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A Note on Normal Ideals

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§1. Introduction

In [3], p. 85 F. Maeda writes $a \triangledown b$ in a lattice L with 0 to denote the fact that $a \wedge b = 0$ and $(a \vee x) \wedge b = x \wedge b$ for all x in L. He then uses this relation to investigate direct sum decompositions of such lattices. If L is modular the relation $\triangledown$ is symmetric and the mapping $S \rightarrow S^{\triangledown} = \{f: s \triangledown f$ for all $s \in S\}$ induces a Galois connection in the lattice $I(L)$ of all ideals of L. The Galois closed objects (i.e., those ideals S such that $S = S^{\triangledown\triangledown}$) are called *normal ideals*. In a *general continuous geometry* (see [3], p. 90) the normal ideals play a role analogous to that played by the center of a continuous geometry. In this note we investigate normal ideals in a more general setting. In §2 we show that in a lattice L with 0, an ideal J is in the center of $I(L)$ if and only if it is a direct summand of L. In §3 we use the fact that the relation $\triangledown$ is symmetric in a relatively complemented lattice with 0 to define normal ideals in such a lattice. We then show that if L is a relatively complemented lattice with 0 and 1, then the center of the completion by cuts $\bar{L}$ of L is precisely the set of normal ideals which are kernels of congruence relations. In the case of a complemented modular lattice, the center of $\bar{L}$ is just the set of normal ideals of L. In §4 these results are extended to the case of an arbitrary relatively complemented lattice with 0.

§2. Direct summands

Let $S_1, S_2, \cdots, S_n$ be subsets of a lattice L with 0. Following the terminology of F. Maeda ([3], p. 85) if

(1°) for any element a of L, $a = a_1 \vee \cdots \vee a_n$ with $a_i \in S_i (i = 1, \cdots, n)$,

(2°) $i \neq j$ implies $S_j \subseteq S_i^{\triangledown}$,

we say that L is a *direct sum* of $S_1, \cdots, S_n$ and write $L = S_1 \oplus \cdots \oplus S_n$. The subsets $S_1, \cdots, S_n$ will be called *direct summands* of L. By [3], Lemma 1.3, p. 86 every direct summand is an ideal of L. We proceed to show that the direct summands are precisely the central elements of $I(L)$.

Theorem 1. *Let L be a lattice with 0. An ideal J of L is a central element of $I(L)$ if and only if it is a direct summand of L.*

PROOF: Assume first that J is a central element of $I(L)$, and let K be its complement. For each $a \in L$, let J_a denote the principal ideal generated by a. Then, working in $I(L)$, we have that $J_a = J_a \cap (J \vee K) = (J_a \cap J) \vee (J_a \cap K)$. This implies that $a = b \vee c$ with $b \in J$ and $c \in K$. Furthermore, if $b \in J$ and $c \in K$, then for arbitrary x in L,

$$(J_b \vee J_x) \cap J_c \subseteq (J \vee J_x) \cap J_c = (J \cap J_c) \vee (J_x \cap J_c) = J_x \cap J_c = J_{x \wedge c}\,.$$

Thus $(b \vee x) \wedge c = x \wedge c$ and since $b \wedge c = 0$ is obvious we see that $b \triangledown c$. A similar argument produces $c \triangledown b$, and we have that $L = J \oplus K$.

Suppose conversely that $L = J \oplus K$. Then $J \cap K = (0)$ and since each a in L can be represented in the form $a = b \vee c$ with $b \in J$ and $c \in K$, it follows that $J \vee K = L$. Thus J and K are complements in $I(L)$. In order to show that J is central it suffices ([2], Theorem 7.2, p. 299) to show that for each ideal I of L the following equations hold:

$$(1) \qquad I = (I \vee J) \cap (I \vee K)$$

$$(2) \qquad I = (I \cap J) \vee (I \cap K).$$

Now let $a \in I$ and write $a = b \vee c$ with $b \in J$, $c \in K$. This puts a in $(I \cap J) \vee (I \cap K)$ and establishes (2). In order to demonstrate (1), we need only show that $(I \vee J) \cap (I \vee K) \subseteq I$. Accordingly, let $a \leq (b_1 \vee c) \wedge (b_2 \vee d)$ with b_1, b_2 in I, c in J and d in K. We may then write $(b_1 \vee c) \wedge (b_2 \vee d) = x \vee y$ where $x \in J$, $y \in K$. But now, since $L = J \oplus K$ we have $d \triangledown x$ so that

$$x = (b_1 \vee c) \wedge (b_2 \vee d) \wedge x = (b_1 \vee c) \wedge b_2 \wedge x \leq b_2\,.$$

This shows that $x \in I$. Similarly, $y \in I$ and we conclude that a is in I, since $a \leq x \vee y$. This completes the proof.

§3. Normal ideals in a relatively complemented lattice

In this section we explore the relation between normal ideals of a relatively complemented lattice with 0 and 1 and central elements of $\bar{L}$, the completion of L by cuts. We first need to know that the relation $\triangledown$ is symmetric.

THEOREM 2. *Let e, f be elements of a relatively complemented lattice with 0. The following conditions are then equivalent:*

(i) $e \triangledown f$.

(ii) $e \vee f \leq a$ *implies f is contained in every complement of e in the interval $L(0, a)$.*

(iii) $e_1 \leq e$, $f_1 \leq f$, *e_1 perspective to f_1 imply that* $e_1 = f_1 = 0$.

(iv) $x = (x \vee e) \wedge (x \vee f)$ *for all x in L.*

Proof: (i)$\Rightarrow$(ii) Let $e\vee f\leq a$, and let y be a complement of e in $L(0,a)$. Then $f=a\wedge f=(e\vee y)\wedge f=y\wedge f$ shows that $f\leq y$.

(ii)$\Rightarrow$(iii) Let $e_1\vee x_1=f_1\vee x_1$ with $e_1\wedge x_1=f_1\wedge x_1=0$. Then with $a=e\vee f\vee x_1$, we may assume the existence of an element x which is a common complement for e_1 and f_1 in the interval $L(0,a)$. Since $e\geq e_1$, we see that $e\vee x=a$ and consequently x dominates an element y which is a complement of e in $L(0,a)$. Invoking (ii), we see that $f\leq y\leq x$, $f_1=f_1\wedge x=0$, $x=a$ and finally also $e_1=0$.

(iii)$\Rightarrow$(iv) For a fixed x in L, set $a=e\vee f\vee[(x\vee e)\wedge(x\vee f)]$. We then choose y so that $y\vee[(x\vee e)\wedge(x\vee f)]=a$ and $y\wedge[(x\vee e)\wedge(x\vee f)]=x$. Then $y\vee e=y\vee f=a$, so there exist elements $e_1\leq e$, $f_1\leq f$ having y as a common complement in a. By (iii), $e_1=f_1=0$ so $y=a$ and $x=(x\vee e)\wedge(x\vee f)$.

(iv)$\Rightarrow$(i) If $x=(x\vee e)\wedge(x\vee f)$ for all x in L, then $0=(0\vee e)\wedge(0\vee f)=e\wedge f$ and for each x in L, $(x\vee e)\wedge f=(x\vee e)\wedge(x\vee f)\wedge f=x\wedge f$.

Corollary 1. *In a relatively complemented lattice with* 0 *the relation* $\bigtriangledown$ *is symmetric.*

Corollary 2. *Let L be a relatively complemented lattice with* 0. *If* $e\bigtriangledown f_\alpha$ *for each* $\alpha\in A$, *and if* $f=\bigvee_{\alpha\in A}f_\alpha$ *exists, then* $e\bigtriangledown f$.

Corollary 3. *Let L be a relatively complemented lattice with* 0. *Then if* $e\vee f\leq a$, $e\bigtriangledown f$ *in the interval* $L(0,a)$ *if and only if f is contained in every complement of e in a.*

It is worth noting that one does not need anything nearly as strong as the fact that L is relatively complemented in order to conclude that the relation $\bigtriangledown$ is symmetric. Indeed if L is a lattice with 0 and 1 having the property that $e<f$ implies the existence of an element $g\neq 1$ such that $f\vee g=1$ and $f\wedge g\geq e$ one can easily show that $e\bigtriangledown f$ is equivalent to the assertion that $x=(x\vee e)\wedge(x\vee f)$ for all x in L. An example of such a lattice is provided by a relatively co-atomic lattice with 0; i.e., a lattice L with 0 and 1 having the property that each $e\neq 1$ is the infimum of the co-atoms that dominate it. Here a *co-atom* denotes an element which is covered by 1.

If L is a lattice with 0 in which the relation $\bigtriangledown$ is symmetric, let us agree to call an ideal J *normal* in case $J=(J^{\bigtriangledown})^{\bigtriangledown}$. The term *homomorphism kernel* will denote an ideal which is the kernel of a congruence relation of L, and we will call J a *normal homomorphism kernel* if J is both a normal ideal and a homomorphism kernel. We are now ready to investigate the center of $\bar{L}$. Suppose J is central in $\bar{L}$ and K is its complement therein. Then J induces a congruence relation on $\bar{L}$ by the formula $I_1\equiv I_2$ if $I_1\vee J=I_2\vee J$. Since $a\rightarrow J_a$ is an isomorphism of L into $\bar{L}$, the relation Θ on L defined by $a\equiv b(\Theta)$ if $J_a\vee J=J_b\vee J$ is evidently a congruence relation on L whose kernel is J. Notice that if $e \in J, f \in K$ then $e \triangledown f$. On the other hand, if $e \triangledown f$ for all f in K then $J_e \cap K = (0)$ and since J is central this implies that $J_e \subseteq J$; i.e., that $e \in J$. Thus J is a normal homomorphism kernel. *Until further notice it will be assumed that* L *is a relatively complemented lattice with* 0 *and* 1. It will be our purpose to show that every normal homomorphism kernel of L is a central element of $\bar{L}$.

LEMMA 3. *Every normal ideal J of L is an element of $\bar{L}$.*

PROOF: If $b \in J^{\triangledown}$, then every complement of b is an upper bound for J. It follows that if a is contained in every upper bound of J, then $a \triangledown b$. But this puts a in $(J^{\triangledown})^{\triangledown} = J$, completing the proof.

An extremely useful observation is provided by

LEMMA 4. *Let J be a normal homomorphism kernel of L. Then $a \in J^{\triangledown}$ if and only if $J_a \cap J = (0)$.*

PROOF: Suppose first that $J_a \cap J = (0)$. Let $b \in J$, $x \in L$, and choose c to be a complement of $x \wedge a$ in $(b \vee x) \wedge a$. Now if J is the kernel of the congruence relation Θ, we may write $c \equiv c \wedge (b \vee x) \equiv c \wedge x \equiv 0(\Theta)$. This implies that $c \in J$ and since $c \leq a$, we have $c = 0$ and $b \triangledown a$. Thus $a \in J^{\triangledown}$. On the other hand, if $a \in J^{\triangledown}$, then $J_a \cap J = (0)$ is obvious, and we are done.

LEMMA 5. *If J is a normal homomorphism kernel of L, the same is true of $J^{\triangledown}$; furthermore, J and $J^{\triangledown}$ are complements in $\bar{L}$.*

PROOF: Let J be the kernel of the congruence relation Θ and let Θ^* denote the pseudo-complement of Θ in the lattice of congruence relations of L. By [1], Lemma 17, p. 163 $a \equiv b(\Theta^*)$ iff $a \vee b \geq c \geq d \geq a \wedge b$ with $c \equiv d(\Theta)$ implies $c = d$. In particular, if $a \equiv 0(\Theta^*)$, then $a \geq c$ with $c \equiv 0(\Theta)$ implies $c = 0$, so that $J_a \cap J = (0)$. By Lemma 4, this puts a in $J^{\triangledown}$. But if $a \in J^{\triangledown}$ and if $a \geq c \geq d$ with $c \equiv d(\Theta)$, then by [4], Hilfsatz 4.5, p. 37 we may write $c = d \vee t$ with $t \in J$. At this point we see that $t = 0$ and $c = d$. This shows that $J^{\triangledown}$ is the kernel of Θ^*. Since $J^{\triangledown}$ is clearly normal, this completes the proof that $J^{\triangledown}$ is a normal homomorphism kernel of L.

In order to show that J and $J^{\triangledown}$ are complements in $\bar{L}$. We need only show that 1 is their only common upper bound in L. To see this, let a be an upper bound for both J and $J^{\triangledown}$ in L. Choosing b as a complement of a in L, we now have that $J_b \cap J = J_b \cap J^{\triangledown} = (0)$. Since J and $J^{\triangledown}$ are both normal homomorphism kernels, two applications of Lemma 4 will now yield the fact that $b \in J \cap J^{\triangledown} = (0)$, whence $b = 0$ and $a = 1$ as claimed.

LEMMA 6. *Let J be a normal homomorphism kernel of L. Then for all K in $\bar{L}$, $K = (K \cap J) \vee (K \cap J^{\triangledown})$.*

PROOF: Let $b \in K$ and suppose that $c \leq b$ is an upper bound for $J \cap J_b$.

Then if d is a complement of c in b, $J_d \cap J = J_d \cap J \cap J_b = (0)$. Applying Lemma 4, we conclude that $d \in J^{\nabla}$. It follows that b is the only common upper bound for $J \cap J_b$ and $J^{\nabla} \cap J_b$ in the interval $L(0, b)$. Now let a be an upper bound for both $K \cap J$ and $K \cap J^{\nabla}$. Then $a \wedge b$ is an upper bound for both $J \cap J_b$ and $J^{\nabla} \cap J_b$ in $L(0, b)$, whence $a \wedge b = b$ and $b \leq a$. Since b was an arbitrary element of K, we conclude that any upper bound for both $K \cap J$ and $K \cap J^{\nabla}$ is also an upper bound for K. Thus $K \subseteq (K \cap J) \vee (K \cap J^{\nabla})$. The reverse inclusion is obvious.

Now by [2], Theorem 7.2, p. 299 if we wish to show that J in the above lemma is a central element of $\bar{L}$, we must show that $K = (K \vee J) \cap (K \vee J^{\nabla})$ for all K in $\bar{L}$. We will demonstrate that this follows by duality. Let us write $e \triangle f$ in case $e \nabla f$ in the dual of L; i.e., if $e \vee f = 1$ and $(e \wedge x) \vee f = x \vee f$ for all x in L. Also, for each ideal J of L, we shall let J^* denote the set of upper bounds of J. Clearly J^* is an element of the completion by cuts of the dual of L.

Lemma 7. *If J is a normal homomorphism kernel of L, then J^* is a normal homomorphism kernel of the dual of L.*

Proof: We have already noted that if J is the kernel of the congruence relation Θ, then J^{∇} is the kernel of Θ^*, the pseudo-complement of Θ in the lattice of congruence relations of L. Given a in J^* and b a complement of a in L, note that $J_b \cap J = (0)$, $b \in J^{\nabla}$, $b \equiv 0(\Theta^*)$ and consequently $a \equiv 1(\Theta^*)$. On the other hand, if $a \equiv 1(\Theta^*)$, then any complement b of a is in J^{∇}. Now if $e \in J$ then $e \nabla b$ implies $e \leq a$ so that $a \in J^*$. Thus $J^* = \{a : a \equiv 1(\Theta^*)\}$.

We next show that $J^{*\triangle} = J^{\nabla *}$. Let $e \in J^*$ and $f \in J^{\nabla *}$. If g is a complement of f, then $J_g \cap J^{\nabla} = (0)$ puts g in J. Thus e is an upper bound for the set of complements of f, and by the dual of Theorem 2, $e \triangle f$. Suppose next that $e \triangle f$ for all e in J^*. We must show that $f \in J^{\nabla *}$. If h is a complement of an element g of J^{∇}, then $h \in J^*$ implies $h \triangle f$ whence $f \geq g$. Thus f is indeed in $J^{\nabla *}$ and we conclude that $J^{*\triangle} = J^{\nabla *}$. If we now make use of the fact that $J = J^{\nabla\nabla}$, we may apply the above argument twice to see that

$$J^* = (J^{\nabla\nabla})^* = (J^{\nabla *})^{\triangle} = J^{*\triangle\triangle}.$$

It is now obvious that the dual of Lemma 6 can be invoked. For if J is a normal homomorphism kernel of L, working in the completion by cuts of the dual of L, we have that for every K in $\bar{L}$, $K^* = (K^* \cap J^*) \vee (K^* \cap J^{\nabla *})$. Now a is a lower bound for $K^* \cap J^*$ if and only if a is contained in every element b which is an upper bound for both K and J. This is equivalent to saying that $a \in K \vee J$. Similarly $a \in K \vee J^{\nabla}$ if and only if a is a lower bound for $K^* \cap J^{\nabla *}$. Thus if $a \in (K \vee J) \cap (K \vee J^{\nabla})$ then a is a lower bound for both $K^* \cap J^*$ and $K^* \cap J^{\nabla *}$. This implies that a is a lower bound for $(K^* \cap J^*) \vee (K^* \cap J^{\nabla *}) = K^*$, whence $a \in K$. It follows that $K = (K \vee J) \cap (K \vee J^{\nabla})$.

Combining all these results, we have

THEOREM 8. *The center of* $\bar{L}$ *coincides with the set of normal homomophism kernels of* L.

We close this section by showing that in a complemented modular lattice L, *every* normal ideal is a central element of $\bar{L}$. In view of [4], Satz 4.5, p. 38 we need only show that a normal ideal is closed under perspectivity.

LEMMA 9. *Let* L *be a relatively complemented modular lattice with* 0. *Then* $e \nabla f$ *and* $b \geq e \vee f$ *imply that* $(e \vee x) \wedge g \nabla (f \vee x) \wedge g$ *for all* x, g *which are complements in the interval* $L(0, b)$.

PRROF: Applying Theorem 2 to the interval $L(0,b)$ we see that $x=(x \vee e) \wedge (x \vee f)$, and if $x \leq a \leq b$, then $a=(a \vee e) \wedge (a \vee f)=(a \vee x \vee e) \wedge (a \vee x \vee f)$. This shows that $e \vee x \nabla f \vee x$ in $L(x,b)$. We now use the fact that $a \rightarrow a \wedge g$ is an isomorphism of $L(x,b)$ onto $L(0,g)$ to conclude that $(e \vee x) \wedge g \nabla (f \vee x) \wedge g$ in $L(0,g)$. Since L is a modular lattice, it is easily seen that this implies $(e \vee x) \wedge g \nabla (f \vee x) \wedge g$ in L.

LEMMA 10. *Let* J *be a normal ideal of a relatively complemented modular lattice with* 0. *Then if* g *is perspective to an element of* J, g *itself is in* J.

PROOF: Since L is modular we may assume the existence of an element f of J such that f and g have a common complement x in $f \vee g$. For arbitrary e in J^{∇}, $e \nabla f$ and $x \wedge (e \vee g) = x \wedge (e \vee g) \wedge (f \vee g) = x \wedge g = 0$. Also, $x \vee (e \vee g) = e \vee f \vee g$ so that x is a complement of $e \vee g$ in $e \vee f \vee g$. Now

$$(e \vee x) \wedge (e \vee g) = e \vee [x \wedge (e \vee g)] = e \vee 0 = e \quad \text{and}$$

$$(f \vee x) \wedge (e \vee g) = (f \vee g) \wedge (e \vee g) = g,$$

so by Lemma 9, $e \nabla g$. Since e was an arbitrary element of J^{∇}, we conclude that g is in J.

We are now ready to state our result.

THEOREM 11. *An ideal* J *of a complemented modular lattice* L *is a central element of* $\bar{L}$ *if and only if it is a normal ideal.*

§4. The general case

Here we shall assume that L *is a relatively complemented lattice with* 0. Our goal will be to extend the results of §3 to such a lattice. Instead of considering $\bar{L}$, it turns out to be appropriate to work in $\tilde{L}$, the set of ideals J such that $J \cap J_a \in \bar{L}$ for all a in L. Since the intersection of an arbitrary family of elements of $\tilde{L}$ falls back in $\tilde{L}$, it is obvious that $\tilde{L}$ is a complete lattice with set inclusion as the partial order and set intersection as the meet operation; furthermore, the mapping $a \to J_a$ embeds L as a sublattice of $\tilde{L}$. In case L happens to have a greatest element, it is worth mentioning the trivial fact that $\tilde{L} = \bar{L}$.

LEMMA 12. *Every normal ideal of* L *is an element of* $\tilde{L}$.

PROOF: This follows with no difficulty from Theorem 2.

LEMMA 13. *Every central element of* $\tilde{L}$ *is a normal homomorphism kernel of* L.

PROOF: The argument is almost identical with the one preceding Lemma 3.

We now proceed to show that the center of $\tilde{L}$ is precisely the set of normal homomorphism kernels of L. *In connection with this, it will prove convenient to let* $\bar{L}_x$ *denote the completion by cuts of the lattice* $L(0, x)$.

LEMMA 14. *For each* x *in* L, $\bar{L}_x$ *is a sublattice of* $\tilde{L}$ *and*

$$\bar{L}_x = \{J \cap J_x : J \in \tilde{L}\} .$$

PROOF: We first observe that if K in $\bar{L}$ has x as an upper bound, then $K \in \bar{L}_x$. This follows from the fact that y is an upper bound for K in L if and only if $y \wedge x$ is an upper bound for K in $L(0, x)$. Thus, if $J \in \tilde{L}$, then $J \cap J_x \in \bar{L}_x$. On the other hand, given $K \in \bar{L}_x$ we claim that $K \in \tilde{L}$. To see this, we must show that $K \cap J_a \in \bar{L}$ for every $a \in L$. Accordingly, let d be contained in all upper bounds of $K \cap J_a$. Then if y is an upper bound for K in $L(0, x)$, surely y is an upper bound for $K \cap J_a$ and we have $d \leq y$. It follows that $d \in K$, and since a is an upper bound for $K \cap J_a$, we also have $d \leq a$. Hence $d \in K \cap J_a$, and we see that $K \cap J_a \in \bar{L}$. This shows that $\bar{L}_x = \{J \cap J_x : J \in \tilde{L}\}$ and that $\bar{L}_x \subseteq \tilde{L}$. Since the infimum operation in both $\bar{L}_x$ and $\tilde{L}$ is set intersection, it is evident that $\bar{L}_x$ is a meet sublattice of $\tilde{L}$. On the other hand, if J, K are elements of $\bar{L}_x$ and M is their join in $\bar{L}_x$, then $M \in \tilde{L}$ and is an upper bound for both J and K in $\tilde{L}$. If $N \in \tilde{L}$ is a common upper bound for J and K, then $N \cap J_x$ is an upper bound in $\bar{L}_x$. It follows that $N \cap J_x \supseteq M$ and consequently that $N \supseteq N \cap J_x \supseteq M$. Thus M is effective as the join of J and K in $\tilde{L}$, thereby completing the proof.

LEMMA 15. *Let* $K \in \tilde{L}$ *and let* J *be a normal homomorphism kernel of* L. *Then* $K \vee J = \bigcup_{a \in L} [(K \cap J_a) \vee (J \cap J_a)]$ *and for each* b *in* L

$$(K \vee J) \cap J_b = (K \cap J_b) \vee (J \cap J_b) .$$

PROOF: Let $M = \bigcup_{a \in L} [(K \cap J_a) \vee (J \cap J_a)]$. Since $K = \bigcup_{a \in L} (K \cap J_a)$ and $J =$

$\bigcup_{a \in L}(J \cap J_a)$ it is evident that M contains both K and J. Furthermore, if $N \in \tilde{L}$ is an upper bound for both K and J, then N contains $K \cap J_a$ and $J \cap J_a$, $N \supseteq (K \cap J_a) \vee (J \cap J_a)$ and finally N contains M. Thus, in order to show that M is the join of K and J in $\tilde{L}$, we need only verify that M is in fact an element of $\tilde{L}$. In order to demonstrate this we must prove that for each b in L, $M \cap J_b \in \bar{L}_b$. This will follow if it can be shown that $M \cap J_b = (K \cap J_b) \vee (J \cap J_b)$. Evidently $M \cap J_b \supseteq (K \cap J_b) \vee (J \cap J_b)$. To obtain the reverse inclusion, choose $x \geq a \vee b$ and work in the interval $L(0, x)$. Suppose $f \leq x$ and $e \nabla f$ for all e in $J \cap J_x$. Then $J_f \cap J = J_f \cap J \cap J_x = (0)$ and by Lemma 4 we see that $f \in J^{\nabla}$. On the other hand, if $f \in J^{\nabla} \cap J_x$ we must clearly have that $e \nabla f$ in $L(0, x)$ for all e in $J \cap J_x$. Thus $(J \cap J_x)^{\nabla}$ as computed in $L(0, x)$ is the ideal $J^{\nabla} \cap J_x$. We thus see that $J \cap J_x$ is a normal homomorphism kernel of $L(0, x)$ and by Theorem 8, it is a central element of $\bar{L}_x$. Hence

$$[(K \cap J_a) \vee (J \cap J_x)] \cap J_b = (K \cap J_a \cap J_b) \vee (J \cap J_x \cap J_b) \subseteq (K \cap J_b) \vee (J \cap J_b).$$

It follows that for each a in L

$$[(K \cap J_a) \vee (J \cap J_a)] \cap J_b \subseteq (K \cap J_b) \vee (J \cap J_b)$$

and therefore that

$$M \cap J_b = \{\bigcup_{a \in L}[(K \cap J_a) \vee (J \cap J_a)]\} \cap J_b$$

$$= \bigcup_{a \in L}\{[(K \cap J_a) \vee (J \cap J_a)] \cap J_b\} \subseteq (K \cap J_b) \vee (J \cap J_b),$$

thus completing the proof.

Now let $K \in \tilde{L}$ and let J be a normal homomorphism kernel of L. We claim first that $K = (K \vee J) \cap (K \vee J^{\nabla})$. In order to see this, we choose an element a of L. Since $J \cap J_a$ is a normal homomorphism kernel of $L(0, a)$ with $(J \cap J_a)^{\nabla}$ as computed in $L(0, a)$ equal to $J^{\nabla} \cap J_a$, we may invoke Lemma 15 and Theorem 8 to see that

$$(K \vee J) \cap (K \vee J^{\nabla}) \cap J_a = [(K \cap J_a) \vee (J \cap J_a)] \cap [(K \cap J_a) \vee (J^{\nabla} \cap J_a)] = K \cap J_a.$$

Since this holds for every a in L, we conclude that $K = (K \vee J) \cap (K \vee J^{\nabla})$. We next show that $K = (K \cap J) \vee (K \cap J^{\nabla})$. Working in the interval $L(0, a)$, we have from Theorem 8 that

$$K \cap J_a = [(K \cap J_a) \cap (J \cap J_a)] \vee [(K \cap J_a) \cap (J^{\nabla} \cap J_a)] \subseteq (K \cap J) \vee (K \cap J^{\nabla}).$$

Hence $K = \bigcup_{a \in L}(K \cap J_a) \subseteq (K \cap J) \vee (K \cap J^{\nabla}) \subseteq K$ and we have equality. By [2], Theorem 7.2, p. 299 we conclude that J is a central elememt of $\tilde{L}$. Combining the above results with Lemma 10, we have

THEOREM 16. *Let L be a relatively complemented lattice with* 0. *An ideal J of L is a central element of* $\tilde{L}$ *if and only if it is a normal homomorphism kernel. In the presence of modularity, the central elements of* $\tilde{L}$ *are precisely the normal ideals of L.*

In connection with the above theorem notice that the partial order in $\tilde{L}$ is given by set inclusion. Since the intersection of an arbitrary family of normal homomorphism kernels is itself a normal homomorphism kernel, we see that the center of $\tilde{L}$ is a complete Boolean sublattice of $\tilde{L}$. As an immediate consequence of these observations we have the following result of F. Maeda ([3], Theorem 3.2, p. 89): *Let L be a conditionally upper continuous, relatively complemented modular lattice with* 0. *The family of normal ideals in L is a complete Boolean algebra, where lattice-order means set-inclusion.*

In closing we mention that in a later paper we shall prove that with L as in F. Maeda's theorem, $\tilde{L}$ is an upper continuous complemented modular lattice. This fact together with Theorem 16 provide considerable insight into the dimension theory of a general continuous geometry as outlined in [3], pp. 90–92.

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