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Two-step Processes by One-step Methods of Order 3 and of Order 4

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1. Introduction

Given a differential equation

$$(1.1) \qquad y' = f(x, y)$$

and the initial condition $y(x_0) = y_0$, where $f(x, y)$ is assumed to be a sufficiently smooth function. We are concerned with the case where the equation (1.1) is integrated numerically by one-step methods of order 3 and of order 4.

It is well known that the one-step methods of order 3 such as Kutta method and those of order 4 such as Runge-Kutta method require three and four evaluations of the derivative respectively. It is also known that, if the same step-size is used twice in succession, an approximate value of the truncation error can be obtained by integrating again with the double step-size. This method of approximating the truncation error requires, per two steps of integration, eight and eleven evaluations of $f(x, y)$ for any one-step method of order 3 and for that of order 4 respectively.

In our previous paper [19][1], it has been shown that there exists a one-step formula of order 4 such that, after two steps of integration with the same step-size, only one additional evaluation of the derivative makes it possible to approximate the truncation error. In that formula, however, four values of $f(x, y)$ evaluated in the first step of integration are not used explicitly in the second step. Thus there remains a possibility of reducing the number of evaluations of the derivative by utilizing all the values of the derivative computed already.

In this paper, it is shown that there exist one-step integration formulas of order 3 and those of of order 4 such that approximate values z_1 and z_2 of $y(x_0+h)$ and $y(x_0+2h)$ and an approximation to their truncation errors can be obtained with five and seven evaluations of $f(x, y)$ respectively. Finally two numerical examples are presented.

1) Numbers in square brackets refer to the references listed at the end of this paper.

2. Preliminaries

Let

(2.1) $$z=y_0+h\sum_{i=1}^{r}p_ik_i \qquad (r\geqq 3),$$

where

(2.2) $$k_1=f(x_0,\ y_0),$$

$$k_i=f(x_0+a_ih,\ y_0+h\sum_{j=1}^{i-1}b_{ij}k_j) \qquad (i=2,\ 3,\ \cdots,\ r),$$

(2.3) $$\sum_{j=1}^{i-1}b_{ij}=a_i,$$

Put, for simplicity [14],

(2.4) $$c_i=\sum_{j=2}^{i+1}a_jb_{i+2\,j},\quad d_i=\sum_{j=2}^{i+1}a_j^2b_{i+2\,j},\quad e_i=\sum_{j=2}^{i+1}a_j^3b_{i+2\,j},\quad (i=1,\ 2,\ \cdots,\ r-2),$$

(2.5) $$l_i=\sum_{j=1}^{i-3}c_jb_{i\,j+2},\quad m_i=\sum_{j=1}^{i-3}d_jb_{i\,j+2} \qquad (i=4,\ 5,\ \cdots,\ r),$$

(2.6) $$x_j=x_0+jh \qquad (j=1,\ 2).$$

Let D be a differential operator defined by the formula [16]

(2.7) $$D=\frac{\partial}{\partial x}+k_1\frac{\partial}{\partial y}$$

and put

(2.8) $$D^jf(x_0,\ y_0)=T^j,\quad D^jf_y(x_0,\ y_0)=S^j \qquad (j=1,\ 2,\ \cdots,\ r-2),$$

(2.9) $$(Df)^2(x_0,\ y_0)=P,\quad (Df_y)^2(x_0,\ y_0)=Q,\quad Df_{yy}(x_0,\ y_0)=R,$$

(2.10) $$f_y(x_0,\ y_0)=f_y,\quad f_{yy}(x_0,\ y_0)=f_{yy}.$$

Then $y(x_0+h)$ and z can be expanded into power series in h as follows:

(2.11) $$y(x_0+h)=y_0+hk_1+\frac{1}{2!}h^2T+\frac{1}{3!}h^3(T^2+f_yT)+\frac{1}{4!}h^4(T^3+3TS+$$

$$+f_yT^2+f_y^2T)+\frac{1}{5!}h^5(T^4+6TS^2+4T^2S+3f_{yy}P+f_yT^3+$$

$$+f_y^2T^2+f_y^3T+7f_yTS)+\frac{1}{6!}h^6(T^5+10TS^3+10T^2S^2+$$

$$+5T^3S+10f_{yy}TT^2+15PR+15TQ+13f_yf_{yy}P+16f_yTS^2+$$

$$+9f_yT^2S+12f_y^2TS+f_yT^4+f_y^2T^3+f_y^3T^2+f_y^4T)+O(h^7),$$

(2.12)
$$z=y_0+hA_1k_1+h^2A_2T+\frac{1}{2!}h^3(A_3T^2+2A_4f_yT)+\frac{1}{3!}h^4(B_1T^3+$$
$$+6B_2TS+3B_3f_yT^2+6B_4f_y^2T)+\frac{1}{4!}h^5(C_1T^4+12C_2TS^2+$$
$$+12C_3T^2S+12C_4f_{yy}P+4C_5f_yT^3+12C_6f_y^2T^2+24C_7f_y^3T+$$
$$+24C_8f_yTS)+\frac{1}{5!}h^6(D_1T^5+20D_2TS^3+30D_3T^2S^2+20D_4T^3S+$$
$$+60D_5f_{yy}TT^2+60D_6PR+120D_7TQ+60D_8f_yf_{yy}P+60D_9f_yTS^2+$$
$$+60D_{10}f_yT^2S+60D_{11}f_y^2TS+5D_{12}f_yT^4+20D_{13}f_y^2T^3+$$
$$+60D_{14}f_y^3T^2+120D_{15}f_y^4T)+O(h^7),$$

where

(2.13)
$$A_1=\sum_{i=1}^{r}p_i,\quad A_2=\sum_{i=2}^{r}a_ip_i,$$

(2.14)
$$A_3=\sum_{i=2}^{r}a_i^2p_i,\quad B_1=\sum_{i=2}^{r}a_i^3p_i,\quad C_1=\sum_{i=2}^{r}a_i^4p_i,\quad D_1=\sum_{i=2}^{r}a_i^5p_i,$$

(2.15)
$$A_4=\sum_{i=3}^{r}c_{i-2}p_i,\quad B_2=\sum_{i=3}^{r}a_ic_{i-2}p_i,\quad B_3=\sum_{i=3}^{r}d_{i-2}p_i,\quad C_2=\sum_{i=3}^{r}a_i^2c_{i-2}p_i,$$
$$C_3=\sum_{i=3}^{r}a_id_{i-2}p_i,\quad C_4=\sum_{i=3}^{r}c_{i-2}^2p_i,\quad D_2=\sum_{i=3}^{r}a_i^3c_{i-2}p_i,$$
$$D_3=\sum_{i=3}^{r}a_i^2d_{i-2}p_i,\quad D_5=\sum_{i=3}^{r}c_{i-2}d_{i-2}p_i,\quad D_6=\sum_{i=3}^{r}a_ic_{i-2}^2p_i,$$

(2.16)
$$B_4=\sum_{i=4}^{r}l_ip_i,\quad C_5=\sum_{i=3}^{r}e_{i-2}p_i,\quad C_6=\sum_{i=4}^{r}m_ip_i,\quad C_7=\sum_{i=5}^{r}(\sum_{j=4}^{i-1}l_jb_{ij})p_i,$$
$$C_8=\sum_{i=4}^{r}(a_il_i+\sum_{j=1}^{i-3}a_{j+2}c_jb_{i\,j+2})p_i,\quad D_4=\sum_{i=3}^{r}a_ie_{i-2}p_i,$$
$$D_7=\sum_{i=4}^{r}a_i(\sum_{j=1}^{i-3}a_{j+2}c_jb_{i\,j+2})p_i,$$

(2.17)
$$D_8=\sum_{i=4}^{r}(2c_{i-2}l_i+\sum_{j=1}^{i-3}c_j^2b_{i\,j+2})p_i,\quad D_9=\sum_{i=4}^{r}(a_i^2l_i+\sum_{j=1}^{i-3}a_{j+2}d_jb_{i\,j+2})p_i,$$
$$D_{10}=\sum_{i=4}^{r}(a_im_i+\sum_{j=1}^{i-3}a_{j+2}d_jb_{i\,j+2})p_i,$$
$$D_{11}=\sum_{i=5}^{r}[\sum_{j=4}^{i-1}(a_il_j+a_jl_j+\sum_{k=1}^{j-3}a_{k+2}c_kb_{j\,k+2})b_{ij}]p_i,$$
$$D_{12}=\sum_{i=3}^{r}(\sum_{j=2}^{i-1}a_j^4b_{ij})p_i,\quad D_{13}=\sum_{i=4}^{r}(\sum_{j=1}^{i-3}e_jb_{i\,j+2})p_i,$$
$$D_{14}=\sum_{i=5}^{r}(\sum_{j=4}^{i-1}m_jb_{ij})p_i,\quad D_{15}=\sum_{i=6}^{r}[\sum_{j=5}^{i-1}(\sum_{k=4}^{j-1}l_kb_{jk})b_{ij}]p_i.$$

Now we impose the condition that

$$p_2=0,\quad c_{i-2}=\frac{1}{2}a_i^2,\quad d_{i-2}=\frac{1}{3}a_i^3 \qquad (i=3,4,\cdots,r). \tag{2.18}$$

Then it follows that

$$A_4=\frac{1}{2}A_3,\quad B_2=\frac{1}{2}B_1,\quad B_3=\frac{1}{3}B_1,\quad C_2=\frac{1}{2}C_1,\quad C_3=\frac{1}{3}C_1, \tag{2.19}$$

$$C_4=\frac{1}{4}C_1,\quad D_2=\frac{1}{2}D_1,\quad D_3=\frac{1}{3}D_1,\quad D_5=\frac{1}{6}D_1,\quad D_6=\frac{1}{4}D_1,$$

$$a_2=\frac{2}{3}a_3 \tag{2.20}$$

$$a_3^2b_{i3}+3\sum_{j=4}^{i-1}a_j(a_j-a_2)b_{ij}=a_i^2(a_i-a_3) \qquad (i=4,5,\cdots,r), \tag{2.21}$$

$$z=y_0+hA_1k_1+h^2A_2T+\frac{1}{2!}h^3A_3(T^2+f_yT)+\frac{1}{3!}h^4[B_1(T^3+3TS+ \tag{2.22}$$

$$+f_yT^2)+6B_4f_y^2T]+\frac{1}{4!}h^5[C_1(T^4+6TS^2+4T^2S+3f_{yy}P)+$$

$$+4C_5f_yT^3+12C_6f_y^2T^2+24C_7f_y^3T+24C_8f_yTS]+\frac{1}{5!}h^6[D_1(T^5+$$

$$+10TS^3+10T^2S^2+10f_{yy}TT^2+15PR)+20D_4T^3S+$$

$$+120D_7TQ+\cdots]+O(h^7),$$

and the equations in (2.14) and (2.16) can be rewritten as follows:

$$\sum_{i=4}^{r}a_i(a_i-a_3)p_i=A_3-a_3A_2, \tag{2.23}$$

$$\sum_{i=5}^{r}a_i(a_i-a_3)(a_i-a_4)p_i=B_1-(a_3+a_4)A_3+a_3a_4A_2, \tag{2.24}$$

$$\sum_{i=6}^{r}a_i(a_i-a_3)(a_i-a_4)(a_i-a_5)p_i=C_1-(a_3+a_4+a_5)B_1+(a_3a_4+a_3a_5+ \tag{2.25}$$

$$+a_4a_5)A_3-a_3a_4a_5A_2,$$

$$\sum_{i=7}^{r}a_i(a_i-a_3)(a_i-a_4)(a_i-a_5)(a_i-a_6)p_i=D_1-(a_3+a_4+a_5+a_6)C_1+ \tag{2.26}$$

$$+(a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6)B_1-(a_3a_4a_5+a_3a_4a_6+$$

$$+a_3a_5a_6+a_4a_5a_6)A_3+a_3a_4a_5a_6A_2,$$

$$\sum_{i=5}^{r}\Big(\sum_{j=4}^{i-1}g_{ij}\Big)p_i=\frac{1}{2}B_1-B_4-\frac{1}{2}a_3A_3, \tag{2.27}$$

(2.28) $$\sum_{i=5}^{r} a_i\Big(\sum_{j=4}^{i-1} g_{ij}\Big)p_i = \frac{3}{2}C_6 - C_8 - \frac{1}{2}(C_1 - a_3B_1),$$

(2.29) $$\sum_{i=5}^{r}\Big(\sum_{j=4}^{i-1} a_j g_{ij}\Big)p_i = 3C_6 - 2a_3B_4,$$

(2.30) $$\sum_{i=6}^{r}\Big[\sum_{j=5}^{i-1}\Big(\sum_{k=4}^{j-1} g_{jk}\Big)b_{ij}\Big]p_i = \frac{3}{2}C_6 - C_7 - a_3B_4,$$

(2.31) $$\sum_{i=5}^{r} a_i\Big(\sum_{j=4}^{i-1} a_j g_{ij}\Big)p_i = D_4 - \frac{2}{9}C_1 - \frac{1}{3}(2C_8 - 3C_6),$$

(2.32) $$3C_6 - C_5 = \frac{2}{9}a_3(6B_4 - B_1),$$

(2.33) $$D_4 - 2D_7 = \frac{2}{9}a_3(C_1 + 9C_6 - 6C_8),$$

where

(2.34) $$g_{ij} = a_j(a_j - a_3)b_{ij}.$$

In the sequel, we assume the condition (2.18).

3. Existence of formulas

3.1 *Formulas of order* 4

To obtain an integration formula of order 4 for $r=4$, we require that

(3.1) $$z_1 - y(x_1) = O(h^5), \quad z_1 = y_0 + h\sum_{i=1}^{4} r_i k_i.$$

Then, since

(3.2) $$A_1 = 1, \quad A_2 = \frac{1}{2}, \quad A_3 = \frac{1}{3}, \quad B_1 = \frac{1}{4}, \quad B_4 = \frac{1}{24},$$

from (2.27), (2.24), (2.23), (2.20), (2.13) and (2.18), it follows that

(3.3) $$a_2 = \frac{1}{3}, \quad a_3 = \frac{1}{2}, \quad a_4 = 1, \quad b_{31} = \frac{1}{8}, \quad b_{32} = \frac{3}{8},$$
$$b_{41} = \frac{1}{2}, \quad b_{42} = -\frac{3}{2}, \quad b_{43} = 2,$$

(3.4) $$r_1 = r_4 = \frac{1}{6}, \quad r_2 = 0, \quad r_3 = \frac{4}{6},$$

and we have

(3.5) $$z_1-y(x_1)=\frac{1}{5!}h^5\left[\frac{1}{24}T^4+\frac{1}{4}TS^2+\frac{1}{6}T^2S+\frac{1}{8}f_{yy}P-\right.$$
$$\left.-\frac{1}{6}f_yT^3-\frac{1}{6}f_y^2T^2-f_y^3T+\frac{1}{2}f_yTS\right]+\frac{1}{6!}h^6\left[\frac{1}{8}T^5+\frac{5}{4}TS^3+\right.$$
$$+\frac{5}{4}T^2S^2-\frac{5}{9}T^3S+\frac{5}{4}f_{yy}TT^2+\frac{15}{8}PR+\frac{21}{8}f_yf_{yy}P+$$
$$+\frac{11}{4}f_yTS^2-\frac{3}{2}f_yT^2S-12f_y^2TS-\frac{3}{8}f_yT^4-\frac{4}{9}f_y^2T^3-$$
$$\left.-f_y^3T^2-f_y^4T\right]+O(h^7).$$

Next, under the condition (3.3), we require that

(3.6) $$z_1^*-y(x_1)=O(h^6),\quad z_1^*=y_0+h\sum_{i=1}^{7}q_ik_i,$$

(3.7) $$z_2^*-y(x_2)=O(h^6),\quad D_1=\frac{32}{3},\quad z_2^*=y_0+h\sum_{i=1}^{7}p_ik_i.$$

Then the following equations must be satisfied:

(3.8) $$g_5p_5+g_6p_6+g_7p_7=\frac{2}{3},$$

(3.9) $$s_5p_5+s_6p_6+s_7p_7=1,$$

(3.10) $$s_5b_{65}p_6+(s_5b_{75}+s_6b_{76})p_7=\frac{4}{15},$$

(3.11) $$g_5b_{65}p_6+(g_5b_{75}+g_6b_{76})p_7=\frac{1}{5},$$

(3.12) $$(a_6-a_5)g_6p_6+(a_7-a_5)g_7p_7=\frac{17}{15}-\frac{2}{3}a_5,$$

(3.13) $$(a_6-a_5)s_6p_6+(a_7-a_5)s_7p_7=\frac{26}{15}-a_5,$$

(3.14) $$(a_7-a_5)(a_7-a_6)s_7p_7=\frac{46}{15}-\frac{26}{15}(a_5+a_6)+a_5a_6,$$

(3.15) $$g_5q_5+g_6q_6+g_7q_7=0,$$

(3.16) $$s_5q_5+s_6q_6+s_7q_7=0,$$

(3.17) $$s_5b_{65}q_6+(s_5b_{75}+s_6b_{76})q_7=\frac{1}{120}$$

(3.18) $$g_5b_{65}q_6+(g_5b_{75}+g_6b_{76})q_7=-\frac{1}{240},$$

(3.19) $$(a_6-a_5)g_6q_6+(a_7-a_5)g_7q_7=\frac{1}{240},$$

(3.20) $$(a_6-a_5)s_6q_6+(a_7-a_5)s_7q_7=-\frac{1}{120},$$

where

(3.21) $$s_j=a_j(a_j-a_3)(a_j-a_4),\quad g_j=\sum_{k=4}^{j-1}g_{jk}\qquad (j=5,\ 6,\ 7).$$

As is easily checked, these equations have a solution

(3.22) $$a_5=\frac{3}{2},\ a_6=2,\ a_7=1,\ b_{54}=\frac{7}{4},\ b_{64}=-2,\ b_{65}=\frac{4}{3},$$
$$b_{74}=-\frac{2}{5}t,\ b_{75}=-\frac{4}{135}t,\ b_{76}=\frac{2}{45}t,\ p_5=\frac{32}{45},\ p_6=\frac{7}{45},$$
$$p_7=\frac{1}{t},\ q_5=\frac{1}{45},\ q_6=-\frac{1}{180},\ q_7=\frac{1}{8t}.$$

For these values, other constants are determined as follows:

(3.23) $$b_{51}=-\frac{7}{8},\ b_{52}=\frac{45}{8},\ b_{53}=-5,\ b_{61}=\frac{8}{3},\ b_{62}=-12,\ b_{63}=12,$$
$$b_{71}=\frac{1}{2}+\frac{46}{135}t,\ b_{72}=-\frac{3}{2}-2t,\ b_{73}=2+\frac{92}{45}t,\ p_1=\frac{7}{45},$$
$$p_2=0,\ p_3=\frac{32}{45},\ p_4=\frac{12}{45}-\frac{1}{t},\ q_1=\frac{29}{180},\ q_2=0,\ q_3=\frac{31}{45},$$
$$q_4=\frac{2}{15}-\frac{1}{8t},$$

and we have

(3.25) $$z_1^*-y(x_1)=\frac{1}{6!}h^6\Big[-\frac{1}{8}T^5-\frac{5}{4}TS^3-\frac{5}{4}T^2S^2-\frac{1}{6}T^3S$$
$$-\frac{5}{4}f_{yy}TT^2-\frac{15}{8}PR-\frac{1}{2}TQ-\frac{21}{8}f_yf_{yy}P-\frac{3}{4}f_yTS^2+\frac{7}{3}f_yT^2S+$$
$$+\frac{44}{3}f_y^2TS+\frac{5}{8}f_yT^4+\frac{1}{6}f_y^2T^3+\frac{1}{6}f_y^3T^2-\frac{31}{6}f_y^4T\Big]+O(h^7),$$

(3.25) $$z_2^*-y(x_2)=\frac{1}{6!}h^6\Big[-\frac{16}{3}T^3S-16TQ+22f_yf_{yy}P+28f_yTS^2+$$
$$+\frac{8}{3}f_yT^2S-\frac{80}{3}f_y^2TS+2f_yT^4-\frac{8}{3}f_y^2T^3-\frac{8}{3}f_y^3T^2-\frac{40}{3}f_y^4T\Big]+$$
$$+O(h^7).$$

Now put

$$m=z_1-z_1^*,\quad z_2=z_2^*+m. \tag{3.26}$$

Then, from (3.5), (3.24) and (3.25), it follows that

$$\begin{aligned}m=&\frac{1}{5!}h^5\Big[\frac{1}{24}T^4+\frac{1}{4}TS^2+\frac{1}{6}T^2S+\frac{1}{8}f_{yy}P-\frac{1}{6}f_yT^3-\\&-\frac{1}{6}f_y^2T^2-f_y^3T+\frac{1}{2}f_yTS\Big]+\frac{1}{6!}h^6\Big[\frac{1}{4}T^5+\frac{5}{2}TS^3+\\&+\frac{5}{2}T^2S^2-\frac{7}{18}T^3S+\frac{5}{2}f_{yy}TT^2+\frac{15}{4}PR+\frac{1}{2}TQ+\frac{21}{4}f_yf_{yy}P-\\&-\frac{7}{2}f_yTS^2-\frac{23}{6}f_yT^2S-\frac{80}{3}f_y^2TS-f_yT^4-\frac{11}{18}f_y^2T^3-\frac{5}{6}f_y^3T^2+\\&+\frac{25}{6}f_y^4T\Big]+O(h^7),\end{aligned}\tag{3.27}$$

$$\begin{aligned}z_2-y(x_2)=&\frac{1}{5!}h^5\Big[\frac{1}{24}T^4+\frac{1}{4}TS^2+\frac{1}{6}T^2S+\frac{1}{8}f_{yy}P-\frac{1}{6}f_yT^3-\\&-\frac{1}{6}f_y^2T^2-f_y^3T+\frac{1}{2}f_yTS\Big]+\frac{1}{6!}h^6\Big[\frac{1}{4}T^5+\frac{5}{2}TS^3+\frac{5}{2}T^2S^2-\\&-\frac{103}{18}T^3S+\frac{5}{2}f_{yy}TT^2+\frac{15}{4}PR-\frac{31}{2}TQ+\frac{209}{4}f_yf_{yy}P+\frac{65}{2}f_yTS^2-\\&-\frac{7}{6}f_yT^2S-\frac{160}{3}f_y^2TS+f_yT^4-\frac{59}{18}f_y^2T^3-\frac{7}{2}f_y^3T^2-\frac{55}{6}f_v^4T\Big]+\\&+O(h^7).\end{aligned}\tag{3.28}$$

Thus the truncation errors of z_1 and z_2 are approximated by m, and z_1^* and z_2^* can also be used as integration formulas of order 5. Finally we choose $t=-8$ so that the coefficient of $\frac{1}{7!}h^7f_y^2f_{yy}P$ in $z_2^*-y(x_2)$ may be small in magnitude.

Summarizing the results, we have the following formulas:

$$k_1=f(x_0,\ y_0),\quad k_2=f\Big(x_0+\frac{h}{3},\ y_0+\frac{h}{3}k_1\Big), \tag{3.29}$$

$$k_3=f\Big(x_0+\frac{h}{2},\ y_0+\frac{h}{8}(k_1+3k_2)\Big),\quad k_4=f\Big(x_0+h,\ y_0+\frac{h}{2}(k_1-3k_2+4k_3)\Big),$$

$$k_5=f\Big(x_0+\frac{3}{2}h,\ y_0+h\Big(-\frac{7}{8}k_1+\frac{45}{8}k_2-5k_3+\frac{7}{4}k_4\Big)\Big),$$

$$k_6=f\left(x_0+2h,\ y_0+h\left(\frac{8}{3}k_1-12k_2+12k_3-2k_4+\frac{4}{3}k_5\right)\right),$$

$$p=8h\left(-\frac{46}{135}k_1+2k_2-\frac{92}{45}k_3+\frac{2}{5}k_4+\frac{4}{135}k_5-\frac{2}{45}k_6\right),$$

$$k_7=f\left(x_0+h,\ y_0+\frac{h}{2}(k_1-3k_2+4k_3)+p\right),$$

$$z_1=y_0+\frac{h}{6}(k_1+4k_3+k_4), \tag{3.30}$$

$$m=\frac{h}{180}(k_1-4k_3+6k_4-4k_5+k_6)+\frac{h}{64}(k_7-k_4),$$

$$z_2=y_0+\frac{h}{45}(7k_1+32k_3+12k_4+32k_5+7k_6)-\frac{h}{8}(k_7-k_4)+m.$$

3.2 *Formulas of order* 3

To obtain a one-step formula of order 3 for $r=3$, we require that

$$z_1-y(x_1)=O(h^4),\quad z_1=y_0+h\sum_{i=1}^{3}r_ik_i. \tag{3.31}$$

Then, from (2.23), (2.13) and (2.18), it follows that

$$a_2=\frac{4}{9},\quad a_3=\frac{2}{3},\quad b_{31}=\frac{1}{6},\quad b_{32}=\frac{1}{2}, \tag{3.32}$$

$$r_1=\frac{1}{4},\quad r_2=0,\quad r_3=\frac{3}{4}, \tag{3.33}$$

and we have

$$z_1-y(x_1)=\frac{1}{4!}h^4\left[-\frac{1}{9}T^3-\frac{1}{3}TS-\frac{1}{9}f_yT^2-f_y^2T\right]+ \tag{3.34}$$

$$+\frac{1}{5!}h^5\left[-\frac{7}{27}T^4-\frac{14}{9}TS^2-\frac{28}{27}T^2S-\frac{7}{9}f_{yy}P-\frac{83}{243}f_yT^3-\right.$$

$$\left.-f_y^2T^2-f_y^3T-7f_yTS\right]+O(h^6).$$

Next, under the condition (3.32), we require that

$$z_1^*-y(x_1)=O(h^5),\quad z_1^*=y_0+h\sum_{i=1}^{5}q_ik_i, \tag{3.35}$$

$$z_2^*-y(x_2)=O(h^5),\quad C_1=\frac{32}{5},\quad z_2^*=y_0+h\sum_{i=1}^{5}p_ik_i. \tag{3.36}$$

Then the following equations must be satisfied:

(3.37) $$a_4(a_4-a_3)p_4+a_5(a_5-a_3)p_5=\frac{4}{3},$$

(3.38) $$a_5(a_5-a_3)(a_5-a_4)p_5=\frac{20}{9}-\frac{4}{3}a_4,$$

(3.39) $$g_{54}p_5=\frac{4}{9},$$

(3.40) $$\frac{56}{15}-\frac{20}{9}(a_4+a_5)+\frac{4}{3}a_4a_5=0,$$

(3.41) $$a_4(a_4-a_3)q_4+a_5(a_5-a_3)q_5=0,$$

(3.42) $$a_5(a_5-a_3)(a_5-a_4)q_5=\frac{1}{36},$$

(3.43) $$g_{54}q_5=-\frac{1}{36}.$$

These equations have a unique solution

(3.44) $$a_4=2,\ a_5=\frac{8}{5},\ b_{54}=\frac{28}{125},\ p_4=\frac{14}{168},\ p_5=\frac{125}{168},$$
$$q_4=\frac{5}{192},\ q_5=-\frac{125}{2688}.$$

For these values, other constants are determined as follows:

(3.45) $$b_{51}=-\frac{20}{125},\ b_{52}=\frac{108}{125},\ b_{53}=\frac{84}{125},\ b_{41}=\frac{7}{2},\ b_{42}=-\frac{27}{2},$$
$$b_{43}=12,\ p_1=\frac{35}{168},\ p_2=0,\ p_3=\frac{162}{168},\ q_1=\frac{637}{2688},\ q_2=0,\ q_3=\frac{351}{448},$$

and we have

(3.46) $$m=\frac{1}{4!}h^4\Big[-\frac{1}{9}T^3-\frac{1}{3}TS-\frac{1}{9}f_yT^2-f_y^2T\Big]+\frac{1}{5!}h^5\Big[-\frac{16}{27}T^4-$$
$$-\frac{32}{9}TS^2-\frac{64}{27}T^2S-\frac{16}{9}f_{yy}P+\frac{160}{243}f_yT^3+\frac{10}{3}f_y^3T-$$
$$-\frac{34}{3}f_yTS\Big]+O(h^6),$$

(3.47) $$z_2-y(x_2)=\frac{1}{4!}h^4\Big[-\frac{1}{9}T^3-\frac{1}{3}TS-\frac{1}{9}f_yT^2-f_y^2T\Big]+$$
$$+\frac{1}{5!}h^5\Big[-\frac{16}{27}T^4-\frac{32}{9}TS^2-\frac{64}{27}T^2S-\frac{16}{9}f_{yy}P+$$

$$+\frac{1024}{243}f_yT^3+\frac{32}{9}f_y^2T^2+\frac{74}{3}f_y^3T-10f_yTS\Big]+O(h^6),$$

where

(3.48) $$m=z_1-z_1^*,\quad z_2=z_2^*+m.$$

Summarizing the results, we have the following formulas:

(3.49) $$k_1=f(x_0,\ y_0),\ k_2=f\Big(x_0+\frac{4}{9}h,\ y_0+\frac{4}{9}hk_1\Big),$$

$$k_3=f\Big(x_0+\frac{2}{3}h,\quad y_0+h\Big(\frac{1}{6}k_1+\frac{1}{2}k_2\Big)\Big),$$

$$k_4=f\Big(x_0+2h,\ y_0+h\Big(\frac{7}{2}k_1-\frac{27}{2}k_2+12k_3\Big)\Big),$$

$$k_5=f\Big(x_0+\frac{8}{5}h,\ y_0+\frac{4}{125}h(-5k_1+27k_2+21k_3+7k_4)\Big),$$

(3.50) $$z_1=y_0+\frac{h}{4}(k_1+3k_3),$$

$$m=\frac{5}{2688}h(7k_1-18k_3-14k_4+25k_5),$$

$$z_2=y_0+\frac{h}{168}(35k_1+162k_3+14k_4+125k_5)+m.$$

4. Numerical examples

To compare m with the truncation error, the following two problems are solved numerically by our formulas:

Example 1

(4.1) $$y'=2xy,\quad y(0)=1,$$

Example 2

(4.2) $$y'=12x^3-8y/x,\quad y(-1)=1.$$

The step-size is halved until the inequality

(4.3) $$|m|\leqq\varepsilon|z_2|\qquad(\varepsilon=0.5\times10^{-7})$$

is satisfied, and x_0 and y_0 are replaced by x_2 and z_2 respectively. Initially h is set equal to 0.05. Computation is carried out in the floating-point arithmetic with 39 bit mantissa and rounding is done by chopping.

The results are given in Tables 1 and 2. In these tables, T denotes the truncation error and u stands for the approximation of T by the usual method, namely

(4.4) $$u=\frac{h}{56}(k_1+6k_3^*-3k_3-k_4-3k_6),$$

where

(4.5) $$k_2=f\Big(x_0+\frac{4}{9}h,\ y_0+\frac{4}{9}hk_1\Big),\quad k_3=f\Big(x_0+\frac{2}{3}h,\ y_0+h\Big(\frac{1}{6}k_1+\frac{1}{2}k_2\Big)\Big),$$

$$z_1=y_0+\frac{h}{4}(k_1+3k_3),\quad k_4=f(x_1,\ z_1),$$

$$k_5=f\Big(x_1+\frac{4}{9}h,\ z_1+\frac{4}{9}hk_4\Big),\quad k_6=f\Big(x_1+\frac{2}{3}h,\ z_1+h\Big(\frac{1}{6}k_4+\frac{1}{2}k_4\Big)\Big),$$

$$k_2^*=f\Big(x_0+\frac{8}{9}h,\ y_0+\frac{8}{9}hk_1\Big),\quad k_3^*=f\Big(x_0+\frac{4}{3}h,\ y_0+h\Big(\frac{1}{3}k_1+k_2^*\Big)\Big).$$

Table 1. $y'=2xy$

x	order 3			order 4	
	m	u	T	m	T
0. 2	−2. 865-08	−2. 906-08	−2. 585-08	1. 619-09	−1. 543-09
0. 4	−4. 138-09	−4. 192-09	−3. 791-09	3. 020-09	−2. 434-09
0. 6	−1. 051-08	−1. 071-08	−9. 415-09	−3. 187-09	−1. 609-08
0. 8	−2. 762-08	−2. 830-08	−2. 424-08	−3. 833-08	−7. 529-08
1. 0	−7. 363-08	−7. 579-08	−6. 342-08	−6. 790-09	−9. 124-09
1. 2	−1. 994-07	−2. 062-07	−1. 680-07	−2. 543-08	−3. 296-08
1. 4	−3. 685-08	−3. 756-08	−3. 360-08	−8. 852-08	−1. 135-07
1. 6	−1. 058-07	−1. 080-07	−9. 540-08	−3. 013-07	−3. 874-07
1. 8	−3. 153-07	−3. 228-07	−2. 822-07	−1. 030-06	−1. 335-06
2. 0	−9. 826-07	−1. 008-06	−8. 693-07	−1. 318-07	−1. 534-07

Table 2. $y'=12x^3-8y/x$

x	order 3			order 4	
	m	u	T	m	T
−0. 9	−2. 687-09	−2. 734-09	−2. 410-09	−1. 149-08	−1. 547-08
−0. 8	−2. 682-09	−2. 736-09	−2. 334-09	−1. 276-08	−1. 775-08
−0. 7	−2. 676-09	−2. 737-09	−2. 260-09	−4. 795-10	−5. 798-10
−0. 6	−2. 667-09	−2. 739-09	−2. 181-09	−5. 531-10	−6. 985-10
−0. 5	−2. 650-09	−2. 735-09	−2. 077-09	−6. 524-10	−8. 500-10
−0. 4	−1. 603-10	−1. 634-10	−1. 397-10	−7. 813-10	−1. 087-09
−0. 3	5. 579-11	6. 127-11	4. 232-11	−1. 009-11	−1. 288-11
−0. 2	1. 195-10	1. 215-10	1. 087-10	1. 576-10	1. 854-10
−0. 1	4. 566-07	4. 717-07	3. 894-07	4. 077-08	4. 819-08

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