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On Certain Properties of Lie Algebras

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Introduction. I. M. Singer [1] has introduced the following condition for a Lie algebra L: (A) Any pair of elements x, y of L such that $[x, [x, y]]=0$ satisfies $[x, y]=0$. M. Sugiura [2] called a Lie algebra satisfying this condition to be an (A)-algebra and proved, among other results, that a Lie algebra L over a field of characteristic 0 is an (A)-algebra if and only if any $x \in L$ such that $(\mathrm{ad}\, x)^k=0$ for some $k \geqq 2$ satisfies $\mathrm{ad}\, x=0$. On the other hand, S. Tôgô [3] has considered a Lie algebra L satisfying the condition that $(\mathrm{ad}\, x)^2=0$ implies $\mathrm{ad}\, x=0$, and has given an example of such a Lie algebra which is solvable but not abelian. This is not an (A)-algebra since any solvable (A)-algebra is abelian ([1], [2]). Thus we are led to consider a Lie algebra which satisfies the condition that $(\mathrm{ad}\, x)^k=0$ implies $\mathrm{ad}\, x=0$ for a fixed integer $k \geqq 2$. We shall call such a Lie algebra to be an (A_k)-algebra. In this paper we shall investigete the properties of (A_k)-algebras.

It will be shown that a solvable (resp. nilpotent) Lie algebra over a field of characteristic 0 is an (A_k)-algebra with $k \geqq 3$ (resp. $k \geqq 2$) if and only if it is abelian. We shall show that an (A_2)-algebra is not always an (A_k)-algebra with $k \geqq 3$, much less an (A)-algebra. As to (A_k)-algebras with $k \geqq 3$, if the basic field is algebraically closed and of characteristic 0, we can show that an (A_k)-algebra is abelian and so an (A)-algebra. A detailed discussion about (A_2)-algebras is also given.

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Notations. We denote by Φ a field of arbitrary characteristic unless otherwise stated and by L a finite dimensional Lie algebra over a field Φ. We denote by $Z(L)$ the center of L and by

$$Z_0(L)=Z(L) \subset Z_1(L) \subset Z_2(L) \subset \cdots \subset Z_n(L) \subset \cdots$$

the ascending central series of L. For a subspace U of L the centralizer of U in L will be denoted by $C(U)$.

1. We start with the following

Definition 1. *For an integer $k \geqq 2$, we call a Lie algebra L over a field Φ*

to be an (A_k)*-algebra provided it has the property*

(A_k): *Any element* x *of* L *such that* $(\operatorname{ad} x)^k=0$ *satisfies* $\operatorname{ad} x=0$.

We also call L *to be an* (A_∞)*-algebra provided it has the property*

(A_∞): *Any element* x *of* L *such that* $(\operatorname{ad} x)^k=0$ *for some* $k\geqq 2$ *satisfies* $\operatorname{ad} x=0$.

According to I. M. Singer [1] and M. Sugiura [2], we shall give the following

Definition 2. *A Lie algebra* L *over a field* Φ *is called an* (A)*-algebra provided it has the property*

(A): *Any pair of elements* x, y *of* L *such that* $[x,[x, y]]=0$ *satisfies* $[x, y]=0$.

Among the Lie algebras given in Definitions 1 and 2 there is the following interrelation.

Proposition 1. *Let* L *be a Lie algebra over a field* Φ. *Then we have the following implications*:

$$(\mathrm{A})\Rightarrow(\mathrm{A}_\infty)\Rightarrow\cdots\Rightarrow(\mathrm{A}_{k+1})\Rightarrow(\mathrm{A}_k)\Rightarrow\cdots\Rightarrow(\mathrm{A}_2).$$

Proof. Assume that L has the property (A). If $x \in L$ satisfies $(\operatorname{ad} x)^k=0$ for some integer $k\geqq 2$, then for any $y\in L$ we have $[x,[x,(\operatorname{ad} x)^{k-2}y]]=0$. Hence $[x,(\operatorname{ad} x)^{k-2}y]=0$, that is, $(\operatorname{ad} x)^{k-1}y=0$. Repeating this procedure, we finally have $(\operatorname{ad} x)y=0$. Therefore $\operatorname{ad} x=0$ and L has the property (A_∞). The statement that (A_{k+1}) implies (A_k) is evident.

2. In this section we shall mainly study the properties of the subalgebras and the ideals of the (A_k)-algebras. We first show

Proposition 2. *Let* L *be a Lie algebra over a field* Φ *and let* H *be any ideal of* L.

(a) *If* L *is an* (A_{k+1})*-algebra, then* H *is an* (A_k)*-algebra.*

(b) *If* L *is an* (A_{k+1})*-algebra and if* $H\subset Z(L)$, *then* L/H *is an* (A_k)*-algebra.*

Proof. (a): Assume that L is an (A_{k+1})-algebra. Suppose that $x\in H$ satisfies $(\operatorname{ad} x)^k H=(0)$, then $(\operatorname{ad} x)^{k+1}L\subset(\operatorname{ad} x)^k H=(0)$, that is, $(\operatorname{ad} x)^{k+1}=0$. Therefore $\operatorname{ad} x=0$. Thus H is an (A_k)-algebra.

(b): Assume that L is an (A_{k+1})-algebra and $H\subset Z(L)$. For any $x\in L$, we denote by $\bar{x}$ the corresponding element of L/H. Suppose that $(\operatorname{ad}\bar{x})^k(L/H)=(0)$. Then $(\operatorname{ad} x)^k L\subset H$. Since $H\subset Z(L)$, we have $(\operatorname{ad} x)^{k+1}L\subset(\operatorname{ad} x)H=(0)$. Since L is an (A_{k+1})-algebra, $(\operatorname{ad} x)L=(0)$. It follows that $(\operatorname{ad}\bar{x})(L/H)=(0)$.

Thus L/H is an (A_k)-algebra.

PROPOSITION 3. *Let L be a Lie algebra over a field Φ and let H be a subalgebra of L. Assume that there is a subspace U such that $L=H+U$ and $(\mathrm{ad}\,x)^2U=(0)$ for any $x \in H$. Then if L is an (A_k)-algebra, H is an (A_k)-algebra.*

PROOF. By using the assumption on H, we have for any $x \in H$

$$(\mathrm{ad}\,x)^kL=(\mathrm{ad}\,x)^{k-2}[x,[x,H+U]]$$
$$=(\mathrm{ad}\,x)^{k-2}[x,[x,H]]=(\mathrm{ad}\,x)^kH.$$

The assertion in the proposition follows from this formula.

COROLLARY. *Let L be an (A_k)-algebra. Then any subalgebra H of L such that $L=H+Z_1(L)$ is an (A_k)-algebra.*

PROOF. This is immediate from Proposition 3, since for any $x \in H$ we have $[x,[x,Z_1(L)]]\subset[x,Z(L)]=(0)$.

LEMMA 1. *Let x be any element of L and let U be any subspace of L such that $[x,U]\subset C(U)$. Then $[x,[U,U]]=(0)$.*

PROOF. For such x and U, we have

$$[x,[U,U]]\subset[[x,U],U]\subset[C(U),U]=(0).$$

PROPOSITION 4. *Let L be the sum of the subalgebras $L_1,\cdots,L_n$ of L. Assume that $[L_i,L_j]\subset Z(L)$ for any $i\neq j$ and $Z(L_i)\subset Z(L)$ for any i. Then L is an (A_k)-algebra (resp. an (A_∞)-algebra) if and only if for each i L_i is an (A_k)-algebra (resp. an (A_∞)-algebra).*

PROOF. Suppose that L is an (A_k)-algebra. For any i, put $U_i=\sum_{j\neq i}L_j$. Then $L=L_i+U_i$ and

$$[L_i,[L_i,U_i]]\subset[L_i,Z(L)]=(0).$$

Hence by Proposition 3 we see that L_i is an (A_k)-algebra.

Conversely, suppose that for each i L_i is an (A_k)-algebra. Let $x \in L$ be such that $(\mathrm{ad}\,x)^kL=(0)$. Then $x=\sum_{i=1}^n x_i$ with $x_i \in L_i$. We assert that $(\mathrm{ad}\,x)^kL_i=(\mathrm{ad}\,x_i)^kL_i$ for each i. In fact,

$$(\mathrm{ad}\,x)^kL_i=(\mathrm{ad}\,x)^{k-1}[x_i+\sum_{j\neq i}x_j,L_i]=(\mathrm{ad}\,x)^{k-1}(\mathrm{ad}\,x_i)L_i,$$

since $[\sum_{j\neq i}x_j,L_i]\subset Z(L)$ and $k-1\geqq 1$. Since $[\sum_{j\neq i}x_j,L_i]\subset C(L_i)$, by Lemma 1 we have $[\sum_{j\neq i}x_j,(\mathrm{ad}\,x_i)L_i]=(0)$. Hence

$$(\mathrm{ad}\, x)^k L_i = (\mathrm{ad}\, x)^{k-2}[x_i + \sum_{j\neq i} x_j, (\mathrm{ad}\, x_i)L_i] \\ = (\mathrm{ad}\, x)^{k-2}(\mathrm{ad}\, x_i)^2 L_i.$$

Repeating this procedure, we have $(\mathrm{ad}\, x)^k L_i = (\mathrm{ad}\, x_i)^k L_i$, as was asserted. Since L_i has the property (A_k), it follows that $x_i \in Z(L_i) \subset Z(L)$. Thus $x \in Z(L)$, that is, $\mathrm{ad}\, x = 0$. Therefore L is an (A_k)-algebra. The statement for the (A_∞)-algebras is immediate from this and the proposition is proved.

As a special case of Proposition 4, we have the following

COROLLARY. *Let L be a Lie algebra over a field Φ. Assume that L is the sum of the subalgebras $L_1, \cdots, L_n$ of L such that $[L_i, L_j] = (0)$ for any $i \neq j$. Then L is an (A_k)-algebra if and only if for each i L_i is an (A_k)-algebra.*

PROOF. For any i, we have $[Z(L_i), \sum_{j\neq i} L_j] = (0)$. Hence $Z(L_i) \subset Z(L)$. The statement now follows from Proposition 4.

3. In this section we study the interrelation among the properties (A_k), (A_∞) and (A) of L in the case where L is solvable.

LEMMA 2. *If L is an (A_k)-algebra, then $Z_1(L) = Z(L)$.*

PROOF. By Proposition 1, L is an (A_2)-algebra. For any $x \in Z_1(L)$, $(\mathrm{ad}\, x)^2 L \subset (\mathrm{ad}\, x) Z(L) = (0)$. Hence $(\mathrm{ad}\, x)^2 = 0$ and therefore $\mathrm{ad}\, x = 0$, that is, $x \in Z(L)$. Thus $Z_1(L) = Z(L)$.

It is known ([1], [2]) that any solvable (A)-algebra over a field of characteristic 0 is abelian. We strengthen this in the following

THEOREM 1. (a) *If L is a nilpotent Lie algebra over Φ, then the properties* $(\mathrm{A}_2), \cdots, (\mathrm{A}_k), \cdots, (\mathrm{A}_\infty)$, (A) *and the property that L is abelian are equivalent.*

(b) *If L is a solvable Lie algebra over a field Φ of characteristic* 0, *then the properties* $(\mathrm{A}_3), \cdots, (\mathrm{A}_k), \cdots, (\mathrm{A}_\infty)$, (A) *and the property that L is abelian are equivalent.*

PROOF. If L is abelian, then L is obviously an (A)-algebra. Hence by Proposition 1, to prove the statement (a) (resp. (b)), it is sufficient for us to show that a nilpotent (A_2)-algebra (resp. a solvable (A_3)-algebra) is abelian.

If L is a nilpotent (A_2)-algebra, then $L = Z_n(L)$ with some n. But Lemma 2 tells us that $Z_1(L) = Z(L)$. Hence $L = Z(L)$ and L is abelian.

Now assume that L is a solvable (A_3)-algebra. Let N denote the nilradical of L. Then there exists some integer n such that $N^n \neq (0)$ and $N^{n+1} = (0)$. Now suppose that $n \geqq 2$. Then $3(n-1) \geqq n+1$. For any $x \in N^{n-1}$, we have

$$(\mathrm{ad}\, x)^3 L \subset (\mathrm{ad}\, x)^2 N^{n-1} \subset N^{3(n-1)} = (0).$$

Since L has the property (A_3), it follows that ad $x=0$. This shows that $N^{n-1} \subset Z(L)$. Hence $N^n=(0)$, which contradicts the choice of n. Thus $n=1$. For any $x \in N$,

$$(\operatorname{ad} x)^2 L \subset (\operatorname{ad} x) N \subset N^2=(0).$$

Since L is an (A_3)-algebra, we have ad $x=0$. From this it follows that $N \subset Z(L)$. Since a field Φ is of characteristic 0, $L^2 \subset N$ and therefore $L^3 \subset [L, N]=0$. Now we can use the statement (a) to conclude that L is abelian.

Thus the proof is complete.

To clarify the connection of the statements (a) and (b) in Theorem 1, we here give an example of solvable Lie algebras which have the property (A_2) but not the property (A_3), although the example given by S. Tôgô [3] is enough for this purpose. Let L be the Lie algebra over the field of real numbers described in terms of a basis e_1, e_2, e_3, e_4 by the following multiplication table:

$$[e_1, e_2]=e_3, \quad [e_1, e_3]=-e_2, \quad [e_2, e_3]=e_4.$$

In addition $[e_i, e_j]=-[e_j, e_i]$ and for $i>j$ $[e_i, e_j]=0$ if it is not in the table above. Then L is solvable but not nilpotent. By computation we see that the set of all the elements x of L such that $(\operatorname{ad} x)^2=0$ is (e_4) and that the set of all the elements x of L such that $(\operatorname{ad} x)^3=0$ is (e_2, e_3, e_4). Hence L is an (A_2)-algebra, but L is not an (A_3)-algebra.

4. This section will be devoted to the study of (A_2)-algebras.

THEOREM 2. *Let L be an (A_2)-algebra over a field of characteristic 0. Let N be the nil-radical of L. Then L is abelian, or L is reductive, or $N \supsetneqq Z(L) \supset N^2 \neq (0)$.*

PROOF. If L is nilpotent, by Theorem 1 (a) L is abelian. So we assume that L is not nilpotent. Let n be the integer such that $N^n \neq (0)$ and $N^{n+1}=(0)$. Suppose that $n \geqq 3$. Then $2(n-1) \geqq n+1$. For any $x \in N^{n-1}$,

$$(\operatorname{ad} x)^2 L \subset [x, N^{n-1}] \subset N^{2(n-1)}=(0).$$

Since L is an (A_2)-algebra, we have ad $x=0$. Thus we see that $N^{n-1} \subset Z(L)$ and therefore $N^n=(0)$, which contradicts the choice of n. Consequently, we have $n \leqq 2$. In the case where $n=1$, we have $N^2=(0)$. For any $x \in N$, $(\operatorname{ad} x)^2 L \subset [x, N]=(0)$ and therefore ad $x=0$. Thus we see that $N \subset Z(L)$. Denoting by R the radical of L, we have $R^3=(0)$ and therefore $R=N=Z(L)$. Since L is not nilpotent by our assumption, L is reductive. In the case where $n=2$, for any $x \in N^2$ we have

$$(\mathrm{ad}\,x)^2L\subset[N^2,\ N^2]\subset N^4=(0).$$

By the property (A_2) of L we see that $\mathrm{ad}\ x=0$. Hence $N^2\subset Z(L)$. Furthermore since $N^2\neq(0)$, we must have $Z(L)\neq N$. Thus the proof is complete.

Corollary. *Let L be a non-abelian solvable (A_2)-algebra over a field of characteristic 0. Then* $\dim L\geqq 4$.

Proof. By Theorem 1 (a), L is not nilpotent. If $\dim L\leqq 3$, the nil-radical N of L must be of dimension $\leqq 2$. It follows that N is abelian, which contradicts Theorem 2. Hence $\dim L\geqq 4$.

Proposition 5. *Let L be a non-solvable (A_2)-algebra over a field of characteristic 0 whose radical R is nilpotent. If L has no non-zero abelian direct summands, then either L is semisimple, or $L=L^2$ and $R^2=Z(L)$.*

Proof. By Theorem 2, either L is reductive or $R\supsetneqq Z(L)\supset R^2\neq(0)$. In the first case, L is semisimple since L has no non-zero abelian direct summands. In the second case, let $L=S+R$ be a Levi decomposition of L. Since $\mathrm{ad}\,S$ is completely reducible and since R and $Z(L)$ are stable under $\mathrm{ad}\,S$, there exists a non-zero subspace U such that

$$R=U+Z(L),\quad U\cap Z(L)=(0)\quad \text{and}\quad [S,\ U]\subset U.$$

It follows that $R^2=U^2$. If $R^2\neq Z(L)$, let V be a subspace of $Z(L)$ complementary to R^2. Since

$$L^2=S+[L,\ R]=S+[S,\ R]+R^2\subset S+U+R^2,$$

we have $V\cap L^2=(0)$ and V is then an abelian direct summand of L. Hence $V=(0)$ and therefore $R^2=Z(L)$. Now we assert that $[S,\ U]=U$. In fact, suppose that $[S,\ U]\neq U$. Since $\mathrm{ad}\,S$ is completely reducible and since U and $[S,\ U]$ are stable under $\mathrm{ad}\ S$, there exists a non-zero subspace U_1 such that

$$U=[S,\ U]+U_1,\quad [S,\ U]\cap U_1=(0)\quad \text{and}\quad [S,\ U_1]\subset U_1.$$

For any non-zero element u_1 of U_1, we have $[u_1,\ S]\subset[S,\ U]\cap U_1=(0)$. Moreover $(\mathrm{ad}\,u_1)R\subset R^2=Z(L)$. Therefore $(\mathrm{ad}\,u_1)^2=0$. By the property (A_2) of L, $\mathrm{ad}\,u_1=0$, that is, $u_1\in Z(L)$. Thus $u_1=0$, which is a contradiction. Hence $[S,\ U]=U$, as was asserted. It is now easy to see that $L=L^2$ and the proof is complete.

Next we shall show a sufficient condition for L to be an (A_2)-algebra.

Proposition 6. *Let L be a non-nilpotent solvable Lie algebra over a field of characteristic 0. Let the nil-radical N of L be such that $N\supsetneqq Z(L)\supset N^2\neq(0)$ and $(\mathrm{ad}\,x)^2\neq 0$ for any $x\in N\setminus Z(L)$. If there exists a subspace U of L complementary to N such that $\mathrm{ad}\,U$ is a commutative set of semisimple elements, then*

L is an (A_2)*-algebra.*

PROOF. Since $\operatorname{ad} U$ is completely reducible, there exists a subspace V such that

$$N=V+Z(L),\quad V\cap Z(L)=(0)\quad \text{and}\quad [U, V]\subset V.$$

Assume that $(\operatorname{ad} x)^2=0$ for $x\in L$. Then x is expressed as

$$x=u+v+z\quad \text{with}\quad u\in U,\quad v\in V\quad \text{and}\quad z\in Z(L).$$

By using the fact that $N^2\subset Z(L)$, we infer for any $v_1\in V$

$$(\operatorname{ad} x)^2 v_1=[u+v, [u, v_1]+[v, v_1]]$$

$$=[u+v, [u, v_1]]=(\operatorname{ad} u)^2 v_1+[v, [u, v_1]],$$

from which it follows that

$$(\operatorname{ad} u)^2 v_1=-[v, [u, v_1]]\in V\cap Z(L)=(0).$$

Hence $(\operatorname{ad} u)^2 V=(0)$. Since $\operatorname{ad} U$ is commutative, $[U, U]\subset Z(L)$. Hence $(\operatorname{ad} u)^2 U\subset(\operatorname{ad} u)Z(L)=(0)$. It follows that $(\operatorname{ad} u)^2=0$. But $\operatorname{ad} u$ is semisimple. Therefore $\operatorname{ad} u=0$, that is, $u\in Z(L)$. So we have $u=0$. If $v\neq 0$, then by our hypothesis $(\operatorname{ad} x)^2=(\operatorname{ad} v)^2\neq 0$. Hence $v=0$ and therefore $x=z\in Z(L)$. Thus we conclude that L is an (A_2)-algebra.

It is to be noted that the conditions in Proposition 6 are satisfied by the example of S. Tôgô [3] and the example in Section 3.

Here we shall give an example of non-solvable (A_2)-algebras. Let L be the Lie algebra over a field of characteristic 0 described in terms of a basis $e_1, e_2, \cdots, e_6$ by the table:

$$[e_1, e_2]=2e_2,\ [e_1, e_3]=-2e_3,\ [e_2, e_3]=e_1,\ [e_1, e_4]=-e_4,$$

$$[e_1, e_5]=e_5,\ [e_2, e_4]=-e_5,\ [e_3, e_5]=-e_4,\ [e_4, e_5]=e_6.$$

In addition $[e_i, e_j]=-[e_j, e_i]$ and for $i<j$ $[e_i, e_j]=0$ if it is not in the table above. The radical $R=(e_4, e_5, e_6)$ is nilpotent. By computation, we see that the set of all $x\in L$ such that $(\operatorname{ad} x)^2=0$ is equal to $Z(L)=(e_6)$. Therefore L is an (A_2)-algebra. L obviously satisfies the conditions that $L=L^2$ and $R\supsetneqq Z(L)=R^2\neq(0)$.

5. In this section we shall give a characterization of (A_k)-algebras for $k\geqq 3$.

THEOREM 3. *Let L be a Lie algebra over a field of characteristic* 0. *Let k be an integer* $\geqq 3$. *Then L is an* (A_k)*-algebra if and only if either L is abelian, or L is reductive and L^2 contains no non-zero elements x such that* $(\operatorname{ad} x)^k=0$.

Proof. By Theorem 1 (b), a solvable Lie algebra is an (A_k)-algebra if and only if it is abelian. Hence we may restrict ourselves to the case where L is not solvable.

Assume that L is an (A_3)-algebra. We denote by R and N respectively the radical and the nil-radical of L. Since L is an (A_2)-algebra by Proposition 1, by using Theorem 2 we see that either L is reductive, or $N \supsetneqq Z(L) \supset N^2 \neq (0)$. In the case where $N \supsetneqq Z(L) \supset N^2 \neq (0)$, for any $x \in N$

$$(\operatorname{ad} x)^3 L \subset (\operatorname{ad} x)^2 N \subset N^3 = (0).$$

It follows that $\operatorname{ad} x = 0$. Hence we have $N \subset Z(L)$, which is a contradiction. Thus L is necessarily reductive. Then L is the direct sum of a semisimple ideal S and the center. It follows that $S = L^2$. By Corollary to Proposition 4 S is an (A_3)-algebra. Since $Z(S) = (0)$, S contains no non-zero elements x such that $(\operatorname{ad} x)^3 = 0$. The converse follows from Corollary to Proposition 4 and Theorem 1 (b).

The proofs of the cases where $k > 3$ can be carried out in the similar way as in the above case.

As a consequence of Theorem 3 we prove the following result of M. Sugiura [2] which was stated in Introduction.

Corollary. *A Lie algebra L over a field of characteristic* 0 *is an* (A)-*algebra if and only if L is an* (A_∞)-*algebra.*

Proof. Assume that L is an (A_∞)-algebra. Then by Proposition 1 L is an (A_k)-algebra for any $k \geqq 2$. Hence by Theorem 3 either L is abelian, or L is reductive and $\operatorname{ad} L$ contains no nilpotent elements. In the first case, L is an (A)-algebra. In the second case, ad L is semisimple and therefore splittable. Hence ad L consists of only the semisimple elements. If any elements x, y of L satisfies $[x, [x, y]] = 0$, then $(\operatorname{ad} x)^2 y = 0$. Since $\operatorname{ad} x$ is semisimple, it follows that $(\operatorname{ad} x) y = 0$. Hence L is an (A)-algebra. Since any (A)-algebra is an (A_∞)-algebra by Proposition 1, the statement is proved.

It has been shown by M. Sugiura [2] that a Lie algebra over an algebraically closed field of characteristic 0 is an (A)-algebra if and only if it is abelian. In order to strengthen this result, we first prove

Lemma 3. *Let L be a split semisimple Lie algebra over a field of characteristic* 0. *Then L is not an* (A_3)-*algebra.*

Proof. Let H be a splitting Cartan subalgebra of L and let $L = H + \sum_{\alpha} L_\alpha$ be the decomposition of L to the root spaces. We can write $L_\alpha = (e_\alpha)$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be the simple system of roots for L relative to H. Then any root α is expressed in the form $\alpha = \sum k_i \alpha_i$ with k_i integers. We choose the root β which is maximal in the lexicographic ordering of the roots determined by $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$. Then $(\mathrm{ad}\, e_\beta)^2 H=(0)$ and $(\mathrm{ad}\, e_\beta)^3 L_\alpha=(0)$ for every root α, since 2β and $3\beta+\alpha$ are not roots. Hence $(\mathrm{ad}\, e_\beta)^3=0$. Thus L is not an (A_3)-algebra.

THEOREM 4. *A Lie algebra L over an algebraically closed field of characteristic* 0 *is an* (A_3)-*algebra if and only if L is abelian.*

PROOF. Assume that L is an (A_3)-algebra. Then by Theorem 3 either L is abelian, or L is reductive and L^2 is an (A_3)-algebra. If L is reductive then L^2 is semisimple and therefore from Lemma 3 it follows that we cannot have the latter case. Hence L is abelian. The converse is evident.

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