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On the Value of Distributions at a Point and the Multiplicative Products

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The theory of multiplication between distributions has been developed by several authors (cf. the references in [5]). Recently M. Itano [5] defined the multiplication satisfying certain reasonable requirements. Such a multiplication was called normal by him. In his theory the notion of a value of a distribution at a point in the sense of S. Łojasiewicz [7] plays an important rôle. On the other hand, in our previous paper [11] the multiplication was defined by using the δ-sequences. The aim of the present paper is to unify these two approaches of defining multiplication. To this end we shall introduce the notion of a δ-sequence in a restricted sense in order to make clear the relationships among different approaches to the theory of multiplication between distributions.

Let T be a distribution defined on R^N. If $\lim\limits_{n\to\infty} < T, \rho_n >$ exists for every δ-sequence $\{\rho_n\}$, T is equivalent to a function bounded in a 0-neighbourhood and continuous at 0 [11, p. 229]. Therefore, to reach the notion of the value of a distribution at a point, it will be necessary to restrict the δ-sequence. This is done in Section 1, where besides the value in the sense of Łojasiewicz we investigate the value introduced by Itano [5, p. 174] in one-dimensional case. Section 2 deals with the multiplicative product defined by Itano. On the model of the methods discussed in our previous paper [11], several equivalent conditions for the existence of the product are established. The various approaches of defining multiplication between distributions are also discussed. In Section 3 we introduce the notion on a $\delta_{\mathscr{H}}$-sequence consisting of functions which belong to a normal space $\mathscr{H}$ of distributions and may have non-compact supports. In virtue of this notion we can generalize the method of defining the multiplication due to H. J. Bremermann and L. Durand [2], H. G. Tillmann [13] and others. These multiplications are not normal in the sense of Itano [5]. Most results in the preceding Section 2 are carried over to these cases by replacing a restricted δ-sequence by $\delta_{\mathscr{H}}$-sequence. The final Section 4 is devoted to presenting some examples concerning the values and multipications.

§1. The value of a distribution at a point

Let R^N be an N-dimensional Euclidean space. Points of R^N are denoted by

$$x=(x_1, \cdots, x_N), \quad y=(y_1, \cdots, y_N) \qquad \text{etc.}$$

We use the notations $x+y=(x_1+y_1, \cdots, x_N+y_N)$, $|x|=(\sum|x_j|^2)^{\frac{1}{2}}$, $\lambda x=(\lambda x_1, \cdots, \lambda x_N)$ for a real number λ, and if $p=(p_1, \cdots, p_N)$, where the p_j are non-negative integers, we will write $|p|=\sum p_j$, $D^p=\left(\frac{\partial}{\partial x_1}\right)^{p_1}\left(\frac{\partial}{\partial x_2}\right)^{p_2}\cdots\left(\frac{\partial}{\partial x_N}\right)^{p_N}$. Similarly we write $x^p=x_1^{p_1}\cdots x_N^{p_N}$.

Recall the definition and some basic facts concerning the value of a distribution at a point introduced by Łojasiewicz [7, p. 15]. Let T be a distribution defined in a neighbourhood of a point $x_0 \in R^N$.

DEFINITION 1 (Łojasiewicz). *We say that T has a value c at x_0 : $T(x_0)=c$, if the distributional limit*

$$\lim_{\lambda\to+0} T(x_0+\lambda x)$$

exists in a neighbourhood of 0 *and if it is a constant function* c.

We now put

$$\phi_\lambda(x)=\frac{1}{\lambda^N}\phi\left(\frac{x}{\lambda}\right), \qquad \lambda \text{ being a positive number,}$$

for any $\phi \in \mathcal{D}(R^N)$ such that $\phi \geqq 0$ and $\int\phi(x)dx=1$. It is clear that T has the value c at x_0 if and only if

$$\lim_{\lambda\to+0} < T, \phi_\lambda(x-x_0)> = c$$

for any ϕ considered above.

It is also known that a necessary and sufficient condition for c to be the value of T at x_0 is that there exist a multi-index $p\geqq 0$ ($p\geqq 0$ means that $p_j\geqq 0$ for $j=1, 2, \cdots, N$), a neighbourhood U of x_0 and a continuous function $F(x)$ in U, for which we can write

$$T=c+D^pF \quad \text{in} \quad U,$$

where $F(x)=o\,(|x-x_0|^{|p|})$ as $|x-x_0|\to 0$.

We shall now extend the notion of the value of a distribution at a point.

DEFINITION 2. *If $\frac{1}{2}(T+\check{T})$ has the value c at* 0, *then we put*

$$T<0>=c.$$

For example, if we take $T=\mathrm{Pf}\frac{1}{x}$ (we assume that $N=1$), then $T<0>$

$=0$. But $T(0)$ does not exist.

One of our objects is to study the value of distribution by using the δ-sequence satisisfying certain conditions which will play an important rôle in what follows. For the sake of brevity, we shall consider the value at the origin.

By a *restricted δ-sequence* we understand every sequence of non-negative functions $\rho_n \in \mathcal{D}(R^N)$ with the following properties:

(i) $\operatorname{supp} \rho_n$ converges to $\{0\}$ as $n \to \infty$;

(ii) $\int \rho_n(x)dx$ converges to 1 as $n \to \infty$;

(iii) $\int |x|^{|p|} |D^p \rho_n(x)| dx \leqq M_p$ (M_p being independent of n),

where the integral is extended over the whole N-dimensional space. In addition, when $\rho_n = \check{\rho}_n$ $(n=1, 2, \cdots)$, the sequence $\{\rho_n\}$ is called a *symmetric restricted δ-sequence.*

We note that if $\{\rho_n\}$ is a restricted δ-sequeace, then $\{\check{\rho}_n\}$ is also a restricted δ-sequence.

We first show

PROPOSITION 1. *T has the value $T(0)=c$ if and only if*

$$\lim_{n\to\infty} < T, \rho_n > = c$$

for every restricted δ-sequence $\{\rho_n\}$.

PROOF. Suppose first that the value $T(0)=c$ exists. Then there exist a continuous function $F(x)$ and a multi-index $p \geqq 0$ such that $T = c + D^p F$ in a 0-neighbourhood and

$$\lim_{|x|\to 0} \frac{|F(x)|}{|x|^{|p|}} = 0.$$

For sufficiently large n we have

$$< T, \rho_n > = c\int \rho_n(x)dx + (-1)^{|p|} \int F(x) D^p \rho_n(x) dx.$$

Then

$$\left|\int F(x) D^p \rho_n(x) dx\right| = \left|\int \frac{F(x)}{|x|^{|p|}} |x|^{|p|} D^p \rho_n(x) dx\right|$$

$$\leqq M_p \sup_{x \in \operatorname{supp} \rho_n} \frac{|F(x)|}{|x|^{|p|}},$$

which proves that $\lim_{n\to\infty} < T, \rho_n >$ exists and is c.

Conversely, suppose that $\lim_{n\to\infty} < T, \rho_n > = c$ exists for every restricted

δ-sequence $\{\rho_n\}$. Let ϕ be any positive function in $\mathcal{D}(R^N)$ such that $\int\phi(x)dx=1$ and $\{\lambda_n\}$ be the sequence of positive numbers such that $\lambda_n \downarrow 0$ as $n\to\infty$. The sequence $\{\phi_{\lambda_n}\}$ satisfies the conditions (i), (ii) and (iii), that is, $\{\phi_{\lambda_n}\}$ is a restricted δ-sequence. Consequently $\lim_{n\to\infty} < T, \phi_{\lambda_n} > = c$ exists. This means that T has the value c at 0, as required.

We note that if $\{\sigma_n\}$ is a sequence of functions $\in \mathcal{D}$ with the properties (i), (iii) and $\lim_{n\to\infty}\int\sigma_n(x)dx=c_0$ (c_0 being a constant), then $\lim_{n\to\infty}< T, \sigma_n> = c_0 c$ when T has the value $T(0)=c$.

Łojasiewicz [7, p. 15] has also introduced the notion of the section of a distribution. Let Ω be a non-empty open subset of $R^N=R_x^{N_1}\times R_y^{N_2}$, $T\in\mathcal{D}'(\Omega)$ and $y_0\in R_y^{N_2}$ such that $\Omega_{y_0}=\{x\in R_x^{N_1}; (x, y_0)\in\Omega\}$ is not empty. If

$$\lim_{\lambda\to+0} T(\hat{x},\ y_0+\lambda\hat{y})=S(\hat{x}),$$

then $S(\hat{x})\in\mathcal{D}(\Omega_{y_0})$ is called the section of T for $y=y_0$. S will be denoted by $T(\hat{x},\ y_0)$.

Then we can show that $T\in\mathcal{D}'(\Omega)$ has the section $S\in\mathcal{D}'(\Omega_{y_0})$ for y_0 if and only if

$$\lim_{n\to\infty} < T(\hat{x},\ y),\ \rho_n(y-y_0)> = S(\hat{x})$$

for every resticted δ-sequence $\{\rho_n\}$ in $\mathcal{D}(R_y^{N_2})$.

As an analogue to Proposition 1 we have

PROPOSITION 2. *T has the value $T<0>=c$ if and only if*

$$\lim_{n\to\infty} < T,\ \rho_n> = c$$

for every symmetric restricted δ-sequence $\{\rho_n\}$.

The proof is omitted, as we can write $< T, \rho_n> = <\frac{1}{2}(T+\check{T}),\ \rho_n>$ so that the proof is carried out in a similar manner as in the proof of Proposition 1.

In the case $N=1$, Itano [5, p. 174] generalized the notion of the value of a distribution at a point. According to Łojasiewicz [6, p. 3], we shall also say that T has the *right* (resp. *left*) *hand limit* c_+ (resp. c_-) at 0 if the distributional limit $\lim_{\lambda\to+0} T(\lambda x)$ exists in a 0-neighbourhood for $x>0$ (resp. $x<0$) and is a constant function c_+ (resp. c_-), and that T *has no mass at* 0 if $\lim_{\lambda\to+0}\lambda T(\lambda x)=0$ [7, p. 23].

DEFINITION 3 (Itano). *If T has the right and left hand limits c_+ and c_-*

at 0 *respectively and moreover* T *has no mass at* 0, *then we put*

$$T[0]=\frac{c_{+}+c_{-}}{2},$$

which will also be referred to as the value of T *at* 0 *in Itano's sense.*

It is known that a necessary and sufficient condition for the existence of $T[0]$ is that there exist a non-negative integer p, a 0-neighbourhood U and a continuous function $F(x)$ in U such that $F(x)=o(|x|^{p})$ as $|x|\to 0$, for which we can write

$$T=c_{+}Y+c_{-}\check{Y}+D^{p}F \quad \text{in} \quad U,$$

where Y denotes the Heaviside function.

we now consider a δ-sequence satisfying the conditions (i), (iii) and

(ii)′ $\int_{-\infty}^{0}\rho_n(x)dx$ and $\int_{0}^{\infty}\rho_n(x)dx$ converge to $\frac{1}{2}$ as $n\to\infty$.

Then we have

PROPOSITION 3. *A necessary and sufficient condition for* T *to have the value* $T[0]=c$ *is that we have*

$$\lim_{n\to\infty}\langle T, \rho_n\rangle = c$$

for every δ*-sequence* $\{\rho_n\}$ *satisfying the conditions* (i), (ii)′ *and* (iii).

PROOF. From the condition of existence of the value $T[0]$, the "only if" part will be proved in an entirely similar way as in the corresponding case of Proposition 1. Thus we shall only need to prove the "if" part. Let $\phi, \psi \in \mathcal{D}(R)$ such that $\phi, \psi \geqq 0$, $\operatorname{supp}\phi\subset(0, \infty)$, $\operatorname{supp}\psi\subset(-\infty, 0)$ and $\int\phi(x)dx=\int\psi(x)dx=1$. Let λ_n, λ_n', λ_n'' be the sequences of positive numbers such that λ_n, λ_n', $\lambda_n''\downarrow 0$ as $n\to\infty$ respectively. $\left\{\frac{1}{2}(\phi_{\lambda_n}+\psi_{\lambda_n''})\right\}$ and $\left\{\frac{1}{2}(\phi_{\lambda_n'}+\psi_{\lambda_n''})\right\}$ are δ-sequences with the properties (i), (ii)′, (iii). It follows therefore that

$$\lim_{n\to\infty}\langle T, \frac{1}{2}(\phi_{\lambda_n}+\psi_{\lambda_n''})\rangle = c$$

and

$$\lim_{n\to\infty}\langle T, \frac{1}{2}(\phi_{\lambda_n'}+\psi_{\lambda_n''})\rangle = c.$$

Consequently

$$\lim_{n\to\infty}(<T, \phi_{\lambda_n}> - <T, \phi_{\lambda'_n}>)=0.$$

Here we can conclude that $\lim_{n\to\infty} <T, \phi_{\lambda_n}>$ exists. In fact, if the contrary is assumed, there would exist a positive number ε such that $|<T, \phi_{\lambda_{m_k}}> - <T, \phi_{\lambda_{n_k}}>| > \varepsilon$ for $\lambda_{m_k}, \lambda_{n_k}$, where $\lambda_{m_k}, \lambda_{n_k} \downarrow 0$ as $k\to\infty$. This is a contradiction as $\lim_{k\to\infty}(<T, \phi_{\lambda_{m_k}}> - <T, \phi_{\lambda_{n_k}}>)=0$. In a similar way, $\lim_{n\to\infty} <T, \phi_{\lambda_n}> = c_+$ is independent of the choice of ϕ. This shows that T has a right hand limit c_+ at 0. Similarly, T has a left hand limit c_- at 0. From the preceding results, we can write with constants a_k $(k=1, 2, \cdots, l)$

$$T=c_+Y+c_-\check{Y}+\sum_{k=1}^{l} a_k\delta^{(k)}+D^pF$$

in a 0-neighbouhood, where F is a continuous function such that $F(x)=o(|x|^p)$ as $|x|\to 0$. Now we have

$$\lim_{n\to\infty}\sum_{k=1}^{l} a_k <\delta^{(k)}, \rho_n> = \lim_{n\to\infty} <T-(c_+Y+c_-\check{Y}+D^pF), \rho_n> = 0$$

for every δ-sequence $\{\rho_n\}$ with the properties (i), (ii)$'$ and (iii). It is not difficult to conclude that $a_1=a_2=\cdots=a_l=0$. Thus T has no mass at 0. This means that $T[0]$ exists and equals $\frac{c_+ + c_-}{2}$. The proof is thereby completed.

§2. The multiplicative products between distributions

Let us recall some definitions and results concerning the multiplicative product of distributions. Let $S, T\in \mathcal{D}'(\Omega)$, Ω being an open subset of R^N. By J. Mikusiński [8, p. 254], the multiplicative product was defined as the distributional limit of the sequence $\{(S*\rho_n)(T*\tilde{\rho}_n)\}$ if it exists for every δ-sequence $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$. On the other hand, by Y. Hirata and H. Ogata ([4], p. 150), if both of the sequences $\{(S*\rho_n)T\}$ and $\{S(T*\rho_n)\}$ have the same distributional limit for every δ-sequence, then the multiplicative product was defined as the limit. However in our previous paper [11, p. 227] collaborated with Itano, it has been proved that the above two definitions are entierly equivalent. Further Itano [5, p. 161] generalized the multiplicative product between distributions:

DEFINITION 4 (Itano). *Let $S, T\in\mathcal{D}'(\Omega)$. If there exists for every $\alpha\in\mathcal{D}(\Omega)$ the value $(\check{S}*(\alpha T))(0)$ at 0, then $W\in\mathcal{D}'(\Omega)$ defined by the formula*

$$<W, \alpha> = (\check{S}*(\alpha T))(0)$$

is called the multiplicative product of S aad T and will be denoted by ST in what follows.

Itano also proved that this definition is equivalent to the following statement:

If the distributional limit

$$\lim_{\lambda\to+0}(S*\phi_\lambda)T$$

exists in $\mathcal{D}'(\Omega)$ for any positive $\phi\in\mathcal{D}(\Omega)$ such that $\int\phi(x)dx=1$ and does not depend on the choice of ϕ, then the limit is called the multiplicative product of S and T. The multiplication just defined is normal in his sense [5, p. 165].

From Proposition 1 and Definition 4 we have immediately

Proposition 4. *Let S, $T\in\mathcal{D}'(\Omega)$. In order that the multiplicative product ST may exist, it is necessary and sufficient that for every restricted δ-sequence $\{\rho_n\}$*

$$\lim_{n\to\infty}(S*\rho_n)T$$

exists. In this situation the limit equals ST.

We also note that if $\{\sigma_n\}$ is a sequence of functions with the properties (i), (iii) and $\lim_{n\to\infty}\int\sigma_n(x)dx=c_0$, then $\lim_{n\to\infty}(S*\sigma_n)T=c_0(ST)$ when ST exists.

Proposition 5. *Let S, $T\in\mathcal{D}'(\Omega)$. The following statements are equivalent*:

(1) $\lim_{n\to\infty}(S*\rho_n)(T*\tilde{\rho}_n)$ *exists for every restricted δ-sequence $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$*;

(2) $\lim_{n\to\infty}(S*\rho_n)T$ *exists for every restricted δ-sequence $\{\rho_n\}$*;

(3) $\lim_{n\to\infty}S(T*\rho_n)$ *exists for every restricted δ-sequence $\{\rho_n\}$.*

In any case the limit equals ST.

Proof. It suffices to prove the proposition in the case where S, $T\in\mathcal{E}'(R^N)$. The implications (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3) will be easily proved, since (1) is equivalent to the condition:

(1)′ $\lim_{n,m\to\infty}(S*\rho_n)(T*\tilde{\rho}_m)$ exists for every restricted δ-sequence $\{\rho_n\}$ and $\{\tilde{\rho}_m\}$.

Thus it remains to prove that (2)$\Rightarrow$(1) holds, since the proof of (3)$\Rightarrow$(1) will be carried out in a similar way. To this end, it is sufficent to show that $\langle(S*\rho_n)(T*\tilde{\rho}_n),\chi\rangle$ converges for any $\chi\in\mathcal{D}(R^N)$ as $n\to\infty$. Let l be a positive integer such that supp S, supp T and supp α are contained in the interior of cube $Q_l=\{x;\ |x_j|\leqq l,\ j=1, 2, \cdots, N\}$. Since for large n $(S*\rho_n)(T*\tilde{\rho}_n)$ vanishes outside the cube Q_l, so we may assume χ to be extended to a periodic function with period $2l$ for each coordinate. Let $\chi(x)=\sum_m c_m e^{i\frac{\pi}{l}\langle m,x\rangle}=\sum_m c_m e(m)$ be the Fourier expansion of χ, where $\sum_m|c_m|(1+|m|)^k<\infty$ for every positive integer k. Now we can write for large n

$$\begin{aligned}<(S*\rho_n)(T*\tilde{\rho}_n), \chi> &= \sum_m c_m <(S*\rho_n)(T*\tilde{\rho}_n), e(m)> \\ &= \sum_m c_m <\big(S*\rho_n*(\check{\tilde{\rho}}_n e(-m))\big)T, e(m)>.\end{aligned}$$

Put $\sigma_n = \rho_n*(\check{\tilde{\rho}}_n e(-m))$. Then $\int \sigma_n(x)dx$ will converge to 1 as $n\to\infty$. In fact, this follows from

$$\begin{aligned}\int \sigma_n(x)dx &= \int \rho_n(x)dx \int \tilde{\rho}_n(-x)e(-m)dx \\ &= \int \rho_n(x)dx \left\{\int \tilde{\rho}_n(x)dx + \int \tilde{\rho}_n(x)\big(e(m)-1\big)dx\right\}\end{aligned}$$

and

$$\left|\int \tilde{\rho}_n(x)\big(e(m)-1\big)dx\right| \leqq \tilde{M}_0 \sup_{x\in \operatorname{supp}\tilde{\rho}_n} |e(m)-1|,$$

$\tilde{M}_0$ being a constant.

On the other hand, by induction over $|p|$ we can write with constants $a_{r,s}$

$$x^{p'}D^p(f*g) = \sum_{|r|+|s|\leqq|p|} a_{r,s}\big((x^{r'}D^r f)*(x^{s'}D^s g)\big), \quad f, g\in \mathcal{D}(\Omega),$$

where $|p|=|p'|$, $|r|=|r'|$ and $|s|=|s'|$. Since $|x|^{|p|} \leqq c'_{|p|} \sum_{|p'|=|p|} |x^{p'}|$ with a positive constant $c'_{|p|}$, we obtain

$$\begin{aligned}\int |x|^{|p|}|D^p\sigma_n(x)|dx &\leqq c'_{|p|} \sum_{|p'|=|p|} \int |x^{p'}D^p\big(\rho_n*(\tilde{\rho}_n e(-m)\big)|dx \\ &\leqq c'_{|p|} \sum_{|r|+|s|\leqq|p|} |a_{r,s}| \int |x|^{|r|}|D^r\rho_n(x)|dx \int |x|^{|s|}|D^s\big(\tilde{\rho}_n e(-m)\big)|dx \\ &\leqq M'_p(1+|m|)^{|p|},\end{aligned}$$

where M'_p is a constant depending on $|p|$. Owing to Proposition 4 the condition (2) implies the existence of the product ST. Thus it follows that $(S*\sigma_n)T$ converges to ST as $n\to\infty$. Since $\{(S*\sigma_n)T\}$ is equicontinuous, we have

$$|<(S*\sigma_n)T, e(m)>| \leqq M(1+|m|)^k,$$

where M is a positive constant and k is a non-negative integer. Consequently

$$\begin{aligned}\lim_{n\to\infty} <(S*\rho_n)(T*\tilde{\rho}_n), \chi> &= \sum c_m \lim_{n\to\infty} <(S*\sigma_n)T, e(m)> \\ &= \sum c_m <ST, e(m)> \\ &= <ST, \chi>,\end{aligned}$$

since $\sum|c_m|(1+|m|)^k<\infty$ as already remarked. Thus the proof is completed.

Remark 1. We may define the multiplicative product $S\times T$ as $\lim_{n\to\infty}(S*\rho_n)\times(T*\rho_n)$, if it exists for every δ-sequence $\{\rho_n\}$. Actually this yields a generalization of the product in the sense of [11] as seen by taking $S=\delta$ and $T=2Y-1$. Similarly, we may also define the multiplicative product $S\triangle T$ as $\lim_{n\to\infty}(S*\rho_n)(T*\rho_n)$, if it exists for every restricted δ-sequence $\{\rho_n\}$. If ST exists, then $S\triangle T$ exists and is equal to ST. However the converse does not hold. Actually we have $\delta\triangle \mathrm{Pf}\frac{1}{x}=-\frac{1}{2}\delta'$, but $\delta\,\mathrm{Pf}\frac{1}{x}$ does not exist. It is to be noted that $\lim_{n\to\infty}(S*\rho_n)(S*\tilde{\rho}_n)$ exists if and only if $\lim_{n\to\infty}(S*\rho_n)(S*\rho_n)$ exists.

Remark 2. Using the symmetric restricted δ-sequences, we may define the multiplicative product. For example, the product is defined as the distributional limit of the sequence $\{(S*\rho_n)T\}$ or $\{S(T*\rho_n)\}$ if it exists for every symmetric restricted δ-sequence $\{\rho_n\}$. But, in general, $\lim_{n\to\infty}(S*\rho_n)T$ does not equal $\lim_{n\to\infty}S(T*\rho_n)$. In fact, if we take $S=\delta$ and $T=\mathrm{Pf}\frac{1}{x}$, then $\lim_{n\to\infty}(S*\rho_n)T=-\delta'$. But $\lim_{n\to\infty}S(T*\rho_n)=0$.

In the case $N=1$, Itano [5, p. 175] also defined an extended multiplicative product $S\times_0 T$ by the formula:

$$<S\times_0 T,\alpha>=((\alpha S)*\check{T})[0],\qquad \alpha\in\mathcal{D}(\Omega),$$

when the right hand side makes sense. And he proved that the multiplication is normal [5, p. 177].

We can show

Proposition 6. *The product* $S\times_0 T$ *exists if and only if* $\lim_{n\to\infty}(S*\rho_n)T$ *or* $\lim_{n\to\infty}S(T*\rho_n)$ *exists for every δ-sequence* $\{\rho_n\}$ *satisfying the conditions* (i), (ii)$'$ *and* (iii), *and the limit is equal to* $S\times_0 T$.

Proof. This is an immediate consequence of Proposition 3 and the definition of $S\times_0 T$, since $<(S*\rho_n)T,\alpha>=<\check{S}*(\alpha T),\rho_n>$ and $<S(T*\rho_n),\alpha>=<(\alpha S)*\check{T},\rho_n>$.

Remark 3. In general, the condition in Proposition 6 for the existence of $S\times_0 T$ is not equivalent to the following condition: $\lim_{n\to\infty}(S*\rho_n)(T*\tilde{\rho}_n)$ exists for every δ-sequence $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$ with the properties (i), (ii)$'$ and (iii). Actually we have $Y\times_0\delta=\frac{1}{2}\delta$, but $\lim_{n\to\infty}(Y*\rho_n)\tilde{\rho}_n$ does not exist.

Finally we note that we can define multiplication between distributions by using any assigned restricted δ-sequences.

§3. $\delta_{\mathscr{H}}$-sequence

Let $\mathscr{H}$ be a normal space of distributions, that is, a linear subspace of $\mathscr{D}'(R^N)$ with locally convex topology such that the injections $\mathscr{D}(R^N)\to\mathscr{H}$, $\mathscr{H}\to\mathscr{D}'(R^N)$ are continuous and $\mathscr{D}(R^N)$ is dense in $\mathscr{H}$. Assume that

(I_1): *$\mathscr{H}$ is stable under every translation τ_h*

and

(I_2): *the linear endomorphism $\tau_h:\mathscr{H}\to\mathscr{H}$ is continuous, where $h=(h_1, \cdots, h_N)\in R^N$.*

We shall consider a sequence of positive functions $g_n(x)$, $g_n\in\mathscr{E}(R^N)$, with the following conditions:

(a) $\int g_n(x)dx$ converges to 1 and $\int_{\varepsilon\leqq|x|} g_n(x)dx$ converges to 0 as $n\to\infty$, ε being any positive number;

(b) $(1-\alpha)g_n$ converges in $\mathscr{H}$ to 0 as $n\to\infty$ for any $\alpha\in\mathscr{D}$ such that α equals 1 in a 0-neighbourhood;

(c) there exists a constant M_p (independent of n) such that for a positive number ε_0

$$\int_{|x|\leqq\varepsilon_0}|x|^{|p|}|D^p g_n(x)|\,dx\leqq M_p.$$

The sequence $\{g_n\}$ equipped with these properties is referred to as a $\delta_{\mathscr{H}}$-*sequence*. Then the condition (b) yields $g_n\in\mathscr{H}$. We use the notations $<\ ,\ >_{\mathscr{H}',\mathscr{H}}$, $<\ ,\ >_{\mathscr{D}',\mathscr{D}}$ to make clear the duality between the spaces of distributions under question.

PROPOSITION 7. *Let $S\in\mathscr{H}'$. S has the value $S(x_0)=c$ at $x_0\in R^N$ if and only if for every $\delta_{\mathscr{H}}$-sequence $\{g_n\}$*

$$\lim_{n\to\infty}<S,\ g_n(x-x_0)>_{\mathscr{H}',\mathscr{H}}=c.$$

PROOF. Since the restricted δ-sequence is a $\delta_{\mathscr{H}}$-sequence, the "if" part is obvious. Suppose the value $S(x_0)=c$ exists. Without loss of generality we may assume that $x_0=0$. Since any subsequence of a $\delta_{\mathscr{H}}$-sequence is again a $\delta_{\mathscr{H}}$-sequence, it is sufficient to prove that $\lim\limits_{n\to\infty}<S,\ g_{j_n}>=c$ for some subsequence $\{g_{j_n}\}$ of $\{g_n\}$. Let $\alpha\in\mathscr{D}$ be equal to 1 in a 0-neighbourhood such that $\operatorname{supp}\alpha\subset\{x:|x|\leqq 1\}$ and $0\leqq\alpha\leqq 1$. Put $\alpha_n(x)=\alpha(nx)$ $(n=1, 2, \ldots)$. By induction we can choose a subsequence $\{g_{j_n}\}$ $(j_1<j_2<\cdots<j_n<\cdots)$ of $\{g_n\}$ such that g_{j_n} is an element of g_k with the following properties: for $k\geqq j_n$

$$\int(1-\alpha_n)g_k dx<\frac{1}{n}$$

and

$$|<S,\ (1-\alpha_n)g_k>_{\mathcal{H}',\mathcal{H}}|<\frac{1}{n}.$$

Putting $\rho_n=\alpha_n g_{j_n}$, we can show that

(1) $\operatorname{supp}\rho_n$ converges to $\{0\}$ as $n\to\infty$,

(2) $\int\rho_n dx=\int g_{j_n}dx-\int(1-\alpha_n)g_{j_n}dx$ converges to 1 as $n\to\infty$,

and

(3)
$$\begin{aligned}&\int_{|x|\leqq\varepsilon_0}|x|^{|p|}|D^p\rho_n(x)|\,dx\\ &\leqq\sum\binom{p}{q}\int|x|^{|q|}n^{|q|}|\alpha^{(q)}(nx)|\,|x|^{|p-q|}|D^{p-q}g_{j_n}(x)|\,dx\\ &\leqq M'_p \qquad (M'_p \text{ being a constant}).\end{aligned}$$

Thus $\{\rho_n\}$ is a restricted δ-sequence. By Proposition 1 we have

$$\lim_{n\to\infty}<S,\ \rho_n>_{\mathcal{D}',\mathcal{D}}=c.$$

Consequently

$$\begin{aligned}\lim_{n\to\infty}<S,\ g_{j_n}>_{\mathcal{H}',\mathcal{H}}&=\lim_{n\to\infty}<S,\ \rho_n>+\lim_{n\to\infty}<S,\ (1-\alpha_n)g_{j_n}>_{\mathcal{H}',\mathcal{H}}\\ &=c.\end{aligned}$$

The proof is completed.

We shall assume, in addition, that

(II_1): *$\mathcal{E}'*\mathcal{H}\subset\mathcal{H}$ and* (II_2): *the bilinear mapping $(W,\ g)\to W*g$ of $\mathcal{E}'\times\mathcal{H}$ into $\mathcal{H}$ is separately continuous.* Let $g\in\mathcal{H}$ and $S\in\mathcal{H}'$. Then the linear forms $W\to<S,\ W*g>_{\mathcal{H}',\mathcal{H}}$ on $\mathcal{E}'$ and $g\to<S,\ W*g>_{\mathcal{H}',\mathcal{H}}$ on $\mathcal{H}$ are continuous, so there exist an $f\in\mathcal{E}$ and a $T\in\mathcal{H}'$ such that $<S,\ W*g>_{\mathcal{H}',\mathcal{H}}=<W,f>_{\mathcal{E}',\mathcal{E}}$ and $<S,\ W*g>_{\mathcal{H}',\mathcal{H}}=<T,\ g>_{\mathcal{H}',\mathcal{H}}$ respectively. We can write $T=S*\check{W}$. We shall also denote f by $S*\check{g}$. As to this notation, if we assume that $\dot{\mathcal{B}}\mathcal{H}\subset\mathcal{H}$ and that $u(B)$ is bounded in $\mathcal{H}$ for any bounded subset $B\subset\mathcal{H}$, where u is a linear mapping $\beta\to\beta g$ of $\dot{\mathcal{B}}$ into $\mathcal{H}$, (Usual normal spaces of distributions satisfy these conditions) we can prove that $S(g*\phi)\in\mathcal{D}'_{L^1}$ for any $\phi\in\mathcal{D}$, and so that $S*\check{g}$ has its proper sense [10, p. 23].

Most of the normal spaces of distributions as referred to in Schwartz [9] satisfy the conditions (I_1), (I_2), (II_1) and (II_2). A space of distributions $\mathcal{K}$ is said to have c-property if any linear map $u:E\to\mathcal{K}$, E being barrelled, is continuous whenever $i\circ u$ is continuous, where i denotes the injection of $\mathcal{K}$ into $\mathcal{D}'$ [10, p. 22]. If $\mathcal{K}$ is barrelled and has c-property, then the condition (I_1) (resp. (II_1)) implies (I_2) (resp. (II_2)).

PROPOSITION 8. *Let $S \in \mathcal{H}'$ and let $\alpha \in \mathcal{D}$ such that $\alpha=1$ in a 0-neighbourhood. Then $S*\check{g}_n$ converges in $\mathcal{D}'$ to S and $S*((1-\alpha)g_n)^\vee$ converges in $\mathcal{E}$ to 0 for every $\delta_{\mathcal{H}}$-sequence $\{g_n\}$ as $n\to\infty$.*

PROOF. For any $\phi \in \mathcal{D}$, we have

$$<S*\check{g}_n, \phi>_{\mathcal{D}',\mathcal{D}}=<S*\check{\phi}, \alpha g_n>_{\mathcal{E},\mathcal{E}'}+<S*\check{\phi}, (1-\alpha)g_n>_{\mathcal{H}',\mathcal{H}}.$$

Since $S*\check{\phi} \in \mathcal{E}$ and $\{\alpha g_n\}$ is a $\delta_{\mathcal{E}'}$-sequence, it follows by Proposision 7 that $\lim_{n\to\infty}<S*\check{\phi}, \alpha g_n>=(S*\check{\phi})(0)$. On the other hand, $<S*\check{\phi}, (1-\alpha)g_n>_{\mathcal{H}',\mathcal{H}}$ converges to 0 as $n\to\infty$. Consequently

$$\lim_{n\to\infty}<S*\check{g}_n, \phi>=(S*\check{\phi})(0)=<S, \phi>.$$

Next we obtain for any $W \in \mathcal{E}'$

$$\begin{aligned}\lim_{n\to\infty}<S*((1-\alpha)g_n)^\vee, W>_{\mathcal{E},\mathcal{E}'}&=\lim_{n\to\infty}<S, W*((1-\alpha)g_n)>_{\mathcal{H}',\mathcal{H}}\\&=\lim_{n\to\infty}<S*\check{W}, (1-\alpha)g_n>_{\mathcal{H}',\mathcal{H}}\\&=0.\end{aligned}$$

Thus the proof is completed.

We shall now turn to some consideration on multiplication between distributions. Let $\mathcal{H}$ and $\mathcal{K}$ be normal spaces of distributions satisfying the conditions (I_1), (I_2), (II_1) and (II_2).

PROPOSITION 9. *For any $S \in \mathcal{H}'$ and $T \in \mathcal{K}'$, the following conditions are equivalent to each other:*

(1) *ST exists;*

(2) *$\lim_{n\to\infty}(S*\check{g}_n)T$ exists for every $\delta_{\mathcal{H}}$-sequence $\{g_n\}$;*

(3) *$\lim_{n\to\infty}S(T*\check{h}_n)$ exists for every $\delta_{\mathcal{K}}$-sequence $\{h_n\}$;*

(4) *$\lim_{n\to\infty}(S*\check{g}_n)(T*\check{h}_n)$ exists for every $\delta_{\mathcal{H}}$-sequence $\{g_n\}$ and $\delta_{\mathcal{K}}$-sequence $\{h_n\}$.*

If one of the equivalent conditions (2), (3) and (4) holds, then the limit is equal to ST.

PROOF. We first note that for any $\phi \in \mathcal{D}$

$$<(S*\check{g}_n)T, \phi>_{\mathcal{D}',\mathcal{D}}=<S*(\phi T)^\vee, g_n>_{\mathcal{H}',\mathcal{H}}$$

and

$$<S(T*\check{h}_n), \phi>_{\mathcal{D}',\mathcal{D}}=<(\phi S)^\vee*T, h_n>_{\mathcal{K}',\mathcal{K}}.$$

From these equalities combined with Definition 1 and Proposition 7, it follows that (1) is equivalent to (2) and (3). Further the implications (4)$\Rightarrow$(2) and (4)$\Rightarrow$(3) are obvious. It remains therefore to show that (1)$\Rightarrow$(4) holds. Let ϕ be any element of $\mathcal{D}$ and α be taken as in the proof of Proposition 7. Then we can choose two subsequences $\{g_{j_n}\}$ and $\{h_{j_n}\}$ so as to satisfy the conditions:

$$\int(1-\alpha_n)g_{j_n}dx<\frac{1}{n},\quad |<S,(1-\alpha_n)g_{j_n}>_{\mathcal{H}',\mathcal{H}}|<\frac{1}{n},$$

$$\int(1-\alpha_n)h_{j_n}dx<\frac{1}{n}\quad\text{and}\quad |<T,(1-\alpha_n)h_{j_n}>_{\mathcal{K}',\mathcal{K}}|<\frac{1}{n}.$$

We put $\rho_n=\alpha_n g_{j_n}$ and $\tilde{\rho}_n=\alpha_n h_{j_n}$. Then we can write

$$\begin{aligned}<(S*\check{g}_{j_n})(T*\check{h}_{j_n}),\phi> &= <(S*\check{\rho}_n)(T*\check{\tilde{\rho}}_n),\phi>\\ &\quad+<S*\check{\rho}_n,\,(T*((1-\alpha_n)h_{j_n})^{\vee})\phi>\\ &\quad+<T*\check{h}_{j_n},\,(S*((1-\alpha_n)g_{j_n})^{\vee})\phi>.\end{aligned}$$

Owing to Proposition 8, it follows that

$$\lim_{n\to\infty}<S*\check{\rho}_n,\,(T*((1-\alpha_n)h_{j_n})^{\vee})\phi>=0$$

and

$$\lim_{n\to\infty}<T*\check{h}_{j_n},\,(S*((1-\alpha_n)g_{j_n})^{\vee})\phi>=0.$$

Since $\{\rho_n\}$ and $\{\tilde{\rho}_n\}$ are restricted δ-sequence, by Proposition 1 it follows that

$$\lim_{n\to\infty}<(S*\check{\rho}_n)(T*\check{\tilde{\rho}}_n),\phi>=<ST,\phi>.$$

Consequently, since a subsequence of $\delta_{\mathcal{H}}$(resp. $\delta_{\mathcal{K}}$)-sequence is also $\delta_{\mathcal{H}}$(resp. $\delta_{\mathcal{K}}$)-sequence, we can conclude that

$$\lim_{n\to\infty}<(S*\check{g}_n)(T*\check{h}_n),\phi>=<ST,\phi>,\qquad \phi\in\mathcal{D}.$$

This completes the proof.

REMARK. 4. In the case $N=1$, we shall also consider a sequence of positive functions g_n, $g_n\in\mathcal{E}(R^N)$, satisfying the conditions (b), (c) and

(ā) $$\int_0^{\infty}g_n(x)dx,\ \int_{-\infty}^{0}g_n(x)dx\quad\text{converge to }\frac{1}{2}$$

and $\int_{\varepsilon\leqq|x|}g_n(x)dx$ converges to 0 as $n\to\infty$, ε being any positive number. As an analogue to Proposition 6 we can prove the following: Let $S\in\mathcal{H}'$ and $T\in\mathcal{K}'$. The product $S\times_0 T$ exists if and only if $\lim_{n\to\infty}(S*\check{g}_n)T$ (resp. $\lim_{n\to\infty}S\,(T*\check{h}_n)$)

exists for every $\delta_{\mathscr{K}}$-sequence $\{g_n\}$ (resp. $\delta_{\mathscr{K}}$-sequence $\{h_n\}$) satisfying the conditions ($\bar{a}$), (b) and (c). The limit is equal to $S\times_{\circ}T$.

§4. Examples

EXAMPLE 1. Let T be a distribution defined on the unit circle $K_1=\{z\in C;\ |z|=1\}$ contained in the complex number field C. By $p_r(\theta)$, $0\leqq r<1$, $-\pi\leqq\theta\leqq\pi$, we understand the Poisson kernel $\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}$. If we put

$$u(z)=u(r,\theta)=\int_{-\pi}^{\pi}T(\tau)p_r(\theta-\tau)d\tau \tag{1}$$

with $z=re^{i\theta}$, then $u(z)$ is harmonic in $|z|<1$ and $u_r(\cdot)=u(r,\theta)$ converges in $\mathscr{D}'(K_1)$ to T as $r\to 1-0$ [12]. Conversly, if $u(z)$ is any harmonic function in $|z|<1$ such that the distributional limit of u_r as $r\to 1-0$ is T, then $u(z)$ is given by the formula (1).

Consider the sequence $\{g_n\}$, where $g_n(\theta)=p_{r_n}(\theta-\theta_n)$ and $r_ne^{i\theta_n}$ converges to 1 from the inside of an angular domain in $|z|<1$ with vertex 1. Then $\{g_n\}$ satisfies the following conditions:

(a′) $\int_{-\pi}^{\pi}g_n(\theta)d\theta=1$ and $\int_{\varepsilon\leqq|\theta|\leqq\pi}g_n(\theta)d\theta$ converges to 0 as $n\to\infty$, ε being any positive number;

(b′) $(1-\alpha)g_n$ converges in $\mathscr{D}(K_1)$ to 0 as $n\to\infty$ for any $\alpha\in\mathscr{D}(K_1)$ such that $\alpha=1$ in a 0-neighbourhood;

(c′) There exists a constant M_k (independent of n) such that

$$\int_{|\theta|\leqq\varepsilon_0}|\theta|^k|D^kg_n(\theta)|d\theta\leqq M_k,$$

where ε_0 is a positive number.
In fact, (a′) and (b′) are clear. By induction over k we can write with constants c_{k_1,k_2}

$$D^kp_r(\theta)=\sum_{k_1+2k_2\leqq k}c_{k_1,k_2}p_r(\theta)\left(\frac{r\sin\theta}{1-2r\cos\theta+r^2}\right)^{k_1}\left(\frac{r\cos\theta}{1-2r\cos\theta+r^2}\right)^{k_2}.$$

Since $(1-2r\cos\theta+r^2)\geqq r^2\sin^2\theta$, $(1-r)^2$ and $\frac{\theta_n}{1-r_n}$ is bounded near $z=1$, it follows from the equation that (c′) holds.

A little modification of the discussions made in the preceding sections allows us to use this restricted δ-sequence to deal with the value of a distribution at a point and multiplication between distributions defined on the unit circle. Thus we obtain an extension of Fatou's theorem [3] which is concerned with the integrable function defined on the unit circle. If the value of $T \in \mathcal{D}'(K_1)$ at $\theta=0$ exists, then $\lim_{z\to 1} u(z)=T(0)$ when $z\to 1$ from the inside of any fixed angular domain in $z<1$ with vertex 1.

Another extension is: If $T \in \mathcal{D}'(K_1)$ has the value $T[0]=\frac{c_+ + c_-}{2}$ at $\theta=0$, and $l_\alpha(1)$ is a curve through 1, making an angle $\alpha \left(-\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right)$ with the radius of K_1 at $z=1$, then $u(z)$ converges to $c_+\left(\frac{1}{2}+\frac{\alpha}{\pi}\right)+c_-\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)$ when $z\to 1$ along $l_\alpha(1)$.

Also we define the multiplicative product of S and T, if the distributional limit $\lim_{r\to 1-0} u(r, \theta)v(r, \theta)$ exists, where we put $u(r, \theta)=\int S(\tau)p_r(\theta-\tau)d\tau$ and $v(r, \theta)=\int T(\tau)p_r(\theta-\tau)d\tau$. We note that if ST exists, the above product also exists and coincides with ST.

Example 2. Consider the case $N=1$. Let $g_{\varepsilon_n}(x)$ be the Abel-Poisson kernel, that is, $g_{\varepsilon_n}(x)=\frac{1}{\pi}\frac{\varepsilon_n}{\varepsilon_n^2+x^2}$, $\varepsilon_n \downarrow 0$. Then $\{g_{\varepsilon_n}(x)\}$ satisfies the conditions (a), (c) and moreover we have with a constant M'_k and a positive number δ

$$\int_{|x|\leqq\delta} |x|^k |g_{\varepsilon_n}^{(k)}(x-y)|\, dx \leqq M'_k,$$

when $(y, \varepsilon_n)\to(0, 0)$ from the inside of an angular domain in $\varepsilon>0$ with vertex $(0, 0)$.

We can select a normal space $\mathcal{H}$ of distributions in various way so that $\{g_{\varepsilon_n}\}$ is a $\delta_{\mathcal{H}}$-sequence. Actually we can take $\mathcal{H}=\mathcal{E}, (1+x^2)\times\mathcal{B}_c, \sqrt{1+x^2}\times\mathcal{D}'_{L^1}$ and $\mathcal{O}_\alpha$ $(\alpha\geqq -1)$. where $\mathcal{O}_\alpha$ is a space introduced by Bremermann [1]. If T has the value $T(0)$, then $\lim_{(h,\varepsilon)\to(0,0)} < T, \tau_h g_\varepsilon > = T(0)$ when $(h, \varepsilon)\to(0, 0)$ from the inside of any fixed angular domain in $\varepsilon>0$ with vertex $(0, 0)$. Let now $l_\alpha(0)$ be a curve through the origin, making an angle $\alpha \left(-\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right)$ with the axis $x=0$ at the origin. If T has the value $T[0]=\frac{c_+ + c_-}{2}$, the $< T, \tau_h g_\varepsilon >$ converges to $c_+\left(\frac{1}{2}+\frac{\alpha}{\pi}\right)+c_-\left(\frac{1}{2}-\frac{\alpha}{\pi}\right)$ as $(h, \varepsilon)\to(0, 0)$ along $l_\alpha(0)$. As to the multiplicative product we can proceed the same way as in Example 1. For detailed discussions, see [2] and [13].

Example 3. Finally we shall consider the Weierstrass kernel $g_n(x)=\frac{1}{(\sqrt{\pi})^N} n^N e^{-n^2|x|^2}$. If we take $\mathcal{H}$ as $\mathcal{S}$, then $\{g_n\}$ is a $\delta_{\mathcal{S}}$-sequence. The same discussions as Example 2 enable us to discuss the value and multiplication for tempered distributions.

In the case $N=1$. we shall consider the function $f(x, t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} T(y)e^{-\frac{(x-y)^2}{4t}}dy$, t being any positive number. Suppose T has the value $T(0)=c$. If $(x, t)\to(0, 0)$ from the inside of any fixed parabola $x^2=4kt$ $(k>0)$, $f(x, t)$ converges to c. Finally we can show that if T has the value $T[0]=\frac{c_+ + c_-}{2}$, then $f(x, t)$ converges to $\frac{c_+ + c_-}{2}+\frac{c_+ - c_-}{\sqrt{\pi}}$ $(\operatorname{sign} x)\int_0^k e^{-x^2}dx$ when $(x, t)\to(0, 0)$ along the parabola $x^2=4k^2t$ $(k>0)$.

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