark applies to (iii).

$S_{\mu_{n'}} \neq \emptyset$. We regard the set of all couples $W=(U, n)$ as a directed set in a natural manner and denote it by E. Let $\lambda_W = \varepsilon_{x(U,n)} \times \mu_{n'}$ correspond to $W=(U, n)$, where ε_x represents the unit point measure at x in general. Thus $\{\lambda_W;\ W \in E\}$ is a net, and converges vaguely to $\varepsilon_{x_0} \times \mu_0$. We have

$$\int \Phi(x_0, y) d\mu_0(y) = \int \Phi\, d(\varepsilon_{x_0} \times \mu_0) \leqq \varliminf_{E} \int \Phi\, d\lambda_W$$

$$= \varliminf_{E} \int \Phi(x(U, n), y) d\mu_{n'}(y) \leqq \varliminf_{E} g(x(U, n)) \leqq g(x_0) .$$

On account of the arbitrary character of $x_0 \in S_{\mu_0}$, we obtain the desired conclusion.

We shall prove one more lemma.

LEMMA 2. *Assume* $\mathscr{M}' \neq \emptyset$ *and* $M' \neq 0, -\infty$, *and assume one of conditions* (i)–(iv). *Suppose there exists a measure* μ_0 *satisfying* $S_{\mu_0} = K$, $\int \Phi(x, y) d\mu_0(y) \leqq g(x)$ *on* K *and* $\int f d\mu_0 = M'$. *Then there is a non-zero measure* ν_0 *which satisfies* $\int \Phi(x, y) d\nu_0(x) \leqq f(y)$ *on* K *and* $\int g d\nu_0 \geqq M'$.

PROOF. We note that $M' < \infty$ if $\mathscr{M}' \neq \emptyset$. We choose $\{\nu_n\}$ such that $\int \Phi(x, y) d\nu_n(x) \geqq f(y)$ on K and $\int g d\nu_n \leqq M' + 1/n$. We have

$$M' + 1/n \geqq \int g d\nu_n \geqq \iint \Phi\, d\mu_0 d\nu_n = \iint \Phi\, d\nu_n d\mu_0 \geqq \int f d\mu_0 = M' . \tag{3}$$

If $\inf g > 0$, then $(\inf g)\nu_n(K) \leqq M' + 1/n$ and it is inferred that $\nu_n(K)$ is bounded. Assuming (ii), we see that $0 < \int \Phi\, d\mu_0 \leqq g$ on K. Let $M_f = \sup f$ on K. If $M_f \leqq 0$, then $\nu \equiv 0$ is optimal for the dual problem on K and hence $M' = 0$, contrary to our assumption. Hence $M_f > 0$. For n with $\inf_{y \in K} \int \Phi(x, y) d\nu_n(x) > M_f$, we consider $\nu_n' = \nu_n M_f / \inf_K \int \Phi\, d\nu_n$. Otherwise we set $\nu_n' = \nu_n$. For each n it holds that $\nu_n' \leqq \nu_n$ and $\int \Phi\, d\nu_n' \geqq f$ on K. It holds also that $\int g d\nu_n' \leqq \int g d\nu_n \leqq M' + 1/n$. We observe that $\nu_n'(K)$ is bounded because

$$(\inf_{K \times K} \Phi)\nu_n'(K) \leqq \inf_K \int \Phi\, d\nu_n' \leqq M_f < \infty .$$

Under (iii) we see easily that $\nu_n(K)$ is bounded. If $\nu_n(K)$ is unbounded under (iv), $f(x) \equiv -\infty$ on K and hence $M' = \int f d\mu_0 = -\infty$ against our assumption. Hence we may suppose that $\nu_n(K)$ is bounded under any one of (i)-(iv).

We choose a vaguely convergent subsequence of $\{\nu_n\}$. Without any confusion we may denote it by $\{\nu_n\}$ again. Let ν_0 be the vague limit. Suppose that there exists y_0 such that $\int \Phi(x, y_0) d\nu_0(x) > f(y_0)$. Given $\delta > 0$, we choose n_0 and a neighborhood U of y_0 such that

$$\int \Phi(x, y) d\nu_n(x) + \delta > \int \Phi(x, y_0) d\nu_0(x)$$

for every $n \geqq n_0$ and at every $y \in U$. This is possible because $\nu_n \times \varepsilon_y$ converges vaguely to $\nu_0 \times \varepsilon_{y_0}$ as $n \to \infty$ and $y \to y_0$. If $f(y_0) > -\infty$, then we may assume that $f(y_0) + \delta > f(y)$ on U. We note that $\mu_0(U) > 0$ because $y_0 \in K = S_{\mu_0}$, and have that

$$\int_U \int \Phi d\nu_n d\mu_0 + \delta\mu_0(U) \geqq \mu_0(U) \int \Phi(x, y_0) d\nu_0(x) > \mu_0(U) f(y_0)$$
$$\geqq \int_U f d\mu_0 - \delta\mu_0(U) \ .$$

If n is large, we have by (3)

$$0 \leqq \int_U \int \Phi d\nu_n d\mu_0 - \int_U f d\mu_0 \leqq \int\int \Phi d\nu_n d\mu_0 - \int f d\mu_0 \leqq 1/n < \delta\mu_0(U) \ .$$

It follows that

$$0 < \left(\int \Phi d\nu_0 - f(y_0) \right) \mu_0(U) \leqq 3\delta\mu_0(U) \ ,$$

which is impossible if δ is small. Next, if $f(y_0) = -\infty$, then we may assume that $-1/\delta > f(y)$ in U. By (3) we have

$$-\infty < \mu_0(U) \int \Phi d\nu_0 \leqq \int_U \int \Phi d\nu_n d\mu_0 + \delta\mu_0(U) \leqq \int_U f d\mu_0 + 2\delta\mu_0(U)$$
$$< -\frac{\mu_0(U)}{\delta} + 2\delta\mu_0(U)$$

for large n. This is impossible. Consequently, $\int \Phi(x, y) d\nu_0(x) \leqq f(y)$ everywhere on K. Furthermore

$$M' = \lim_{n \to \infty} \int g d\nu_n \leqq \int g d\nu_0 \ .$$

Finally we shall show $\nu_0 \neq 0$. Since $\int \Phi d\mu_0 \leqq g$ on K and Φ is bounded below on K, g is bounded on K. Therefore if $\nu_n \to \nu_0 \equiv 0$, then $M' = \lim_n \int g d\nu_n = 0$, contrary to our assumption.

Now we give

Proof of Theorem 1. We divide the proof into three steps.

I. As the first step we shall establish $N^+ \leqq \check{N}^+$. We choose $\{\mu_n\}$ in $\mathcal{N}^+$ such that each $\int f d\mu_n$ is finite and tends to N^+ as $n \to \infty$. Naturally $\int f d\mu_n \leqq N^+(S_{\mu_n}) \leqq N^+$, whence $\lim_{n\to\infty} N^+(S_{\mu_n}) = N^+$. By our assumption, $N^+ \neq 0$ and accordingly $N^+(S_{\mu_n})$ may be assumed to be non-zero for all n. Let us show that we may assume further that f is bounded on S_{μ_n}. We need not consider the case subject to (i). If f is unbounded on S_{μ_n}, then we can find a large number $p > 0$ such that $\int_F f d\mu_n$ is close to $\int f d\mu_n$ and $\mu_n(S_{\mu_n} - F)$ is small, where $F = \{x \in S_{\mu_n}; f(x) \geqq -p\}$ is a closed set. Under (ii) the restriction $\mu_n|_F$ of μ_n to F belongs to $\mathcal{N}^+$ and $\int f d(\mu_n|_F) = \int_F f d\mu_n$. Under (iii) or (iv) we may replace μ_n by $a\mu_n|_F$, where a is a number greater than but close to 1. Hence we assume from the beginning that f is bounded on S_{μ_n} for each n.

For each n, we choose $\{\mu_n^{(k)}\}$, $k = 1, 2, \cdots$, in $\mathcal{N}^+(S_{\mu_n})$ such that $\int f d\mu_n^{(k)} \to N^+(S_{\mu_n})$ as $k \to \infty$. As agreed before, $\mathcal{N}^+(S_{\mu_n})$ and $N^+(S_{\mu_n})$ mean the class $\mathcal{N}^+$ and the value N^+ respectively when S_{μ_n} is regarded as the basic space. If $\inf f > 0$ or $\Phi > 0$ or $f < 0$, $\mu_n^{(1)}(K)$, $\mu_n^{(2)}(K)$, $\cdots$, are bounded. Let us see that we may assume the boundedness under (iv). First we note that $f \leqq 0$ on S_{μ_n} because, otherwise, there exists $x_0 \in S_{\mu_n}$ with $f(x_0) > 0$ and $\mu_n + p\varepsilon_{x_0}$ belongs to $\mathcal{N}^+(S_{\mu_n})$ for any $p > 0$, so that

$$N^+ \geqq N^+(S_{\mu_n}) \geqq \int f d(\mu_n + p\varepsilon_{x_0}) = \int f d\mu_n + pf(x_0) \to \infty \qquad \text{as } p \to \infty$$

against our assumption. Therefore $N^+(S_{\mu_n}) \leqq 0$. If $\mu_n^{(k)}(K) \to \infty$ as $k \to \infty$, there is k_0 such that $\mu_n^{(k)}(K) > (-\inf_{S_{\mu_n}} g)^+ / (-\sup_{K \times K} \Phi) + 1$ for every $k \geqq k_0$. We denote the value on the right hand side by b. Then $b \int \Phi d\mu_n^{(k)} / \mu_n^{(k)}(K) \leqq g$ on $S_{\mu_n^{(k)}}$ and, since $f \leqq 0$ on S_{μ_n}, $\int f d\mu_n^{(k)} \leqq b \int f d\mu_n^{(k)} / \mu_n^{(k)}(K) \leqq N^+(S_{\mu_n})$ for $k = k_0$, $k_0 + 1$, $\cdots$. Therefore we may assume from the beginning that $\mu_n^{(1)}(K)$, $\mu_n^{(2)}(K)$, $\cdots$ are bounded under any one of (i)–(iv).

We extract a vaguely convergent subsequence of $\{\mu_n^{(k)}\}$. We denote it again by $\{\mu_n^{(k)}\}$ and let λ_n be the limit. We have

(4) $$N^+(S_{\mu_n})=\lim_{k\to\infty}\int f\,d\mu_n^{(k)}\leqq\int f\,d\lambda_n$$

on account of the upper semicontinuity of f. We shall show that $\lambda_n\not\equiv 0$ for all n. We have seen above that we may assume f to be bounded on S_{μ_n}. If $\mu_n^{(k)}\to\lambda_n\equiv 0$, then $N^+(S_{\mu_n})=\lim_{k\to\infty}\int f\,d\mu_n^{(k)}=0$. This contradicts the assumption $N^+(S_{\mu_n})\neq 0$ made at the beginning of our proof. Therefore $\lambda_n\not\equiv 0$ for all n. Consequently $\lambda_n\in\mathscr{N}^+(S_{\mu_n})$ by Lemma 1 and hence $\int f\,d\lambda_n=N^+(S_{\mu_n})$ on account of (4). We obtain $N^+(S_{\mu_n})=N^+(S_{\lambda_n})$ easily.

We shall verify that $N^+(S_{\lambda_n})=M(S_{\lambda_n})$. Since $\lambda_n\in\mathscr{M}(S_{\lambda_n})$, $N^+(S_{\lambda_n})\leqq M(S_{\lambda_n})$. Let μ be any non-zero measure of $\mathscr{M}(S_{\lambda_n})$. Then $\mu\in\mathscr{N}^+(S_{\lambda_n})$ and hence $\int f\,d\mu\leqq N^+(S_{\lambda_n})$. Under (i), $N^+(S_{\lambda_n})\geqq 0$ and hence $M(S_{\lambda_n})\leqq N^+(S_{\lambda_n})$. Thus $M(S_{\lambda_n})=N^+(S_{\lambda_n})$ under (i). If there exists a point $x_0\in S_{\lambda_n}$ with $f(x_0)<0$ under (ii), then there is a neighborhood U of x_0 on which f is negative. The restriction of λ_n to $S_{\lambda_n}-U$ belongs to $\mathscr{N}^+(S_{\lambda_n})$ and gives a greater value for the integral of f. This contradicts $\int f\,d\lambda_n=N^+(S_{\lambda_n})$. Therefore $f\geqq 0$ on S_{λ_n}, whence $N^+(S_{\lambda_n})\geqq 0$. It is thus inferred that $M(S_{\lambda_n})=N^+(S_{\lambda_n})$ is true under (ii) too. The same equality is true under (iii) because $0\notin\mathscr{M}(S_{\lambda_n})$. If (iv) is assumed and $0\in\mathscr{M}(S_{\lambda_n})$, any measure $\mu\not\equiv 0$ belongs to $\mathscr{N}^+(S_{\lambda_n})$ and $N^+(S_{\lambda_n})\geqq 0$ is concluded. The equality follows in this case too.

By a duality theorem (Theorem 4 in [5]) $\mathscr{M}'(S_{\lambda_n})=\left\{\nu;\int\Phi(x,y)d\nu(x)\geqq f(y)\text{ on }S_{\lambda_n}\right\}$ is not empty and $M'(S_{\lambda_n})$ is equal to $M(S_{\lambda_n})=N^+(S_{\lambda_n})$. We apply Lemma 2 and find a non-zero measure π_n with $S_{\pi_n}\subset S_{\lambda_n}$ such that $\int\Phi(x,y)\,d\pi_n(x)\leqq f(y)$ on S_{π_n} and $\int g\,d\pi_n\geqq M'(S_{\lambda_n})$. It belongs to $\check{\mathscr{N}}^+$ and it follows that

$$N^+(S_{\lambda_n})=M'(S_{\lambda_n})\leqq\int g\,d\pi_n\leqq\check{N}^+\ .$$

Since $N^+(S_{\lambda_n})=N^+(S_{\mu_n})$ as already obtained and $N^+=\lim_{n\to\infty}N^+(S_{\mu_n})$, the inequality $N^+\leqq\check{N}^+$ follows.

II. As the second step we shall prove (1).[3] Let us denote both sides of (1) by L and $\check{L}$. By adding a positive constant to Φ if necessary, we may assume in this step that Φ is positive on $K\times K$. First, we consider the case where L is finite. Take $\mu\in\mathscr{U}$ for which $V(\mu)=\sup_{x\in S_\mu}\int\Phi(x,y)d\mu(y)$ is finite.

3) This step shows that Theorem 1 implies (1).

For $\mu'=\mu/V(\mu)$ it holds that $V(\mu')=\sup_{x\in S_{\mu'}}\int\Phi(x, y)d\mu'(y)=1$ and $\mu'(K)=1/V(\mu)$. The class $\tilde{\mathscr{N}}^+=\left\{\mu\neq 0;\ \int\Phi(x, y)d\mu(y)\leqq 1 \text{ on } S_\mu\right\}$ is not empty and it is seen that $\tilde{N}^+=\sup\{\mu(K);\ \mu\in\tilde{\mathscr{N}}^+\}$ equals $1/L$. By our first step, $\check{\tilde{\mathscr{N}}}^+=\left\{\nu\neq 0;\ \int\Phi(x, y)d\nu(x)\leqq 1 \text{ on } S_\nu\right\}$ is not empty and $\tilde{N}^+\leqq\check{\tilde{N}}^+=\sup\{\nu(K);\ \nu\in\check{\tilde{\mathscr{N}}}^+\}$. It follows also that $\check{L}$ is finite and $\check{\tilde{N}}^+=1/\check{L}$. Since $\check{\tilde{\mathscr{N}}}^+\neq\emptyset$ and $0<\check{\tilde{N}}^+<\infty$, $\check{\tilde{N}}^+\leqq\tilde{N}^+$ holds for the same reason as at the first step. Thus $\tilde{N}^+=\check{\tilde{N}}^+$ and hence $L=\check{L}$ is concluded in case L is finite. We obtain the same conclusion if we start from the assumption $\check{L}<\infty$. The only remaining case is that $L=\check{L}=\infty$.

III. As the last step we shall show $\check{N}^+\leqq N^+$. If $\check{N}^+\neq 0,\ \pm\infty$, we start from $\check{\mathscr{N}}^+$ and $\check{N}^+$ and obtain $\check{N}^+\leqq N^+$ as in the first step. Since $N^+\leqq\check{N}^+$, $\check{N}^+\neq-\infty$ is assured. First we shall see that $\check{N}^+\neq 0,\ \infty$ under any one of (i)-(iii). Under (ii), both N^+ and $\check{N}^+$ are finite and $0<N^+$ implies $0<\check{N}^+$ because $N^+\leqq\check{N}^+$. We have $\check{N}^+=\sup_{\nu\in\check{\mathscr{N}}^+}\int gd\nu<0$ under (iii), because $(\inf\Phi)\nu(K)\leqq\sup f<0$ and $\nu(K)$ has a positive lower bound.

Next we assume (i). The assumption $0<N^+$ yields $0<\check{N}^+$ because $N^+\leqq\check{N}^+$. We shall show that $\check{N}^+=\infty$ implies $N^+=\infty$, whence $N^+<\infty$ implies $\check{N}^+<\infty$.[4] We choose $\{\nu_n\}$ in $\check{\mathscr{N}}^+$ such that $\int gd\nu_n$ tends to ∞. Since g is bounded above, $\nu_n(K)$ tends to infinity. Using Lemma 1, we infer from $\int\Phi d\nu_n\leqq f$ that $\int\Phi d\nu_0'\leqq 0$ on $S_{\nu_0'}$, where ν_0' is the vague limit in $\mathscr{U}$ of some subsequence of $\{\nu_n/\nu_n(K)\}$. By (1), we have

$$\inf_{\mu\in\mathscr{U}}\sup_{x\in S_\mu}\int\Phi(x, y)d\mu(y)=L=\check{L}\leqq\sup_{y\in S_{\nu_0'}}\int\Phi(x, y)d\nu_0'(x)\leqq 0\ .$$

Using Lemma 1 again, we observe that there is $\mu_0'\in\mathscr{U}$ which satisfies $\int\Phi(x, y)d\mu_0'(y)\leqq L\leqq 0$ on $S_{\mu_0'}$. Hence $k\mu_0'\in\mathscr{N}^+$ for any $k>0$ and hence $N^+\geqq k\int fd\nu_0'\to\infty$ as $k\to\infty$ under (i). Thus $N^+=\infty$.

Finally, under the assumption of (iv), we can observe easily that the assumption $N^+<\infty$ implies $N^+<0$; see the proof of $N^+(S_{\mu_n})\leqq 0$ in the first step. We choose $\{\nu_n\}$ in $\check{\mathscr{N}}^+$ such that $\int gd\nu_n$ is finite for each n and $\int gd\nu_n\to\check{N}^+$ as $n\to\infty$. As in the first step, we may assume that g is bounded on each S_{ν_n}. Evidently $\int gd\nu_n\leqq\check{N}^+(S_{\nu_n})\to\check{N}^+$ as $n\to\infty$. We shall show that condition (iii)

4) We can show similarly that $N^+=\infty$ implies $\check{\mathscr{N}}^+\neq\emptyset$ and $\check{N}^+=\infty$ under (i).

is fulfilled on S_{ν_n} for each n. Since $\int \Phi d\nu_n \leqq f$ on S_{ν_n}, f is bounded on S_{ν_n}. If there is $x_0 \in S_{\nu_n}$ with $g(x_0) \geqq 0$, then ε_{x_0}/p belongs to $\mathscr{N}^+$ with any $p>0$ and hence

$$0 > N^+ \geqq \int f d\varepsilon_{x_0}/p = f(x_0)/p \to 0 \qquad \text{as } p \to \infty \ .$$

This is impossible. Therefore $g<0$ on S_{ν_n}. Next, if there is $y_0 \in S_{\nu_n}$ with $f(y_0) \geqq 0$, then the measure $p\varepsilon_{y_0}$ belongs to $\mathscr{N}^+$ for large p and

$$0 > N^+ \geqq p \int f d\varepsilon_{y_0} = pf(y_0) \geqq 0 \ .$$

This is absurd. Now (iii) being valid, we have $\check{N}^+(S_{\nu_n}) \leqq N^+(S_{\nu_n}) \leqq N^+$ for every n. Hence

$$N^+ = \lim_{n\to\infty} \check{N}^+(S_{\nu_n}) \leqq N^+ \ .$$

REMARK 1. It does not happen that $N^+=0$ under either one of (i) and (iii). If $\Phi \equiv 1$, $g \equiv 1$ and $f \equiv -1$, then $N^+=0$ and $\check{\mathscr{N}}^+$ is empty. Hence the condition $N^+ \neq 0$ is necessary besides (ii). If $\Phi \equiv -1$, $g \equiv 1$ and $f \equiv 0$, then $N^+=0$ and $\check{N}^+=\infty$. Hence the condition $N^+ \neq 0$ is necessary in addition to (iv).

REMARK 2. We would check the case $N^+ = -\infty$. Under (i), $N^+>0$ if $\mathscr{N}^+ \neq \emptyset$. If $\Phi \equiv 1$, $g \equiv 1$ and $f \equiv -\infty$, then $N^+ = -\infty$ and $\check{\mathscr{N}}^+ = \emptyset$. Hence $N^+ > -\infty$ is to be assumed in addition to (ii). If $\Phi \equiv -1$, $g \equiv -1$ and $f \equiv -\infty$, then $N^+ = -\infty$ and $\check{\mathscr{N}}^+ = \emptyset$. Hence $N^+ > -\infty$ is necessary in addition to (iii) and (iv) too.

REMARK 3. Next we want to treat the case $N^+=\infty$. This does not happen under any one of (ii) and (iii). If $\Phi \equiv -1$, $g \equiv 0$ and $f \equiv 1$, then $N^+=\infty$ but $\check{N}^+=0$. Hence the condition $N^+<\infty$ is to be assumed in addition to (iv). As remarked at footnote 4), $N^+=\infty$ implies $\check{\mathscr{N}}^+ \neq \emptyset$ and $\check{N}^+=\infty$ under (i).

Let us next examine whether $N^+=N$ or not. We note that $N=0$ if $\mathscr{N}^+ = \emptyset$, that $N^+=N$ if $\mathscr{N}^+ \neq \emptyset$ and $N^+ \geqq 0$ and that $N^+ < N = 0$ if $\mathscr{N}^+ \neq \emptyset$ and $N^+<0$. If $\mathscr{N}^+ \neq \emptyset$ under (i), then $0<N^+$ and hence $N^+=N$. Under (ii), it is easily seen that $N^+ \geqq 0$ unless $N^+ = -\infty$. Accordingly, $N^+=N$ unless $N^+ = -\infty$. If we assume (iii), then $N^+ \leqq 0$ and hence $N=0$. Under (iv) we have $N^+ \leqq 0 = N$ unless $N^+=N=\infty$, as shown in the proof of Theorem 1.

Next we shall see relation between N and $\check{N}$.

THEOREM 2. *Under the assumptions of Theorem* 1 *we have* $N=\check{N}$ *except for the case where* $N=0$ *and* $\check{N}=\infty$ *or the case where* $\check{N}=0$ *and* $N=\infty$, *which can really arise only under* (iv).

PROOF. It will suffice to verify $N \leqq \check{N}$. As remarked in the first paragraph in § 1, both N and $\check{N}$ are non-negative. First we assume (i). If $\mathscr{N}^{+}=\emptyset$, then $N=0 \leqq \check{N}$. If $\mathscr{N}^{+} \neq \emptyset$, then $\check{\mathscr{N}}^{+} \neq \emptyset$ and $N^{+}=\check{N}^{+}$ by Theorem 1 and footnote 4). Hence $0<N=N^{+}=\check{N}^{+}=\check{N}$. Next we assume (ii). As stated in Remark 3 of Theorem 1, $N^{+}<\infty$. If $N^{+} \leqq 0$, then $N=0 \leqq \check{N}$. If $N^{+}>0$, then $N=N^{+}=\check{N}^{+}=\check{N}$ by Theorem 1. Under (iii) we have $N=\check{N}=0$. Finally assume $0<N^{+}<\infty$ under (iv). Then by Theorem 1, $N=N^{+}=\check{N}^{+}=\check{N}$. This is the same if $0<\check{N}^{+}<\infty$. Thus the exceptions for $N=\check{N}$ arise only when $N=N^{+}=0$ and $\check{N}=\check{N}^{+}=\infty$ or when $N=\infty$ and $\check{N}=0$. These exceptional cases really arise as the example in Remark 3 of Theorem 1 shows.

§ 2. Upper semicontinuous kernel

In this section we are interested in upper semicontinuous kernels which are bounded above.

LEMMA 3. *Let $D=\{\kappa\}$ be a directed set, $\{\Psi_{\kappa}\}$ be a net of upper semicontinuous functions on $K \times K$ decreasing to Φ which is bounded above, and $\{g_{\kappa}\}$ be a net of lower semicontinuous functions increasing to g which is bounded below. Then for any non-zero μ satisfying $\int \Phi(x, y) d\mu(y) \leqq g(x)$ on S_{μ},*

$$\varliminf_{D} \inf_{x \in S_{\mu}} \left\{ g_{\kappa}(x) - \int \Psi_{\kappa}(x, y) d\mu(y) \right\}$$

is non-negative.

PROOF. Suppose, to the contrary, that there are a directed subset $D' \subset D$ and a constant $a>0$ such that, for every $\kappa \in D'$, there exists $x_{\kappa} \in S_{\mu}$ satisfying

$$g_{\kappa}(x_{\kappa}) - \int \Psi_{\kappa}(x_{\kappa}, y) d\mu(y) < -a \ .$$

We may assume that x_{κ} converges to a point $x_0 \in S_{\mu}$ along D'. Fix $\kappa_0 \in D'$ for a moment. We have

$$g_{\kappa_0}(x_0) - \int \Psi_{\kappa_0}(x_0, y) d\mu(y) \leqq \varliminf_{D'} \left\{ g_{\kappa_0}(x_{\kappa}) - \int \Psi_{\kappa_0}(x_{\kappa}, y) d\mu(y) \right\}$$

$$\leqq \varliminf_{D'} \left\{ g_{\kappa}(x_{\kappa}) - \int \Psi_{\kappa}(x_{\kappa}, y) d\mu(y) \right\} \leqq -a \ .$$

On account of the arbitrariness of $\kappa_0 \in D'$ we infer that

$$g(x_0) + a \leqq \int \Phi(x_0, y) d\mu(y) \ .$$

This is a contradiction.

We define $\mathscr{N}$, $\mathscr{N}^+$, $\check{\mathscr{N}}$, $\check{\mathscr{N}}^+$, N, N^+, $\check{N}$, $\check{N}^+$ as in §1.

THEOREM 3. *Let Φ be an upper semicontinuous function bounded above on $K\times K$, and g and f be lower semicontinuous functions bounded below on K. Assume one of conditions* (i)–(iv) *given in Theorem* 1. *If $\mathscr{N}^+\neq\emptyset$ and $N^+\neq 0$, $\pm\infty$, then $\check{\mathscr{N}}^+\neq\emptyset$ and $N^+=\check{N}^+$.*

PROOF. First we consider the case where g and f are continuous. We denote by D the directed set of all continuous functions Ψ on $K\times K$ such that $\Psi\geqq\Phi$. Let $\mathscr{N}_\Psi$ ($\mathscr{N}_\Psi^+$ resp.) be the class of measures (non-zero measures resp.) μ satisfying $\int\Psi\,d\mu\leqq g$ on S_μ and set $N_\Psi=\sup\left\{\int f\,d\mu;\ \mu\in\mathscr{N}_\Psi\right\}$ ($N_\Psi^+=\sup\left\{\int f\,d\mu;\ \mu\in\mathscr{N}_\Psi^+\right\}$ if $\mathscr{N}_\Psi^+\neq\emptyset$ resp.). Evidently $\mathscr{N}_\Psi\subset\mathscr{N}$ for each $\Psi\in D$ and hence $N_\Psi\leqq N$. Similarly $N_\Psi^+\leqq N^+$ if $\mathscr{N}_\Psi^+\neq\emptyset$.

Assume $\mathscr{N}^+\neq\emptyset$ and fix $\mu\in\mathscr{N}^+$ for a moment. For $\varepsilon>0$, there is $\Psi_\varepsilon\in D$ such that

$$\int\Psi(x,\,y)d\mu(y)\leqq g(x)+\varepsilon \qquad \text{on } S_\mu$$

for every $\Psi\in D$ not greater than Ψ_ε by Lemma 3. Under (i) or (ii) we see $\min_{S_\mu} g>0$. Hence, given $\eta>0$, there exists $\varepsilon>0$ such that $g(x)+\varepsilon\leqq(1+\eta)g(x)$ on S_μ. Thus $\mu/(1+\eta)$ belongs to $\mathscr{N}_\Psi$ if $\Psi\in D$ and $\Psi\leqq\Psi_\varepsilon$, and hence

$$N_\Psi^+\geqq\frac{1}{1+\eta}\int f\,d\mu \qquad \text{if } \Psi\in D \text{ and } \Psi\leqq\Psi_\varepsilon\ .$$

It follows that $\lim_D N_\Psi^+\geqq(1+\eta)^{-1}\int f\,d\mu$, whence $\lim_D N_\Psi^+\geqq N^+$ on account of the arbitrariness of $\eta>0$ and $\mu\in\mathscr{N}^+$. The equality is derived because of the inverse inequality obtained already.

Let us assume (iii) next. Given $\eta>0$, we can find $\varepsilon>0$ such that $g(x)+\varepsilon<g(x)/(1+\eta)$ on S_μ. Under (iv) we choose $\Psi_0\in D$ such that $a_0=\sup_{K\times K}\Psi_0$ is negative. Given $\eta>0$, take $\varepsilon>0$ smaller than $-a_0\eta\mu(K)$. We may assume that Ψ_ε chosen above is not greater than Ψ_0. Then $\eta\int\Psi\,d\mu<-\varepsilon$ for every $\Psi\in D$, $\Psi\leqq\Psi_\varepsilon$, and $(1+\eta)\int\Psi\,d\mu\leqq g$ on S_μ. Thus under either one of (iii) and (iv), $(1+\eta)\mu$ belongs to $\mathscr{N}_\Psi$ for every $\Psi\in D$, $\Psi\leqq\Psi_\varepsilon$. We obtain $\lim_D N_\Psi^+=N^+$ as above. We note that this identity is true even if $N^+=0$ or ∞ or $-\infty$.

By our assumption we may assume $N_\Psi^+\neq 0$, $\pm\infty$ for every $\Psi\in D$, $\Psi\leqq\Psi_\varepsilon$. We apply Theorem 1 and see $\check{\mathscr{N}}_\Psi^+\neq\emptyset$ and $N_\Psi^+=\check{N}_\Psi^+$. Since $\check{\mathscr{N}}_\Psi^+\subset\check{\mathscr{N}}^+$, we can derive $\lim_D\check{N}_\Psi^+=\check{N}^+$ as above. Now $N^+=\check{N}^+$ follows.

Next we consider the case where g is lower semicontinuous and f is continuous. We denote by H the directed set of all continuous functions h satisfying $h \leqq g$. Let $\mathscr{N}_h$ be the class of measures μ satisfying $\int \Phi d\mu \leqq h$ on S_μ and set $N_h = \sup \int f d\mu$ for $\mu \in \mathscr{N}_h$. In the same way as above we have $\lim_H N_h = N$. Consider $\check{N}_h = \sup \int h d\nu$ for $\nu \in \check{\mathscr{N}}$. Naturally $\check{N}_h \leqq \check{N}$. On the other hand, given $\nu \in \check{\mathscr{N}}$,

$$\int g d\nu = \sup_{h \in H} \int h d\nu \leqq \sup_{h \in H} \sup_{\nu \in \mathscr{N}} \int h d\nu = \sup_{h \in H} \check{N}_h \ .$$

Thus we have $\lim_H \check{N}_h = \check{N}$. Since $N_h = \check{N}_h$ for each $h \in H$, $N = \check{N}$ follows in this case too. Finally we consider the general case and can complete the proof easily.

We change the signs of Φ, f and g and obtain

COROLLARY. *Let Φ be a lower semicontinuous function bounded above on $K \times K$, and g and f be upper semicontinuous functions bounded below on K. Under any one of* (i)–(iv) *we have*

$$\inf\left\{\int f d\mu;\ \mu \neq 0,\ \int \Phi(x, y) d\mu(y) \geqq g(x) \text{ on } S_\mu\right\}$$
$$= \inf\left\{\int g d\nu;\ \nu \neq 0,\ \int \Phi(x, y) d\nu(x) \geqq f(y) \text{ on } S_\nu\right\}$$

provided the left hand side is well-defined and equal to none of $0, \infty, -\infty$.

We remarked at footnote 3) that Theorem 1 implies (1). Likewise we can show that this Corollary implies (2).

The following theorem corresponds to Theorem 2.

THEOREM 4. *Under the assumptions of Theorem* 3 *we have $N = \check{N}$ except for the case where $N = 0$ and $\check{N} = \infty$ or the case where $\check{N} = 0$ and $N = \infty$, which can really arise only under* (iv).

We can obtain a corollary corresponding to the Corollary of Theorem 3.

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