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Note on Kishi's Theorem for Capacitability

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1. Let Ω be a locally compact Hausdorff space and $\Phi(x, y)$ be a kernel on Ω, i.e. a lower semicontinuous function on $\Omega\times\Omega$ with values in $[0, +\infty]$. A measure μ will be always a non-negative Radon measure on Ω. The potential of μ is defined by $\Phi(x, \mu)=\int\Phi(x, y)d\mu(y)$ and the mutual energy of μ and ν is defined by $(\nu, \mu)=\int\Phi(x, \mu)d\nu(x)$. We call (μ, μ) simply the energy of μ. Let $\mathscr{E}$ be the class of all measures with finite energy and $\mathscr{E}_A$ be the class of all measures of $\mathscr{E}$ whose supports are compact and contained in the given set A. The support of μ will be denoted by $S\mu$. For a compact set K, we define

$$e(K)=\sup\{2\mu(K)-(\mu, \mu);\ \mu\in\mathscr{E}_K\} \qquad \text{if } \mathscr{E}_K\neq\{0\}$$

and

$$e(K)=0 \qquad \text{if } \mathscr{E}_K=\{0\}.$$

For any set A, we define an inner quantity and an outer quantity as follows:

$$e_i(A)=\sup\{e(K);\ K \text{ is compact and } K\subset A\}$$

and

$$e_0(A)=\inf\{e_i(G);\ G \text{ is open and } G\supset A\}.$$

The problem of capacitability is to discuss when $e_i(A)$ coincides with $e_0(A)$.

M. Kishi [4] proved the equality $e_i(A)=e_0(A)$ for every analytic set A (Theorem 13 in [4]) under the hypotheses that Ω is a locally compact separable metric space, that $\Phi>0$ and that Φ is of positive type (§3) and satisfies the continuity principle (footnote 6), condition (*) (§4) and a regularity condition (§7).

The object of this note is to improve his theorem. Namely, we shall show that Kishi's regularity condition, the condition $\Phi>0$ and a restriction on Ω can be omitted. The reasoning is analogous to that of B. Fuglede [3] and the author [7], but our quantities are different from theirs (footnote 14).

Our problem was also studied by B. Fuglede [2] and M. Ohtsuka [6] under different additional conditions. The differences will be illustrated in §8.

2. We recall the quantities related to the capacity which were introduced by Ohtsuka [6]. For a measure $\mu \neq 0$, we put $V(\mu)=\sup\{\Phi(x, \mu); x \in S\mu\}$. We define $V_i(A)$ by $\inf\{V(\mu); \mu \in \mathscr{U}_A\}$ if $A \neq \phi$ and set $V_i(\phi)=\infty$, where ϕ denotes the empty set and $\mathscr{U}_A=\{\mu;\ S\mu$ is compact, $S\mu \subset A$ and $\mu(\Omega)=1\}$. We define also $V_e(A)=\sup\{V_i(G); G$ is open and $G \supset A\}$. We shall say that a property holds *nearly everywhere* or *n.e.* (*quasi everywhere* or *q.e.* resp.) on A if the V_i-value (V_e-value resp.) of the exceptional set in A is infinite.

The following propositions, which will be often used later, were obtained by Ohtsuka.

PROPOSITION 1.[1] *Let* $\{B_n\}$ *be a sequence of sets which are measurable for every measure on* Ω *and* A *be an arbitrary set. Then we have* $V_i(\bigcup A \cap B_n)^{-1} \leqq \sum_n V_i(A \cap B_n)^{-1}$.

PROPOSITION 2.[2] *Let* $\{A_n\}$ *be a sequence of arbitrary sets. Then we have*

$$V_e(\bigcup_n A_n)^{-1} \leqq \sum_n V_e(A_n)^{-1}.$$

As an immediate consequence of this, we have

COROLLARY. *Let* A_1 *and* A_2 *be arbitrary sets. If* $V_e(A_2)=\infty$, *then* $V_e(A_1-A_2)=V_e(A_1)=V_e(A_1 \cup A_2)$.

PROPOSITION 3.[3] *For any compact set* K, *we have* $V_i(K)=V_e(K)$.

3. We assume hereafter that the kernel Φ is *of positive type*, i.e. $\Phi(x, y)=\Phi(y, x)$ for all $x, y \in \Omega$ and $(\mu, \mu)+(\nu, \nu)-2(\nu, \mu) \geqq 0$ for all $\mu, \nu \in \mathscr{E}$. The following theorem is well-known.

THEOREM 1. *For any compact set* K, *we have* $e(K)=V_i(K)^{-1}$. *In case* $V_i(K)>0$, *there is a measure* $\mu_K \in \mathscr{E}_K$ *which minimizes* $I(\nu)=(\nu, \nu)-2\nu(K)$ *for* $\nu \in \mathscr{E}_K$. *This measure* μ_K *has the following properties*: (1) $\Phi(x, \mu_K) \leqq 1$ *on* $S\mu_K$, (2) $\Phi(x, \mu_K) \geqq 1$ *n.e. on* K *and* (3) $\mu_K(\Omega)=(\mu_K, \mu_K)=e(K)$.

COROLLARY 1. *For any set* A, *it holds that*

$$e_i(A)=V_i(A)^{-1} \quad \textit{and} \quad e_0(A)=V_e(A)^{-1}.$$

Consequently we see that $e_0(K)=e_i(K)=e(K)$ for any compact set K and that $e_0(A)=0$ is equivalent to $V_e(A)=\infty$.

COROLLARY 2. *If* $V_e(N)=\infty$, *then* $e_0(A-N)=e_0(A)=e_0(A \cup N)$ *for any set* A.

For an arbitrary set A, we introduce two classes of measures:

1) [6], Proposition 1, p. 139.
2) [6], Proposition 2, p. 140.
3) [6], Theorem 1.14, p. 207.

$$\Gamma^i_A=\{\nu\in\mathscr{E};\ \Phi(x,\nu)\geqq 1 \text{ n.e. on } A\}$$

and

$$\Gamma^e_A=\{\nu\in\mathscr{E};\ \Phi(x,\nu)\geqq 1 \text{ q.e. on } A\}.$$

In case $\Gamma^i_A\neq\phi$, we define $c^i(A)$ by $\inf\{(\nu,\nu);\ \nu\in\Gamma^i_A\}$, and in case $\Gamma^i_K=\phi$, we set $c^i(A)=\infty$. For the class Γ^e_A, $c^e(A)$ is defined in the same way.[4)]

These quantities $c^i(A)$ and $c^e(A)$ are increasing set functions of A and it holds in general that $c^i(A)\leqq c^e(A)$ for any set A. In case A is a compact set K, we have

THEOREM 2. $c^i(K)=c^e(K)$.

PROOF. It is enough to show $\Gamma^i_K=\Gamma^e_K$. This follows from Propositions 2 and 3.

In case $c^i(A)(e_i(A)$ resp.) coincides with $c^e(A)$ ($e_0(A)$ resp.), we denote by $c(A)$ ($e(A)$ resp.) the common value. We observe that this $e(K)$ is equal to the original $e(K)$ in §1 and $c^i(K)=c^e(K)=c(K)$ for any compact set K and $e_i(G)=e_0(G)=e(G)$ for any open set G. The relation between $c(K)$ and $e(K)$ is given by

THEOREM 3.[5)] *$e(K)=c(K)$ for any compact set K.*

PROOF. Let $\nu\in\Gamma^i_K$ and $\mu\in\mathscr{E}_K$. Then it follows that $(\nu,\mu)\geqq\mu(K)$ and

$$(\nu,\nu)+(\mu,\mu)-2\mu(K)\geqq(\nu,\nu)+(\mu,\mu)-2(\nu,\mu)\geqq 0.$$

Therefore $c(K)\geqq e(K)$.

The converse inequality is led by Theorem 1. In fact, in case $e(K)<\infty$, the measure μ_K belongs to Γ^i_K.

COROLLARY. *$e_i(A)\leqq c_i(A)$ for any set A.*

4. Now we assume in the sequel that the kernel Φ satisfies *the continuity principle.*[6)] In case Ω is not compact, we further assume that the kernel Φ fulfils the following *condition* $(*)$ of Kishi: for any compact set K and any positive number ε, there is a compact set K_ε such that $\Phi(x,y)<\varepsilon$ on $K\times(\Omega-K_\varepsilon)$.

Kishi proved in [4]

LEMMA 1. *Suppose that a sequence $\{\mu_n\}$ of measures converges vaguely to*

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4) Fuglede considered Γ^i_A, Γ^e_A, $c^i(A)$ and $c^e(A)$ in [2]. Kishi introduded a family of measures similar to Γ^e_A, but the meaning of "q.e." was different. Kishi defined that a set A is $\mathscr{E}$-polar if and only if there is a measure $\nu\in\mathscr{E}$ such that $\Phi(x,\nu)=\infty$ on A. He said that a property holds q.e. on A if the exceptional set in A is $\mathscr{E}$-polar.

5) [4], p. 107.

6) Continuity principle: If the potential $\Phi(x,\mu)$ of a measure μ with compact support $S\mu$ is finite and continuous as a function on $S\mu$, then $\Phi(x,\mu)$ is continuous in Ω.

μ_0 and that the total masses $\mu_n(\Omega)$ are bounded. Let ν be a measure which has a compact support and whose potential is finite and continuous in Ω. Then we have $\lim_{n\to\infty}(\nu, \mu_n)=(\nu, \mu_0)$.

We shall consider a quantity $I_A^i=\inf\{(\nu, \nu)-2\nu(\Omega);\ \nu\in\mathscr{E}_A\}$ for the class of measures $\mathscr{E}_A=\{\nu\in\mathscr{E};\ S\nu$ is compact and $S\nu\subset A\}$. By the obvious relations $I_K^i=-e(K)$ for any compact set K and $I_A^i=\inf\{I_K^i;\ K$ is compact and $K\subset A\}$, we have

THEOREM 4. *$I_A^i=-e_i(A)$ for any set A.*

The relation between $e_i(A)$ and $c^i(A)$ is given by

THEOREM 5.[7] *$e_i(A)=c^i(A)$ for any set A. If $e_i(A)$ is finite, then there is a measure μ_0 supported by $\bar{A}$ (the closure of A) such that $\Phi(x, \mu_0)\geqq 1$ n.e. on A, $\Phi(x, \mu_0)\leqq 1$ on $S\mu_0$ and $\mu_0(\Omega)=(\mu_0, \mu_0)=e_i(A)$.*

PROOF. On account of Corollary to Theorem 3, we may assume that $e_i(A)$ is finite. In this case $V_i(K)\geqq V_i(A)>0$ for any compact set $K\subset A$. Let $\{K_n\}$ be an increasing sequence of compact sets contained in A such that $I_{K_n}^i$ tends to I_A^i. For each n, let μ_n be the measure μ_{K_n} obtained in Theorem 1. Then $\mu_n(\Omega)=(\mu_n, \mu_n)$ tends to $-I_A^i=e_i(A)$ (Theorem 4), $\Phi(x, \mu_n)\geqq 1$ n.e. on K_n and $\Phi(x, \mu_n)\leqq 1$ on $S\mu_n$. Since the total masses $\mu_n(\Omega)$ are bounded, we can find a vaguely convergent subsequence of $\{\mu_n\}$. Denote it by $\{\mu_n\}$ again and let μ_0 be the vague limit. Naturally, it is suppoted by $\bar{A}$. We set $N=\{x\in A;\ \Phi(x, \mu_0)<1\}$ and prove $V_i(N)=\infty$. If we deny this, then there would be a unit measure $\nu\in\mathscr{E}_N\subset\mathscr{E}_A$ such that $\Phi(x, \nu)$ is finite and continuous in Ω by the continuity principle; see for instance [6], for the proof of Lemma 1.4, p. 190. Obviously we have $(\nu, \mu_0)<1$. Since $\mu_n+t\nu\in\mathscr{E}_A$ for any positive number t, it holds that

$$I_A^i\leqq(\mu_n+t\nu,\ \mu_n+t\nu)-2(\mu_n+t\nu)(\Omega)$$
$$=(\mu_n, \mu_n)-2\mu_n(\Omega)+2t(\mu_n, \nu)+t^2(\nu, \nu)-2t\nu(\Omega).$$

Letting $n\to\infty$, we have by Lemma 1

$$I_A^i\leqq I_A^i+2t(\nu, \mu_0)+t^2(\nu, \nu)-2t.$$

Cancelling I_A^i, dividing the rest by t and letting $t\to 0$, we obtain $(\nu, \mu_0)\geqq 1$. This is a contradiction. Hence $\Phi(x, \mu_0)\geqq 1$ n.e. on A and $(\mu_0, \mu_0)\leqq\varliminf_{n\to\infty}(\mu_n, \mu_n)$ $=e_i(A)$. Consequently $\mu_0\in\Gamma_A^i$ and $c^i(A)\leqq(\mu_0, \mu_0)\leqq e_i(A)$. Therefore (μ_0, μ_0) $=e_i(A)=c^i(A)$.

Let x_0 be any point of $S\mu_0$, and $\mathscr{N}$ be the directed set of neighborhoods of x_0. For every couple (U, n) of $U\in\mathscr{N}$ and n, we select any point $x(U, n)$ in $U\cap S\mu_{n'}$, where n' is the smallest integer satisfying $n'\geqq n$ and $U\cap S\mu_{n'}\neq\phi$.

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7) cf. [4], Lemma 12, p. 107 and [2], Theorem 4.1, p. 175.

We regard the set of all couples $W=(U, n)$ as a directed set in a natural manner and denote it by D. Let $\lambda_W=\varepsilon_{x(U,n)}\times\mu_{n'}$ correspond to $W=(U, n)$, where ε_x represents the unit point measure at x in general. Thus $\{\lambda_W;\ W\in D\}$ is a net, and converges vaguely to $\varepsilon_{x_0}\times\mu_0$. We have

$$\Phi(x_0, \mu_0)=\int\Phi d(\varepsilon_{x_0}\times\mu_0)\leqq\varliminf_{D}\int\Phi d\lambda_W$$

$$=\varliminf_{D}\int\Phi(x(U, n),\ y)d\mu_{n'}(y)\leqq 1.$$

By the arbitrariness of $x_0\in S\mu_0$, we conclude $\Phi(x, \mu_0)\leqq 1$ on $S\mu_0$. Hence $e_i(A)=(\mu_0, \mu_0)\leqq\mu_0(\Omega)$. On the other hand, $\mu_0(\Omega)\leqq\varliminf_{n\to\infty}\mu_n(\Omega)=e_i(A)$ and hence $\mu_0(\Omega)=e_i(A)$, This completes the proof.

5. In this section, we shall establish the equality $e_0(A)=c^e(A)$. First we shall prove

THEOREM 6.[8] *$e_0(A)\leqq c^e(A)$ for any set A.*

PROOF. We may assume $c^e(A)<\infty$. Given $\varepsilon>0$, there is a measure ν such that $(\nu, \nu)<c^e(A)+\varepsilon$ and $\Phi(x, \nu)\geqq 1$ q.e. on A. For any positive number $t<1$, put $G_t=\{x\in\Omega;\ \Phi(x, \nu)>t\}$ and $N=\{x\in A;\ \Phi(x, \nu)<1\}$. Then G_t is open and contains $A-N$. Since ν/t belngs to $\Gamma^i_{G_t}$ and $V_e(N)=\infty$, we have $e_0(A)=e_0(A-N)\leqq e_i(G_t)=c^i(G_t)\leqq(\nu, \nu)/t^2<[c^e(A)+\varepsilon]/t^2$ by Theorem 5 and Corollary 2 to Theorem 1. Letting $t\to 1$, we have $e_0(A)\leqq c^e(A)+\varepsilon$ and hence obtain the inequality.

Our problem is to see the converse inequality. In what follows, if we assume that any open set in Ω is a K_σ-set, then our reasoning becomes simplier. However, we do not assume this and we follow the method of Fuglede [3], making use of quasi topology but without referring to this terminology explicitly (cf. [7]). We shall prepare several lemmas, which were proved for $V_i^*(A)^{-1}$ and $V_e^*(A)^{-1}$ in [3] (see footnote 14). We have

LEMMA 2. *Assume that a set A has the following property: for any $\varepsilon>0$, there is a set B_ε such that $e_i(B_\varepsilon)=e_0(B_\varepsilon)$, $e_0(A-B_\varepsilon)<\varepsilon$ and $e_0(B_\varepsilon-A)<\varepsilon$. Then it holds that $e_i(A)=e_0(A)$.*

PROOF. This is an immediate consequence of Proposition 2 and the inequality $e_i(A\cup B)\leqq e_i(A)+e_0(B)$.

LEMMA 3.[9] *Take $A\subset\Omega$ and $\mu, \nu\in\mathscr{E}$.* If *$\Phi(x, \mu)\geqq\Phi(x, \nu)+t$ on A for a positive number t, then we have $V_i(A)\geqq t^2\|\mu-\nu\|^{-2}$, where $\|\mu-\nu\|=[(\mu, \mu)+(\nu, \nu)-2(\mu, \nu)]^{1/2}$.*

8) cf. [4], Lemma 13, p. 108 and [2], Lemma 4.3.2, p. 181.

9) [6], Lemma 3.4, p. 298.

LEMMA 4.[10] *Let $\mu\in\mathscr{E}$. Given $\varepsilon>0$, there exists a compact set K_ε such that the potential of μ_ε, the restriction of μ to K_ε, is finite and continuous in Ω and $\|\mu-\mu_\varepsilon\|<\varepsilon$.*

Like in [4] and [6], by using Lemmas 3 and 4 we can prove

LEMMA 5.[11] *The potential of a measure with finite energy is quasi continuous, i.e. for any $\varepsilon>0$, there is an open set G_ε such that $V_i(G_\varepsilon)>1/\varepsilon$ and the potential is finite and continuous as a function on $\Omega-G_\varepsilon$*

LEMMA 6. *Let G be an open set and ν be a measure with finite energy. Then $\Phi(x,\nu)\geqq 1$ n.e. on G implies $\Phi(x,\nu)\geqq 1$ q.e. on G.*

PROOF. It is enough to show that the e_0-value of the exceptional set $N=\{x\in G;\ \Phi(x,\nu)<1\}$ is zero. Given $\varepsilon>0$, by Lemma 5, there is an open set G_ε such that $e_i(G_\varepsilon)<\varepsilon$ and $\Phi(x,\nu)$ is finite and continuous as a function on $\Omega-G_\varepsilon$. Put $N_\varepsilon=\{x\in\Omega-G_\varepsilon;\ \Phi(x,\nu)<1\}$ and $B_\varepsilon=(N_\varepsilon\cup G_\varepsilon)\cap G$. Then $N\subset B_\varepsilon\subset N\cup G_\varepsilon$ and $N_\varepsilon\cup G_\varepsilon$ is open. Therefore, it follows that B_ε is open, $e_0(N-B_\varepsilon)=e_0(\phi)=0$ and $e_0(B_\varepsilon-N)\leqq e_i(G_\varepsilon)<\varepsilon$. Obviously $e_i(B_\varepsilon)=e_0(B_\varepsilon)$. We see by Lemma 2 that $e_0(N)=e_i(N)=0$.

REMARK. We need not condition $(*)$ in Lemmas 2–6.

THEOREM 7.[12] $e_0(A)=c^e(A)$ *for any set A.*

PROOF. For any open set $G\supset A$, it holds by Lemma 6 and Theorem 5 that $c^e(A)\leqq c^e(G)=c^i(G)=e_i(G)$. Thus we have $c^e(A)\leqq e_0(A)$. The converse inequality was shown in Theorem 6.

Next, we shall give

LEMMA 7.[13] *Let F be a closed set with $e_0(F)<\infty$. Then we have $e_i(F)=e_0(F)$.*

PROOF. We may suppose $e_0(F)>0$. On account of Lemma 2 and Proposition 3 it is enough to show that, for any $\varepsilon>0$, we can find a compact set $F_\varepsilon\subset F$ such that $e_0(F-F_\varepsilon)<\varepsilon$. Take an open set G such that $G\supset F$ and $e_i(G)<\infty$. We can find, by Theorem 5, a measure μ such that $\mu(G)=(\mu,\mu)=e_i(G)>0$ and $\Phi(x,\mu)\geqq 1$ n.e. on G. By Lemma 6 $\Phi(x,\mu)\geqq 1$ q.e. on G. Put $N=\{x\in G;\ \Phi(x,\mu)<1\}$ and $H=\{x\in\Omega;\ \Phi(x,\mu)\geqq 1\}$. Then $H\cup N\supset F$ and $e_0(N)=e_i(N)=0$. Given $\varepsilon>0$, there is a compact set K_ε such that $(\nu_\varepsilon,\nu_\varepsilon)<\varepsilon/4$, where $\nu_\varepsilon=\mu-\mu_\varepsilon$ and μ_ε is the restriction of μ to K_ε. On account of condition $(*)$, there exists a compact set L_ε such that $\Phi(x,y)\leqq 1/[2\mu(\Omega)]$ on $(\Omega-L_\varepsilon)\times K_\varepsilon$. Consequently $\Phi(x,\mu_\varepsilon)\leqq 1/2$ for any $x\in\Omega-L_\varepsilon$ and $H\cap(\Omega-L_\varepsilon)\subset A_\varepsilon=\{x\in\Omega;\ \Phi(x,\nu_\varepsilon)\geqq 1/2\}$. Since $2\nu_\varepsilon$ belongs to $\Gamma^e_{A_\varepsilon}$, we have

10) [6], Lemma 1.4, p. 190.

11) cf. [4], Lemma 10, p, 150 and [6], Theorem 1.13, p. 206.

12) cf. [4], Lemma 13, p. 108.

13) [2], Lemma 4.22, p. 179 and [3], Corollaire du Lemme 7.1, p. 81.

$$e_0(H-L_\varepsilon)\leqq e_0(A_\varepsilon)=c^e(A_\varepsilon)\leqq 4(\nu_\varepsilon,\nu_\varepsilon)<\varepsilon.$$

Take $F_\varepsilon=F\cap L_\varepsilon$. Then F_ε is compact and $F-F_\varepsilon\subset(H-L_\varepsilon)\cup N$. It follows that $e_0(F-F_\varepsilon)\leqq e_0(H-L_\varepsilon)+e_0(N)<\varepsilon$.

We use this fact in the following theorem.

THEOREM 8.[14] *Suppose that a sequence $\{\mu_n\}$ of measures in $\mathscr{E}$ converges vaguley to μ_0 and that the total masses $\mu_n(\Omega)$ are bounded. Then we have $\varliminf_{n\to\infty}\Phi(x,\mu_n)\leqq\Phi(x,\mu_0)$ q.e. in Ω.*

PROOF. If we put $h_n(x)=\inf\{\Phi(x,\mu_k);\,k\geqq n\}$, then $h_n(x)$ increases to $\varliminf_{n\to\infty}\Phi(x,\mu_n)$. Given $\varepsilon>0$, we can find, by Lemma 5, an open set G_ε such that $e_i(G_\varepsilon)<\varepsilon$ and the restriction of $\Phi(x,\mu_n)$ to $\Omega-G_\varepsilon$ is finite and continuous for each n $(n=1,2,\dots)$. For $t>0$, we put

$$E_n(t)=\{x\in\Omega;\,h_n(x)-\Phi(x,\mu_0)\geqq t\},$$

$$E_n(\varepsilon,t)=\{x\in\Omega-G_\varepsilon;\,h_n(x)-\Phi(x,\mu_0)\geqq t\}$$

and

$$A_n(t)=\{x\in\Omega;\,\Phi(x,\mu_n)\geqq t\}.$$

Then since the restriction of $h_n(x)$ to $\Omega-G_\varepsilon$ is upper semicontinuous, $E_n(\varepsilon,t)$ is a closed set in Ω. We observe that $E_n(\varepsilon,t)\subset A_n(t)$ and $e_0(E_n(\varepsilon,t))\leqq e_0(A_n(t))=c^e(A_n(t))\leqq(\mu_n,\mu_n)/t^2<\infty$. Consequently, by Lemma 7, $e_i(E_n(\varepsilon,t))=e_0(E_n(\varepsilon,t))$. If $e_i(E_n(\varepsilon,t))$ were positive, we could find a unit measure ν such that $S\nu$ is compact, $S\nu\subset E_n(\varepsilon,t)$ and $\Phi(x,\nu)$ is finite and continuous in Ω on account of the continuity principle. It would follow that

$$\begin{aligned} t&\leqq\int[h_n(x)-\Phi(x,\mu_0)]d\nu(x)=\int h_n(x)\,d\nu(x)-\int\Phi(x,\mu_0)d\nu(x)\\ &\leqq\int\Phi(x,\mu_k)d\nu(x)-\int\Phi(x,\mu_0)d\nu(x)\\ &=\int\Phi(x,\nu)d\mu_k(x)-\int\Phi(x,\nu)d\mu_0(x)\qquad\qquad(k\geqq n),\end{aligned}$$

and the right side tends to 0 as $k\to\infty$. This is a contradiction. Therefore $e_0(E_n(\varepsilon,t))=e_i(E_n(\varepsilon,t))=0$. It follows that

14) Ohtsuka [6] proved Theorem 8 in case Ω is compact. Kishi [4] and Fuglede [3] obtained similar results. However the latter two authors used the terminology "q.e." in a sense different from ours. Namely, set $V^*(\mu)=\sup\{\Phi(x,\mu);\,x\in\Omega\}$, $V_i^*(A)=\inf\{V^*(\mu);\,\mu\in\mathscr{U}_A\}$ if $A\neq\phi$, $V_i^*(\phi)=\infty$ and $V_e^*(A)=\sup\{V_i^*(G);\,G$ is open and $G\supset A\}$. Obviously $V_i(A)\leqq V_i^*(A)$ and $V_e(A)\leqq V_e^*(A)$. Fuglede said that a property holds n.e. (q.e. resp.) on A if the exceptional set in A has infinite V_i^*-value (V_e^*-value resp.).

$$0 \leqq e_0(E_n(t)) \leqq e_0(E_n(\varepsilon, t) \cup G_\varepsilon)$$
$$\leqq e_0(E_n(\varepsilon, t)) + e_0(G_\varepsilon) < \varepsilon.$$

Thus $e_0(E_n(t))=0$. By the relation

$$N = \{x \in \Omega;\ \overline{\lim_{n\to\infty}} \Phi(x, \mu_n) - \Phi(x, \mu_0) > 0\} = \bigcup_k \bigcup_n E_n(1/k),$$

we see $e_0(N)=0$. Namely $\overline{\lim_{n\to\infty}} \Phi(x, \mu_n) \leqq \Phi(x, \mu_0)$ q.e. in Ω.

Now we shall prove

THEOREM 9.[15] *If $e_0(A)$ is finite, then there is a measure μ_0 such that $\Phi(x, \mu_0) \geqq 1$ q.e. on A, $\Phi(x, \mu_0) \leqq 1$ on $S\mu_0$ and $\mu_0(\Omega)=(\mu_0, \mu_0)=e_0(A)$.*

PROOF. We can find a sequence $\{G_n\}$ of open sets such that $e_i(G_n)$ is finite, $\lim_{n\to\infty} e_i(G_n)=e_0(A)$ and $G_n \supset G_{n+1} \supset A$. For each G_n, combining Theorem 5 with Lemma 6, we can find a measure μ_n such that $S\mu_n \subset \bar{G}_n$, $\Phi(x, \mu_n) \geqq 1$ q.e. on $G_n \supset A$, $\Phi(x, \mu_n) \leqq 1$ on $S\mu_n$ and $\mu_n(\Omega)=(\mu_n, \mu_n)=e_i(G_n)$. Since the total masses $\mu_n(\Omega)$ are bounded, there is a subsequence $\{\mu_{n_j}\}$ which converges vaguely to some measure μ_0. On account of Theorem 8, we see $\Phi(x, \mu_0) \geqq 1$ q.e. on A and $(\mu_0, \mu_0) \leqq \underline{\lim}_{j\to\infty} (\mu_{n_j}, \mu_{n_j}) = e_0(A)$. Consequently μ_0 belongs to Γ_A^e and hence $(\mu_0, \mu_0) \geqq c^e(A)=e_0(A)$ (Theorem 7). Thus $(\mu_0, \mu_0)=e_0(A)$. The rest of the proof is carried out in the same way as that of Theorem 5.

THEOREM 10. *Let $\{A_n\}$ be an increasing sequence of arbitrary sets and $A=\bigcup_{n=1}^{\infty} A_n$. Then we have*

$$e_0(A) = \lim_{n\to\infty} e_0(A_n).$$

PROOF. Since $e_0(A_n) \leqq e_0(A_{n+1}) \leqq e_0(A)$, it holds that $\lim_{n\to\infty} e_0(A_n) \leqq e_0(A)$. It is enough to prove the converse inequality in case $\lim_{n\to\infty} e_0(A_n)$ is finite. For each n, we can find, by Theorem 9, a measure μ_n such that $\mu_n(\Omega)=(\mu_n, \mu_n)= e_0(A_n) \leqq \lim_{n\to\infty} e_0(A_n) < \infty$ and $\Phi(x, \mu_n) \geqq 1$ q.e. on A_n. We may suppose that $\{\mu_n\}$ converges vaguely to a measure μ_0 by selecting a subsequence if necessary. Making use of Theorem 8, we see that μ_0 belongs to Γ_A^e and hence $e_0(A)= c^e(A) \leqq (\mu_0, \mu_0) \leqq \lim_{n\to\infty} (\mu_n, \mu_n) = \lim_{n\to\infty} e_0(A_n)$. This completes the proof.

6. Because of Corollary 1 to Theorem 1 and Theorem 10 we can apply Choquet's theorem (Théorème 30.1 in [1]). Thus we have

THEOREM 11. *It holds that $e_i(A)=e_0(A)$, or equivalently $V_i(A)=V_e(A)$, for every analytic set A.*

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15) cf. [2], Theorem 4.3, p. 182.

7. We shall show that our Theorem 11 is a generalization of Kishi's theorem (Theorem 13 in [4]). Kishi postulated further that the kernel Φ is strictly positive and regular, i.e. for any point x_0 and any neighborhood δ_{x_0} of x_0, there is a positive constant t depending only on x_0, and a unit measure μ such that $\mu\in\mathscr{E}$, $S\mu\subset\delta_{x_0}$, $\Phi(x, \mu)\leqq t\Phi(x, x_0)$ in Ω.

We shall give a kernel which satisfies all our conditions except for this regularity.

EXAMPLE. Let Ω be the 3-dimensional Euclidean space, $f(x)=\inf(|x|, |x|^{-1/2})$ and $\Phi(x, y)=\dfrac{1}{|x-y|}+f(x)f(y)$. Then Φ satisfies both the continuity principle and condition (*) and is of positive type. We observe that Φ is not regular in the above sense.

In fact, if we take $x_0=0=$the origin, then $\Phi(x, x_0)=1/|x|$. For any nonzero measure $\mu\in\mathscr{E}$, $\alpha_\mu=\int f(x)d\mu(x)>0$. If there were a positive constant t, depending only on x_0, such that

$$\Phi(x, \mu)\leqq t\Phi(x, x_0) \qquad \text{in } \Omega,$$

then we should have

$$\int\frac{1}{|x-y|}\,d\mu(y)+\alpha_\mu f(x)\leqq\frac{t}{|x|} \qquad\qquad \text{in } \Omega.$$

Namely $\alpha_\mu|x|f(x)\leqq t<\infty$ in Ω. This is impossible because $|x|f(x)\to\infty$ as $|x|\to\infty$.

8. The quantities $e_i(A)$ and $e_0(A)$ were also studied by Fuglede [2] and Ohtsuka [6] under different conditions. We shall explain the differences. If Φ satisfies the continuity principle and condition (*) in case Ω is not compact, we denote $\Phi\in[K]$[16].

If the kernel is of positive type, the pseudo-norm $\|\mu-\nu\|=[(\mu, \mu)+(\nu, \nu)-2(\nu, \mu)]^{1/2}$ defines the strong topology on $\mathscr{E}$. Fuglede called a kernel consistent if it is of positive type and any strong Cauchy net converging vaguely to a measure converges strongly to the same measure. If Φ is consistent, we denote $\Phi\in[F]$[16].

If $\mathscr{E}$ is strongly complete (i.e. complete with respect to the strong topology), we denote $\Phi\in[O]$[16].

The range within which their theory is available will be illustrated by the following examples. For simplicity, we assume in what follows except in Example 3 that Ω is the 3-dimensional Euclidean space.

EXAMPLE 1. $\Phi\notin[K]\cup[F]\cup[O]$.

16) The symbols K, F, O are after Kishi, Fuglede, Ohtsuka respectively.

Let $f(x)$ be the characteristic function of $\{|x|<1\}$ and $\Phi(x, y)=\frac{1}{|x-y|}+f(x)f(y)$. Fuglede observed $\Phi\notin[F]\cup[O]$. Since this kernel does not satisfy the continity principle, $\Phi\notin[K]$.

EXAMPLE 2. $\Phi\in[K]$ and $\Phi\notin[F]\cup[O]$.
Let $f(x)=\inf(1, |x|^{-1/4})$ and $\Phi(x, y)=\frac{1}{|x-y|}+f(x)f(y)$. It is clear that $\Phi\in[K]$. In order to show that $\Phi\notin[F]\cup[O]$, it is sufficient to prove that our kernel is not consistent. Let μ_n be the uniform measure on $\{|x|=n\}$ with $\mu_n(\Omega)=n^{1/4}$. Then $\|\mu_n-\mu_m\|^2\leqq(n^{-1/4}+m^{-1/4})^2$. Therefore $\{\mu_n\}$ is a strong Cauchy sequence. It converges vaguely to 0. However $\|\mu_n\|^2=n^{-1/4}+1$ does not approach 0. Namely, Φ is not consistent.

EXAMPLE 3. (Fuglede [2], p. 210) $\Phi\in[F]\cap[K]$ and $\Phi\notin[O]$.
Let Ω be the interval [0, 1] in the real line and $\Phi(x, y)=xy/(2-xy)$. Fuglede proved that $\Phi\in[F]$ and $\Phi\notin[O]$. It is obvious that $\Phi\in[K]$.

EXAMPLE 4. $\Phi\in[O]$ and $\Phi\notin[K]\cup[F]$.
Let $\Phi(x, y)\equiv1$. Then our assertion is easily verified.

EXAMPLE 5. $\Phi\notin[K]$ and $\Phi\in[F]\cap[O]$.
See Ohtsuka's example (Example 2 in [5]).

EXAMPLE 6. $\Phi\notin[F]$ and $\Phi\in[K]\cap[O]$.
Let $f(x)=\inf(1, 1/|x|)$ and $\Phi(x, y)=f(x)f(y)$.

Fuglede proved that $\mathscr{E}$ is strongly complete if Φ is consistent and $\Phi>0$ (Lemma 3.3.1 in [2], p. 167). Therefore we have $[F]\subset[O]$ if $\Phi>0$.

In case Ω is a compact space, we see $[K]\subset[F]$. The author does not know a kernel such that $\Phi\in[F]$ and $\Phi\notin[K]\cup[O]$.

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