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Geometrical Association Schemes and Fractional Factorial Designs

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1. Summary

In this paper an attempt is made to throw light on the algebraic structure of symmetrical s^{k-p}-fractional factorial designs, where s is not necessary 2 but a prime power. For such purpose a geometrical factorial association scheme of PG$(k-1, s)$-type and the corresponding s^{k-p}-fractional factorial association scheme are introduced in sections 2 and 3 respectively. The corresponding association algebras $\mathfrak{A}(\mathrm{PG}(k-1, s))$ and $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ are also introduced there.

Mutually orthogonal idempotents of those algebras are given in section 4. The notion of fractionally similar mapping is introduced in section 5 and the relationship between $\mathfrak{A}(\mathrm{PG}(k-1, s))$ and $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ is investigated there. A general definition of the classical notion of aliases is given in section 6. Blocking of the fractional factorial designs is discussed in section 7 in relation to the notion of partial confounding and the pseudo-block factors.

The following notation is used throughout this paper:

I_n : The unit matrix of order n.

G_n : An $n\times n$ matrix whose elements are all unity.

A' : Transpose of a matrix A.

$A\otimes B$: Kronecker product of the matrices $A=||a_{ij}||$ and B, i.e., $A\otimes B=||a_{ij}B||$.

$[A_i;\ i=1, \cdots, m]$: An algebra generated by the linear closure of those matrices indicated in the [ ].

GF(s) : A finite field consists of $s\ (=q^u)$ elements, where q is a prime integer and u is a positive integer. An element a in GF(s) is represented by the coordinate representation or polynomial representation, i.e., $a=\langle a^{(1)}, \cdots, a^{(u)}\rangle$ where $a^{(i)}$ is an element of GF(q), $i=1, 2, \cdots, u$.

EG(k, s) : A k-dimensional Euclidean space over GF(s).

PG$(k-1, s)$: A $k-1$-dimensional projective space over GF(s).

$\mathfrak{P}(A)$: A subspace of PG$(k-1, s)$ generated by the linear closure of column vectors of a matrix A.

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$\overline{\mathrm{PG}}(k-1, s)$: A space containing $\mathrm{PG}(k-1, s)$ and an additional null vector $\mathbf{0}$ in $\mathrm{EG}(k, s)$, i.e., $\overline{\mathrm{PG}}(k-1, s) = \mathrm{PG}(k-1, s)^{\vee}\{\mathbf{0}\}$.

$\overline{\mathfrak{P}}(A)$: $\overline{\mathfrak{P}}(A) = \mathfrak{P}(A)^{\vee}\{\mathbf{0}\}$

$\mathfrak{P}^{\vee}\mathfrak{Q}$: The smallest subspace containing both subspaces $\mathfrak{P}$ and $\mathfrak{Q}$ in $\mathrm{PG}(k-1, s)$.

$\mathfrak{P}_{\wedge}\mathfrak{Q}$: The largest subspace contained in both subspaces $\mathfrak{P}$ and $\mathfrak{Q}$ in $\mathrm{PG}(k-1, s)$.

$\boldsymbol{a} = (a_1, \cdots, a_k)'$: A point in $\mathrm{EG}(k, s)$. Latin letters are used exclusively for the points in $\mathrm{EG}(k, s)$.

$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k)'$: A point in $\mathrm{PG}(k-1, s)$. Greek letters are used exclusively for the points in $\mathrm{PG}(k-1, s)$.

2. $\mathrm{PG}(k-1, s)$-association scheme

Suppose that there are $v_k = s^k$ objects or treatments $\phi(\boldsymbol{a})$ indexed by the points $\boldsymbol{a}$ in $\mathrm{EG}(k, s)$. Among those v_k treatments an association of geometrical type is defined as follows:

Definition: Two treatments $\phi(\boldsymbol{a})$ and $\phi(\boldsymbol{b})$ are $\boldsymbol{\alpha}$-th associates, if the difference of these indices $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfies the relation

$$\boldsymbol{a} - \boldsymbol{b} = \rho\boldsymbol{\alpha} \tag{2.1}$$

where $\rho \in \mathrm{GF}(s)$, $\rho \neq 0$, and $\boldsymbol{\alpha} \in \mathrm{PG}(k-1, s)$. Each treatment is the $\boldsymbol{0}$-th associate of itself.

The association defined above satisfies three conditions of the association scheme with $m_k = (s^k - 1)/(s-1)$ associate classes:

(i) Any two treatments are either $\boldsymbol{\alpha}_1$-th, $\boldsymbol{\alpha}_2$-th, $\cdots$, or $\boldsymbol{\alpha}_{m_k}$-th associates, where $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_{m_k} \in \mathrm{PG}(k-1, s)$, the relation of association being symmetrical.

(ii) Each treatment $\phi(\boldsymbol{a})$ has

$$n_{\boldsymbol{\alpha}} = s - 1 \tag{2.2}$$

$\boldsymbol{\alpha}$-th associates, the number $n_{\boldsymbol{\alpha}}$ being independent of $\phi(\boldsymbol{a})$.

(iii) If any two treatments $\phi(\boldsymbol{a})$ and $\phi(\boldsymbol{b})$ are $\boldsymbol{\alpha}$-th associates, then the number of treatments which are $\boldsymbol{\beta}$-th associates of $\phi(\boldsymbol{a})$ and at the same time $\boldsymbol{\gamma}$-th associates of $\phi(\boldsymbol{b})$ is $p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\boldsymbol{\gamma}}$ and it is independent of the pair of $\boldsymbol{\alpha}$-th associates $\phi(\boldsymbol{a})$ and $\phi(\boldsymbol{b})$. In fact, when $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are in $\mathrm{PG}(k-1, s)$,

$$p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta}, \boldsymbol{\gamma};\ \boldsymbol{\beta} \neq \boldsymbol{\gamma} \text{ and } \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \text{ are} \\ & \text{collinear in } \mathrm{PG}(k-1, s) \\ s-2 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

and when some of the $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are zero,

$$p^{\mathbf{0}}_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \begin{cases} s-1 & \text{if } \boldsymbol{\beta}=\boldsymbol{\gamma}\neq\mathbf{0} \\ 1 & \text{if } \boldsymbol{\beta}=\boldsymbol{\gamma}=\mathbf{0} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

$$p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\mathbf{0}} = p^{\boldsymbol{\alpha}}_{\mathbf{0}\boldsymbol{\beta}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}=\boldsymbol{\beta} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We call this association scheme PG$(k-1,\ s)$-*association scheme* with $(s^k-1)/(s-1)$ associate classes.

In the case $k=2$, this scheme is an association of orthogonal Latin square type or OL$_r$ type with $s+1$ associate classes [5], where $r=s+1$. After numbering v_k indices in some way but once for all, the association matrices can be expressed as follows:

$$A_{\boldsymbol{\alpha}} = \|a^{\boldsymbol{b}}_{\boldsymbol{a}\boldsymbol{\alpha}}\| \tag{2.6}$$

where

$$a^{\boldsymbol{b}}_{\boldsymbol{a}\boldsymbol{\alpha}} = \begin{cases} 1 & \text{if the treatments } \phi(\boldsymbol{a}) \text{ and } \phi(\boldsymbol{b}) \text{ are} \\ & \boldsymbol{\alpha}\text{-th associates} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the treatments $\phi(\boldsymbol{a})$ are numbered lexicographically by indices $\boldsymbol{a}$, i.e., numbered with respect to the coordinates in EG$(k,\ s)$ and each coordinate element is numbered with respect to the coordinates in GF(s), then the association matrices $A_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha}\in$ PG$(k-1,\ s)$ have the following forms:

$$A_{\boldsymbol{\alpha}} = \sum_{\substack{\rho\in \mathrm{GF}(s)\\ \rho\neq 0}} P^{\rho\boldsymbol{\alpha}} \tag{2.7}$$

where P is a $q\times q$ permutation matrix, $P^{\rho\boldsymbol{\alpha}} = \prod_{i=1}^{k}\otimes P^{\rho\alpha_i}$, $P^{\rho\alpha_i} = \prod_{l=1}^{u}\otimes P^{\gamma_i^{(l)}}$ and $\rho\alpha_i = <\gamma_i^{(1)},\ \cdots,\ \gamma_i^{(u)}>$, $\gamma_i^{(l)}\in$ GF(q), $i=1,\ \cdots,\ k$, $l=1,\ \cdots,\ u$.

$A_{\boldsymbol{\alpha}}$'s are all symmetric $v_k\times v_k$ matrices and satisfy the following relations:

$$\begin{cases} \displaystyle\sum_{\boldsymbol{\alpha}\in\overline{\mathrm{PG}}(k-1,s)} A_{\boldsymbol{\alpha}} = G_{v_k} \\ A_{\boldsymbol{\alpha}}A_{\boldsymbol{\beta}} = A_{\boldsymbol{\beta}}A_{\boldsymbol{\alpha}} = \displaystyle\sum_{\boldsymbol{\gamma}\in\overline{\mathrm{PG}}(k-1,s)} p^{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}A_{\boldsymbol{\gamma}} \end{cases} \tag{2.8}$$

From (2.3), (2.4), (2.5) and (2.8), we have

$$\begin{cases} A_{\mathbf{0}}A_{\boldsymbol{\alpha}} = A_{\boldsymbol{\alpha}}A_{\mathbf{0}} = A_{\boldsymbol{\alpha}} & \text{for } \boldsymbol{\alpha}\in\overline{\mathrm{PG}}(k-1,\ s) \\ A_{\boldsymbol{\alpha}}^2 = (s-1)A_{\mathbf{0}} + (s-2)A_{\boldsymbol{\alpha}} & \text{for } \boldsymbol{\alpha}\in \mathrm{PG}(k-1,\ s) \\ A_{\boldsymbol{\alpha}}A_{\boldsymbol{\beta}} = A_{\boldsymbol{\beta}}A_{\boldsymbol{\alpha}} = \displaystyle\sum_{\boldsymbol{\gamma}\in\mathfrak{P}(\boldsymbol{\alpha},\boldsymbol{\beta}),\boldsymbol{\gamma}\neq\boldsymbol{\alpha},\boldsymbol{\beta}} A_{\boldsymbol{\gamma}} & \text{for } \boldsymbol{\alpha},\ \boldsymbol{\beta}\in \mathrm{PG}(k-1,\ s) \text{ and } \boldsymbol{\alpha}\neq\boldsymbol{\beta}. \end{cases} \tag{2.9}$$

From (2.9), we have

$$\begin{cases} (A_0+A_\alpha)^2 = s(A_0+A_\alpha) & \text{for} \quad \alpha \in \mathrm{PG}(k-1, s) \\ (A_0+A_\alpha)(A_0+A_\beta) = A_0 + \sum_{\gamma \in \mathfrak{P}(\alpha,\beta)} A_\gamma & \text{for} \quad \alpha, \beta \in \mathrm{PG}(k-1, s) \text{ and } \alpha \neq \beta. \end{cases} \tag{2.9'}$$

We call this association algebra generated by the above association matrices, PG$(k-1, s)$-association algebra and denote it as

$$\mathfrak{A}(\mathrm{PG}(k-1, s)) = [A_\alpha;\ \alpha \in \overline{\mathrm{PG}}(k-1, s)].$$

3. s^{k-p}-fractional factorial association scheme

Let $F=\|f_{ij}\|$ be a $p\times k(p<k)$ matrix over GF(s), whose rank is p. All points which satisfy the consistent and independent linear equation

$$Fx = f \tag{3.1}$$

form a $(k-p)$-dimensional subspace, or $(k-p)$-flat in EG(k, s) where $x=(x_1, \cdots, x_k)' \in \mathrm{EG}(k, s)$ and $f=(f_1, \cdots, f_p)' \in \mathrm{EG}(p, s)$. We denote the $(k-p)$-flat in EG(k, s) as

$$\mathfrak{F}^{k-p} = \{x \mid Fx = f,\ x \in \mathrm{EG}(k, s)\}. \tag{3.2}$$

Among those v_k treatments $\{\phi(a)\}$, if we select and consider a fractional set of $v=s^{k-p}$ treatments $\{\phi(x)\}$ indexed by the points x in $\mathfrak{F}^{k-p}$, we can introduce into this fractional set of treatments an association induced by the PG$(k-1, s)$-association scheme in such a way that any two treatments $\phi(x)$ and $\phi(y)$ are α-th associates if two indices x and y $(x, y \in \mathfrak{F}^{k-p})$ are α-th associate in original PG$(k-1, s)$-association scheme.

If $\phi(x)$ and $\phi(y)$ $(x, y \in \mathfrak{F}^{k-p})$ are α-th $(\alpha \neq 0)$ associates, since α must satisfy the relation $F\alpha = 0$, the set of all such α forms a $(k-1-p)$-flat in PG$(k-1, s)$, i.e.,

$$\mathfrak{P}^{k-1-p} = \{\alpha \mid F\alpha = 0,\ \alpha \in \mathrm{PG}(k-1, s)\}.$$

It can be seen that the relation of association thus introduced in the fractional set of treatments satisfies three conditions of the association scheme with $m_{k-p}=(s^{k-p}-1)/(s-1)$ associate classes:

(i) Any two treatments $\phi(x)$ and $\phi(y)$ $(x, y \in \mathfrak{F}^{k-p})$ are either α_1-th, α_2-th, $\cdots$, $\alpha_{m_{k-p}}$-th associates, where $\alpha_1, \cdots, \alpha_{m_{k-p}} \in \mathfrak{P}^{k-1-p}$, the relation of association being symmetrical.

(ii) Each treatment $\phi(x)$ has

$$n_\alpha = s-1 \tag{3.3}$$

α-th associates, the number n_α being independent of $\phi(x)$.

(iii) If any two treatments $\phi(\boldsymbol{x})$ and $\phi(\boldsymbol{y})$ are $\boldsymbol{\alpha}$-th associates, then the number of treatments which are $\boldsymbol{\beta}$-th associates of $\phi(\boldsymbol{x})$ and at the same time $\boldsymbol{\gamma}$-th associates of $\phi(\boldsymbol{y})$ is $p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta\gamma}}$ and it is independent of the pair of $\boldsymbol{\alpha}$-th associates $\phi(\boldsymbol{x})$ and $\phi(\boldsymbol{y})$. In fact, when $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are in $\mathfrak{P}^{k-1-p}$,

$$p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta\gamma}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta}, \boldsymbol{\gamma};\ \boldsymbol{\beta} \neq \boldsymbol{\gamma} \text{ and } \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \text{ are collinear in } \mathfrak{P}^{k-1-p} \\ s-2 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

and when some of the $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are zero,

$$p^{\mathbf{0}}_{\boldsymbol{\beta\gamma}} = \begin{cases} s-1 & \text{if } \boldsymbol{\beta} = \boldsymbol{\gamma} \neq \mathbf{0} \\ 1 & \text{if } \boldsymbol{\beta} = \boldsymbol{\gamma} = \mathbf{0} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

and

$$p^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\mathbf{0}} = p^{\boldsymbol{\alpha}}_{\mathbf{0}\boldsymbol{\beta}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

We call this scheme *s^{k-p}-fractional factorial association scheme* with m_{k-p} associate classes induced by the PG$(k-1, s)$-association scheme. Thus we have the following theorem.

THEOREM 1. *The induced relation of association of the* PG$(k-1, s)$*-association scheme defined among a fractional set of s^{k-p} treatments* $\{\phi(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathfrak{F}^{k-p}\}$, *satisfies the three conditions of the association scheme with m_{k-p} associate classes. The parameters of the s^{k-p}-fractional factorial association scheme are given in* (3.3), (3.4), (3.5) *and* (3.6).

After numbering v indices $\boldsymbol{x}$'s in some way but once for all, the association matrices can be expressed as follows:

$$B_{\boldsymbol{\alpha}} = \|a^{\boldsymbol{y}}_{\boldsymbol{x\alpha}}\| \tag{3.7}$$

where

$$a^{\boldsymbol{y}}_{\boldsymbol{x\alpha}} = \begin{cases} 1 & \text{if the treatments } \phi(\boldsymbol{x}) \text{ and } \phi(\boldsymbol{y}) \text{ are } \boldsymbol{\alpha}\text{-th associates} \\ 0 & \text{otherwise.} \end{cases}$$

$B_{\boldsymbol{\alpha}}$'s are all symmetric $v \times v$ matrices and satisfy the following relations:

$$\begin{cases} \sum_{\boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p}} B_{\boldsymbol{\alpha}} = G_v \\ B_{\boldsymbol{\alpha}} B_{\boldsymbol{\beta}} = B_{\boldsymbol{\beta}} B_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma} \in \mathfrak{P}^{k-1-p}} p^{\boldsymbol{\gamma}}_{\boldsymbol{\alpha\beta}} B_{\boldsymbol{\gamma}} \end{cases} \tag{3.8}$$

From (3.4), (3.5), (3.6) and (3.8), we have

$$\begin{cases} B_0B_{\boldsymbol{\alpha}}=B_{\boldsymbol{\alpha}}B_0=B_{\boldsymbol{\alpha}} & \text{for} \quad \boldsymbol{\alpha}\in\mathfrak{P}^{k-1-p} \\ B_{\boldsymbol{\alpha}}^2=(s-1)B_0+(s-2)B_{\boldsymbol{\alpha}} & \text{for} \quad \boldsymbol{\alpha}\in\mathfrak{P}^{k-1-p} \\ B_{\boldsymbol{\alpha}}B_{\boldsymbol{\beta}}=B_{\boldsymbol{\beta}}B_{\boldsymbol{\alpha}}=\sum\limits_{\boldsymbol{\gamma}\in\mathfrak{P}(\boldsymbol{\alpha},\boldsymbol{\beta}),\boldsymbol{\gamma}\neq\boldsymbol{\alpha},\boldsymbol{\beta}} B_{\boldsymbol{\gamma}} & \text{for} \quad \boldsymbol{\alpha},\,\boldsymbol{\beta}\in\mathfrak{P}^{k-1-p} \text{ and } \boldsymbol{\alpha}\neq\boldsymbol{\beta}. \end{cases} \tag{3.9}$$

From (3.9), we have

$$\begin{cases} (B_0+B_{\boldsymbol{\alpha}})^2=s(B_0+B_{\boldsymbol{\alpha}}) & \text{for} \quad \boldsymbol{\alpha}\in\mathfrak{P}^{k-1-p} \\ (B_0+B_{\boldsymbol{\alpha}})(B_0+B_{\boldsymbol{\beta}})=B_0+\sum\limits_{\boldsymbol{\gamma}\in\mathfrak{P}(\boldsymbol{\alpha},\boldsymbol{\beta})} B_{\boldsymbol{\gamma}} & \text{for } \boldsymbol{\alpha},\,\boldsymbol{\beta}\in\mathfrak{P}^{k-1-p} \text{ and } \boldsymbol{\alpha}\neq\boldsymbol{\beta}. \end{cases} \tag{3.9$'$}$$

We call this association algebra generated by the above association matrices s^{k-p}-fractional factorial association algebra and denote it as

$$\mathfrak{A}(s^{k-p}-\mathrm{Fr})=[B_{\boldsymbol{\alpha}};\,\boldsymbol{\alpha}\in\mathfrak{P}^{k-1-p}].$$

4. Mutually orthogonal idempotents of $\mathfrak{A}(\mathrm{PG}(k-1,\,s))$ and $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$

It is well-known that an association algebra is commutative and completely reducible and each of its minimum two sided ideals is linear over the field containing all characteristic roots of the matrices [2], [3].

We wish to find the principal idempotent matrices $A_{\boldsymbol{\alpha}}^{\#}$ and $B_{\boldsymbol{\beta}}^{\#}$ of the minimum two sided ideals of the association algebras $\mathfrak{A}(\mathrm{PG}(k-1,\,s))$ and $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ defined in sections 2 and 3.

Let $\tilde{F}$ be the $(k-p)\times k$ matrix over $\mathrm{GF}(s)$ such that the rank of the matrix $(F';\,\tilde{F}')$ is equal to k, for the matrix F defined in section 3.

The following two lemmas are useful in obtaining the mutually orthogonal principal idempotents $A_{\boldsymbol{\alpha}}^{\#}$ and $B_{\boldsymbol{\beta}}^{\#}$ the algebras $\mathfrak{A}(\mathrm{PG}(k-1,\,s))$ and $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$, respectively.

LEMMA 1.

(i) *The matrix* $s^{-1}(A_0+A_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha}\in\mathrm{PG}(k-1,\,s))$ *is idempotent.*

(ii) *For any* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ $(\neq\boldsymbol{\alpha})$ *in* $\mathrm{PG}(k-1,\,s)$,

$$s^{-2}(A_0+A_{\boldsymbol{\alpha}})(A_0+A_{\boldsymbol{\beta}})=\prod_{\boldsymbol{\gamma}\in\mathfrak{P}(\boldsymbol{\alpha},\boldsymbol{\beta})}(s^{-1}(A_0+A_{\boldsymbol{\gamma}})). \tag{4.1}$$

(iii) *When* $\mathfrak{P}$ *is a flat generated by* l *linearly independent points* $\boldsymbol{\alpha}_1,\,\cdots,\,\boldsymbol{\alpha}_l$ *in* $\mathrm{PG}(k-1,\,s)$, i.e., $\mathfrak{P}=\mathfrak{P}(\boldsymbol{\alpha}_1,\,\cdots,\,\boldsymbol{\alpha}_l)$, *we have*

$$\prod_{i=1}^{l}(s^{-1}(A_0+A_{\boldsymbol{\alpha}_i}))=\prod_{\boldsymbol{\gamma}\in\mathfrak{P}}(s^{-1}(A_0+A_{\boldsymbol{\gamma}}))=s^{-l}(A_0+\sum_{\boldsymbol{\gamma}\in\mathfrak{P}}A_{\boldsymbol{\gamma}}).$$

In particular, when $\mathfrak{P}=\mathrm{PG}(k-1,\,s)$, *we have* (4.2)

$$\prod_{\boldsymbol{\alpha}\in\mathrm{PG}(k-1,s)}(s^{-1}(A_0+A_{\boldsymbol{\alpha}}))=v_k^{-1}G_{v_k} \tag{4.3}$$

(iv) *For any two flats* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *in* $\mathrm{PG}(k-1,\,s)$, *we have*

$$\prod_{\alpha\in\mathfrak{P}}(s^{-1}(A_0+A_\alpha))\cdot\prod_{\beta\in\mathfrak{Q}}(s^{-1}(A_0+A_\beta))=\prod_{\alpha\in\mathfrak{P}\cup\mathfrak{Q}}(s^{-1}(A_0+A_\alpha)). \tag{4.4}$$

PROOF. (i) is an immediate consequence of (2.9′). The latter half of (2.9′) shows that the product of $s^{-1}(A_0+A_\alpha)$ and $s^{-1}(A_0+A_\beta)$ is independent of the pair $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Hence we have

$$s^{-1}(A_0+A_\alpha)s^{-1}(A_0+A_\beta)s^{-1}(A_0+A_\gamma)=s^{-1}(A_0+A_\alpha)s^{-1}(A_0+A_\beta)$$

for any $\boldsymbol{\gamma}\in\mathfrak{P}(\boldsymbol{\alpha},\boldsymbol{\beta})$. This implies (ii).

The former half of (iii) can be proved inductively by using (ii). The latter half of (iii) can be proved by induction with respect to the number of independent points in $\mathfrak{P}$. In fact, (2.9′) shows that the formula certainly holds for $l=2$. Assuming the formula holds for $l=n-1$ and using (2.9), we have

$$\begin{aligned}\prod_{i=1}^{n}(s^{-1}(A_0+A_{\alpha_i}))&=\prod_{i=1}^{n-1}(s^{-1}(A_0+A_{\alpha_i}))\cdot s^{-1}(A_0+A_{\alpha_n})\\&=s^{-(n-1)}(A_0+\sum_{\alpha\in\mathfrak{P}(\alpha_1,\dots,\alpha_{n-1})}A_\alpha)\cdot s^{-1}(A_0+A_{\alpha_n})\\&=s^{-n}(A_0+\sum_{\alpha\in\mathfrak{P}(\alpha_1,\dots,\alpha_n)}A_\alpha)\end{aligned}$$

for $l=n$, thus we have (4.2).

When $\mathfrak{P}=\mathrm{PG}(k-1,s)$, the formula (2.8) shows that the formula (4.3) holds.

(iv) can be proved from (iii) by selecting independent base points which generate $\mathfrak{P}\cap\mathfrak{Q}$ first and then selecting the remaining base points in order to generate $\mathfrak{P}$ and $\mathfrak{Q}$, respectively.

LEMMA 2.

(i) *For any flat* $\mathfrak{P}$ *in* PG$(k-1, s)$, *the matrix*

$$\prod_{\alpha\in\mathfrak{P}}(s^{-1}(A_0+A_\alpha))-v_k^{-1}G_{v_k}\quad\textit{is idempotent.}$$

(ii) *For any flats* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *in* PG$(k-1, s)$, *the matrices*

$$\prod_{\alpha\in\mathfrak{P}}(s^{-1}(A_0+A_\alpha))-v_k^{-1}G_{v_k}\quad\textit{and}\quad\prod_{\beta\in\mathfrak{Q}}(s^{-1}(A_0+A_\beta))-v_k^{-1}G_{v_k}$$

are mutually orthogonal if and only if $\mathfrak{P}\cup\mathfrak{Q}=\mathrm{PG}(k-1, s)$.

PROOF. From (4.3) and (4.4), we have (i). The sufficiency of (ii) is obvious and the necessity of (ii) follows from the linear independency of the association matrices.

Duality of the projective space shows that the correspondence $\boldsymbol{\alpha}\leftrightarrow\mathfrak{P}_\alpha$ is one-to-one for any $\boldsymbol{\alpha}\in\mathrm{PG}(k-1, s)$ and a $(k-2)$-flat $\mathfrak{P}_\alpha$ defined by

$$\mathfrak{P}_\alpha=\{\boldsymbol{\beta}\,|\,\boldsymbol{\alpha}'\boldsymbol{\beta}=0,\ \boldsymbol{\beta}\in\mathrm{PG}(k-1, s)\}.$$

Utilizing the correspondence, if we define a set of m_k idempotent matrices as

$$A_{\boldsymbol{\alpha}}^{\#}=\prod_{\gamma \in \mathfrak{P}_{\boldsymbol{\alpha}}}\left(s^{-1}(A_{\mathbf{0}}+A_{\gamma})\right)-v_k^{-1} G_{v_k} \tag{4.5}$$

and define $A_{\mathbf{0}}^{\#}=v_k^{-1} G_{v_k}$, then we have the theorem.

THEOREM 2. *The set of idempotent matrices* $\{A_{\boldsymbol{\alpha}}^{\#} \mid \boldsymbol{\alpha} \in \overline{\mathrm{PG}}(k-1, s)\}$ *is the set mutually orthogonal principal idempotent matrices of the minimum two sided ideals of* $\mathrm{PG}(k-1, s)$*-association algebra.*

PROOF. Orthogonality of those matrices can easily be proved by using (ii) of Lemma 2. Using (2.9) and (2.9′), we can prove that each $[A_{\boldsymbol{\alpha}}^{\#}]$ $(\boldsymbol{\alpha} \in \overline{\mathrm{PG}}(k-1, s))$ is a principal two sided ideal of $\mathfrak{A}(\mathrm{PG}(k-1, s))$.

Using (4.2) and (2.8) we have the following formula:

$$\begin{cases} A_{\boldsymbol{\alpha}}^{\#}=v_k^{-1}\{(s-1) \sum\limits_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}} A_{\boldsymbol{\beta}}-\sum\limits_{\boldsymbol{\beta} \notin \mathfrak{P}_{\boldsymbol{\alpha}}} A_{\boldsymbol{\beta}}\} & \boldsymbol{\alpha} \in \mathrm{PG}(k-1, s) \\ A_{\mathbf{0}}^{\#}=v_k^{-1} G_{v_k}. & \end{cases} \tag{4.6}$$

Since s^{k-p}-fractional factorial association algebra has the same algebraic structure as that of $\mathrm{PG}(k-1, s)$-association algebra except some small changes such as the range of indexing points of the former is restricted to a $(k-1-p)$-flat $\mathfrak{P}^{k-1-p}$ in $\mathrm{PG}(k-1, s)$, etc.. We, therefore, do not describe the versions of Lemmas 1 and 2 here.

For any $\boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}')$, i.e., for any point in $\mathrm{PG}(k-1, s)$ which is independent of all points in $\mathfrak{P}(F')$, there corresponds a $(k-2-p)$-flat

$$\mathfrak{P}_{\boldsymbol{\beta}}^{k-2-p}=\{\boldsymbol{\alpha} \mid \boldsymbol{\beta}'\boldsymbol{\alpha}=0, \quad \boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p}\}$$

in $\mathfrak{P}^{k-1-p}$. It can be seen that the correspondence

$$\boldsymbol{\beta} \leftrightarrow \mathfrak{P}_{\boldsymbol{\beta}}^{k-2-p}$$

is one-to-one.

Utilizing the correspondence, if we define a set of m_{k-p} idempotent matrices as

$$B_{\boldsymbol{\beta}}^{\#}=\prod_{\boldsymbol{\alpha} \in \mathfrak{P}_{\boldsymbol{\beta}}^{k-2-p}}\left(s^{-1}(B_{\mathbf{0}}+B_{\boldsymbol{\alpha}})\right)-v^{-1} G_v$$

and define

$$B_{\mathbf{0}}^{\#}=v^{-1} G_v.$$

Then, we have the theorem.

THEOREM 3. *The set of idempotent matrices* $\{B_{\boldsymbol{\beta}}^{\#} \mid \boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}')\}$ *is the set of mutually orthogonal idempotent matrices of the minimum two sided ideals of* s^{k-p}*-fractional factorial association algebra.*

The proof of this theorem is quite similar to that of Theorem 2 and is omitted here.

In this case, the formula similar to that of (4.6) is

$$\begin{cases} B_{\boldsymbol{\beta}}^{\#} = v^{-1}\{(s-1) \sum\limits_{\boldsymbol{\alpha} \in \mathfrak{P}_{\boldsymbol{\beta}}^{k-2-p}} B_{\boldsymbol{\alpha}} - \sum\limits_{\boldsymbol{\alpha} \notin \mathfrak{P}_{\boldsymbol{\beta}}^{k-2-p}} B_{\boldsymbol{\alpha}}\}, & \boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}') \\ B_{\mathbf{0}}^{\#} = v^{-1} G_v. \end{cases} \tag{4.7}$$

It is well-known [4] that each of the association matrices A_i $(i=0, 1, \cdots, m)$ of an association algebra $\mathfrak{A}=[A_0, \cdots, A_m]$ can be expressed as a linear combination of the principal idempotent matrices $A_j^{\#}$ $(j=0, 1, \cdots, m)$ of the minimum two-sided ideals of $\mathfrak{A}$, i.e.,

$$A_i = \sum_{j=0}^{m} z_{ji} A_j^{\#}$$

and that each of the $A_i^{\#}$ can also be expressed as a linear combination of the A_j, i.e.,

$$A_i^{\#} = \sum_{j=0}^{m} z^{ij} A_j.$$

It is also known [5] that there exists a simple relation between those coefficients z^{ij} and z_{ij}, i.e.,

$$z^{ij} = r_i z_{ij}/(v n_j)$$

where v is the number of treatments, n_j is the number of the j-th associates and $r_i = \text{rank}(A_i^{\#})$. In our case, since $z_{\boldsymbol{\alpha\beta}} = v_k z^{\boldsymbol{\beta\alpha}}$ in (4.6), we have

$$\begin{cases} A_{\boldsymbol{\alpha}} = (s-1) \sum\limits_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}} A_{\boldsymbol{\beta}}^{\#} - \sum\limits_{\boldsymbol{\beta} \notin \mathfrak{P}_{\boldsymbol{\alpha}}} A_{\boldsymbol{\beta}}^{\#} & \text{for} \quad \boldsymbol{\alpha} \in \text{PG}(k-1, s) \\ A_{\mathbf{0}} = \sum\limits_{\boldsymbol{\beta} \in \text{PG}(k-1, s)} A_{\boldsymbol{\beta}}^{\#} \end{cases} \tag{4.8}$$

and since $z_{\boldsymbol{\alpha\beta}} = v z^{\boldsymbol{\beta\alpha}}$ in (4.7), we have

$$\begin{cases} B_{\boldsymbol{\alpha}} = (s-1) \sum\limits_{\boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}'), \boldsymbol{\beta}'\boldsymbol{\alpha}=0} B_{\boldsymbol{\beta}}^{\#} - \sum\limits_{\boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}'), \boldsymbol{\beta}'\boldsymbol{\alpha} \neq 0} B_{\boldsymbol{\beta}}^{\#} & \text{for} \quad \boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p} \\ B_{\mathbf{0}} = \sum\limits_{\boldsymbol{\beta} \in \mathfrak{P}(\tilde{F}')} B_{\boldsymbol{\beta}}^{\#}. \end{cases} \tag{4.9}$$

5. Relations between $\mathfrak{A}(\text{PG}(k-1, s))$ and $\mathfrak{A}(s^{k-p}-\text{Fr})$

Taking into account the interrelationship between the geometric structure of the association algebra $\mathfrak{A}(\text{PG}(k-1, s))$ and that of induced algebra $\mathfrak{A}(s^{k-p}-\text{Fr})$, we define a $v \times v_k$ matrix $\boldsymbol{\Phi}$ giving a linear mapping from v_k dimensional vector space to v dimensional vector space as a function of F and f, such that

$$\boldsymbol{\Phi} = \|\varphi_{\boldsymbol{x}\boldsymbol{a}}\|, \qquad \boldsymbol{x} \in \mathfrak{F}^{k-p}, \qquad \boldsymbol{a} \in \text{EG}(k, s) \tag{5.1}$$

where

$$\varphi_{xa} = \begin{cases} 1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \neq a. \end{cases}$$

The linear mapping Φ naturally induces the following linear mapping σ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$:

$$\sigma: \mathfrak{A}(\mathrm{PG}(k-1, s)) \ni A \longrightarrow \Phi A \Phi'.$$

From the definition of A_{α}'s, B_{β}'s and Φ, we have

$$\Phi A_{\alpha} \Phi' = \begin{cases} B_{\alpha} & \text{if} \quad \alpha \in \overline{\mathfrak{P}}^{k-1-p} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

We, therefore, have

$$\mathfrak{A}(s^{k-p} - \mathrm{Fr}) = \{\Phi A \Phi' \mid A \in \mathfrak{A}(\mathrm{PG}(k-1, s))\}. \tag{5.3}$$

Since $\mathfrak{P}(\tilde{F}') \cap \overline{\mathfrak{P}}(F')$ is the empty set and $\mathrm{PG}(k-1, s) = \mathfrak{P}(\tilde{F}') \cup \mathfrak{P}(F')$, every α of $\overline{\mathrm{PG}}(k-1, s)$ has a unique representation as a linear combination

$$\alpha = \xi \alpha_1 + \alpha_2 \tag{5.4}$$

where $\alpha_1 \in \mathfrak{P}(\tilde{F}')$, $\alpha_2 \in \overline{\mathfrak{P}}(F')$ and $\xi \in \mathrm{GF}(s)$. The condition $\alpha \in \overline{\mathfrak{P}}(F')$ is equivalent to the condition $\xi = 0$ and the condition $\alpha \notin \overline{\mathfrak{P}}(F')$ is equivalent to $\xi \neq 0$. From the definition $\mathfrak{P}_{\alpha}$ and $\mathfrak{P}^{k-1-p}$, we have

$$\overline{\mathfrak{P}}_{\alpha} \cap \overline{\mathfrak{P}}^{k-1-p} = \begin{cases} \overline{\mathfrak{P}}_{\alpha_1}^{k-2-p} & \text{if} \quad \xi \neq 0 \quad \text{or} \quad \alpha \notin \overline{\mathfrak{P}}(F') \\ \overline{\mathfrak{P}}^{k-1-p} & \text{if} \quad \xi = 0 \quad \text{or} \quad \alpha \in \overline{\mathfrak{P}}(F') \end{cases}$$

Thus we have the following lemma.

Lemma 3. *With respect to the linear mapping σ defined by the matrix Φ, we have*

(i)
$$\Phi A_0^{\sharp} \Phi' = s^{-p} B_0^{\sharp} \tag{5.5}$$

(ii)
$$\Phi A_{\alpha}^{\sharp} \Phi' = \begin{cases} s^{-p} B_{\alpha_1}^{\sharp} & \text{if} \quad \xi \neq 0 \quad \text{or} \quad \alpha \notin \overline{\mathfrak{P}}(F') \\ s^{-p}(s-1) B_0^{\sharp} & \text{if} \quad \xi = 0 \quad \text{or} \quad \alpha \in \overline{\mathfrak{P}}(F') \end{cases} \tag{5.6}$$

for any $\alpha = \xi \alpha_1 + \alpha_2 \in \mathrm{PG}(k-1, s)$, where $\alpha_1 \in \mathfrak{P}(\tilde{F}')$, $\alpha_2 \in \overline{\mathfrak{P}}(F')$ and $\xi \in \mathrm{GF}(s)$.

Proof. (i) can be easily derived from (5.2) by using (4.6) and (4.7). (ii) can also be derived from (5.2) by using (4.6) and (4.7). In fact, for $\alpha = \xi \alpha_1 + \alpha_2 \notin \overline{\mathfrak{P}}(F')$ or $\xi \neq 0$, we have

$$\begin{aligned}\Phi A_{\boldsymbol{\alpha}}^{\#}\Phi' &= v_k^{-1}\{(s-1)\sum_{\boldsymbol{\beta}\in\mathfrak{P}_{\boldsymbol{\alpha}_1}^{k-2-p}} B_{\boldsymbol{\beta}} - \sum_{\boldsymbol{\beta}\notin\mathfrak{P}_{\boldsymbol{\alpha}_1}^{k-2-p}} B_{\boldsymbol{\beta}}\} \\ &= s^{-p}B_{\boldsymbol{\alpha}_1}^{\#},\end{aligned}$$

and for $\boldsymbol{\alpha}\in\mathfrak{P}(F')$ or $\xi=0$, we have

$$\Phi A_{\boldsymbol{\alpha}}^{\#}\Phi' = v_k^{-1}(s-1)\sum_{\boldsymbol{\beta}\in\mathfrak{P}^{k-1-p}} B_{\boldsymbol{\beta}} = s^{-p}(s-1)B_{\mathbf{0}}^{\#}.$$

Although a linear mapping σ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ may be naturally defined by an arbitrary matrix Ψ such as

$$\sigma:\quad \mathfrak{A}(\mathrm{PG}(k-1, s)) \longrightarrow \{\Psi A\Psi' \mid A\in\mathfrak{A}(\mathrm{PG}(k-1, s))\},$$

the image of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ is not necessarily an algebra.

Among those linear mappings, the linear mapping of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ induced by Φ defined in (5.1) has some implications. We call a linear mapping σ induced by Ψ a *fractionally similar mapping* of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ onto $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ when it has the properties (i) and (ii) described in Lemma 3.

We have the following theorem.

THEOREM 4. *The linear mapping σ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ onto $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ defined by Φ described in* (5.1) *is fractionally similar. It carries those $(s^p-1)/(s-1)+1$ idempotents $A_{\boldsymbol{\alpha}}^{\#}(\boldsymbol{\alpha}\in\bar{\mathfrak{P}}(F'))$ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ to the idempotent $B_{\mathbf{0}}^{\#}$ of $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ exclusive of the scale factors, and those s^p idempotents $A_{\xi\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}^{\#}$, for fixed $\boldsymbol{\alpha}_1\in\mathfrak{P}(\tilde{F}')(\boldsymbol{\alpha}_2\in\bar{\mathfrak{P}}(F'), \xi\neq 0)$ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ to an idempotent $B_{\boldsymbol{\alpha}_1}^{\#}$ of $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ exclusive of a common scale factor.*

6. The aliases pattern

For every pair of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\overline{\mathrm{PG}}(k-1, s)$, we say that they are $\bar{\mathfrak{P}}(F')$-equivalent and write $\boldsymbol{\alpha}\equiv\boldsymbol{\beta}$ (mod $\bar{\mathfrak{P}}(F')$) when and only when $\boldsymbol{\alpha}-\boldsymbol{\beta}\in\bar{\mathfrak{P}}(F')$. It is easily verified that $\bar{\mathfrak{P}}(F')$-equivalence is indeed an equivalence relation. Thus we can decompose $\overline{\mathrm{PG}}(k-1, s)$ into the set of equivalence classes under the relation as follows:

$$\overline{\mathrm{PG}}(k-1, s)/\bar{\mathfrak{P}}(F') = \{\mathfrak{P}^{\boldsymbol{\alpha}}(F') \mid \boldsymbol{\alpha}\in\bar{\mathfrak{P}}(\tilde{F}')\} \tag{6.1}$$

where $\mathfrak{P}^{\boldsymbol{\alpha}}(F') = \{\boldsymbol{\beta} \mid \boldsymbol{\beta}\equiv\boldsymbol{\alpha}\ (\mathrm{mod}\ \bar{\mathfrak{P}}(F'))\}$.

In the fractionally similar mapping σ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ onto $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ defined by Φ, two idempotents $A_{\boldsymbol{\alpha}}^{\#}$ and $A_{\boldsymbol{\beta}}^{\#}$ are called *aliases to each other* if and only if $\boldsymbol{\alpha}\equiv\boldsymbol{\beta}$ (mod $\bar{\mathfrak{P}}(F')$). In this case, the set of all mutually orthogonal idempotents $A_{\boldsymbol{\alpha}}^{\#}$ of $\mathfrak{A}(\mathrm{PG}(k-1, s))$ is decomposed into the union of the aliases classes:

$$\{A_{\boldsymbol{\alpha}}^{\#} \mid \boldsymbol{\alpha}\in\overline{\mathrm{PG}}(k-1, s)\} = \bigcup_{\boldsymbol{\beta}\in\mathfrak{P}(\tilde{F}')}\{A_{\boldsymbol{\alpha}}^{\#} \mid \boldsymbol{\alpha}\in\mathfrak{P}^{\boldsymbol{\beta}}(F')\} \tag{6.2}$$

where any two idempotents belonging to the same class are aliases to each other. We call the formula (6.2) the aliases pattern of $\{A_{\boldsymbol{\alpha}}^{\#} \mid \boldsymbol{\alpha} \in \overline{\mathrm{PG}}(k-1, s)\}$ with respect to the fractionally similar mapping σ.

Theorem 4 shows that any two idempotents which are aliases to each other correspond to the single idempotent of $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ under the linear mapping σ defined by $\boldsymbol{\Phi}$ described in (5.1).

It can be seen that the independent set of consistent equation $\boldsymbol{Fx}=\boldsymbol{f}$ which define a fractional set of treatments corresponds to the generalization of the set of defining relations introduced by Box and Hunder [1] in the 2^{k-p}-fractional factorial designs.

The implications of the general definition of aliases pattern given in this section can also be understood in connection with the definition introduced by Box and Hunter [1].

7. Block design and Relationship algebra

Let $B_1, B_2, \cdots, B_r$ be $l\times k$ matrices over GF(s), respectively, and suppose $F, B_1, \cdots, B_r$ satisfy the condition

$$\mathrm{rank}(F'; B_1'; \cdots; B_r') = p + rl \leqq k. \tag{7.1}$$

Consider rs^l flats $\mathfrak{B}_{iu}$ of $(k-p-l)$-dimension in EG(k, s) defined by

$$\mathfrak{B}_{i\boldsymbol{u}} = \{\boldsymbol{x} \mid F\boldsymbol{x} = \boldsymbol{f},\ \ B_i\boldsymbol{x} = \boldsymbol{u}\} \tag{7.2}$$

where $i=1, \cdots, r$ and $\boldsymbol{u} \in \mathrm{EG}(l, s)$. Evidently, there are $\kappa = s^{k-p-l}$ points in each of these flats. Corresponding to each flat $\mathfrak{B}_{i\boldsymbol{u}}$, we define a block $\phi(\mathfrak{B}_{i\boldsymbol{u}})$ consists of all treatments whose indexing points lie in the flat, i.e.,

$$\phi(\mathfrak{B}_{i\boldsymbol{u}}) = \{\phi(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathfrak{B}_{i\boldsymbol{u}}\} \tag{7.3}$$

With respect to a fractional set of $v=s^{k-p}$ treatments $\phi(\boldsymbol{x})$ ($\boldsymbol{x} \in \mathfrak{F}^{k-p}$) defined in section 3 by F and $\boldsymbol{f}$, we consider a design

$$\mathfrak{D}\{\phi(\mathfrak{B}_{i\boldsymbol{u}}) \mid i=1, \cdots, r,\ \ \boldsymbol{u} \in \mathrm{EG}(l, s)\}$$

composed of all $b=rs^l$ blocks $\phi(\mathfrak{B}_{i\boldsymbol{u}})$ $\big(i=1, \cdots, r,\ \ \boldsymbol{u} \in \mathrm{EG}(l, s)\big)$.

It can be easily seen that the number of replications of the design is r and that the block size is $\kappa=s^{k-p-l}$. Since any two treatments $\phi(\boldsymbol{x})$ and $\phi(\boldsymbol{y})$ which are $\boldsymbol{\alpha}$-th associates can occur together in the same block $\mathfrak{B}_{i\boldsymbol{u}}$ if and only if $B_i\boldsymbol{x}=B_i\boldsymbol{y}=\boldsymbol{u}$ or $B_i\boldsymbol{\alpha}=\boldsymbol{0}$ ($\boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p}$), the number $\lambda_{\boldsymbol{\alpha}}$ of such blocks depends only on the index of the associate class $\boldsymbol{\alpha}$. The design is, therefore, a partially balanced incomplete block design (PBIBD) with parameters $v=s^{k-p}$, $b=rs^l$, $r=r$, $\kappa=s^{k-p-l}$ and $\lambda_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p}$).

Let N be the treatment-block incidence matrix of the design such that

$$N = \|n_{\boldsymbol{x},\, i\boldsymbol{u}}\| \tag{7.4}$$

where

$$n_{x,\, iu} = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in \mathfrak{B}_{iu} \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$NN' = rB_0 + \sum_{\boldsymbol{\alpha} \in \mathfrak{P}^{k-1-p}} \lambda_{\boldsymbol{\alpha}} B_{\boldsymbol{\alpha}}. \tag{7.5}$$

The formula (7.5) shows that the association algebra induced by the incidence matrix of the PBIBD mentioned above is a subalgebra of the s^{k-p}-fractional factorial association algebra $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$. Thus the relationship algebra of the design preserves partial similarity to the s^{k-p}-fractional factorial association algebra when the block effects are eliminated [3].

As is indicated in [4], (7.5) can be expressed in terms of the mutually orthogonal idempotents $B^{\#}_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathfrak{P}(\tilde{F}'))$ of $\mathfrak{A}(s^{k-p}-\mathrm{Fr})$ as

$$NN' = \sum_{\boldsymbol{\alpha} \in \mathfrak{P}(\tilde{F}')} \mu_{\boldsymbol{\alpha}} B^{\#}_{\boldsymbol{\alpha}} \tag{7.6}$$

where

$$\begin{cases} \mu_0 = r\kappa = r + (s-1) \displaystyle\sum_{\boldsymbol{\beta} \in \mathfrak{P}^{k-1-p}} \lambda_{\boldsymbol{\beta}} \\ \mu_{\boldsymbol{\alpha}} = r + (s-1) \displaystyle\sum_{\boldsymbol{\beta} \in \mathfrak{P}^{k-2-p}_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\beta}} - \displaystyle\sum_{\boldsymbol{\beta} \notin \mathfrak{P}^{k-2-p}_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\beta}} \qquad (\boldsymbol{\alpha} \neq 0). \end{cases} \tag{7.7}$$

The formula (7.7) may also be written as

$$\mu_{\boldsymbol{\alpha}} = s \sum_{\boldsymbol{\beta} \in \mathfrak{P}^{k-2-p}_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\beta}} - r(\kappa - s)(s-1)^{-1} \qquad \text{for} \quad \boldsymbol{\alpha} \in \mathfrak{P}(\tilde{F}'). \tag{7.8}$$

According to the general theory of the analysis of the relationship algebra of PBIBD developed in [4], the behavior of the component sum of squares corresponding to $B^{\#}_{\boldsymbol{\alpha}}$ is determined by the corresponding characteristic root $\mu_{\boldsymbol{\alpha}}$ of NN', i.e., the component is orthogonal to the block space if and only if $\mu_{\boldsymbol{\alpha}}=0$, partially confounded with the block space if and only if $0<\mu_{\boldsymbol{\alpha}}<r\kappa$ and totally confounded with the block space if and only if $\mu_{\boldsymbol{\alpha}}=r\kappa$.

As to the classification of the behavior of the component sum of spuares, we have the following theorem.

THEOREM 5. *In the* PBIBD

$$\mathfrak{D}\{\phi(\mathfrak{B}_{iu}) \mid i = 1, \cdots, r, \ \boldsymbol{u} \in \mathrm{EG}(l, s)\},$$

the characteristic root $\mu_{\boldsymbol{\alpha}}$ *corresponding to* $\mathrm{B}^{\#}_{\boldsymbol{\alpha}}$ *is*

$$\mu_{\boldsymbol{\alpha}} = \begin{cases} 0 & \textit{if} \quad \boldsymbol{\alpha} \notin \mathfrak{P}(B_i') \quad \textit{for all} \quad i=1, \cdots, r. \\ \kappa & \textit{if there exists a } \mathfrak{P}(B_i') \textit{ such that } \boldsymbol{\alpha} \in \mathfrak{P}(B_i'). \end{cases}$$

The classification of the behavior of the component sum of squares are as follows:

(i) $\mu_{\boldsymbol{\alpha}}=0$ *(Orthogonal case) if and only if* $\boldsymbol{\alpha} \notin \mathfrak{P}(B_i')$ *for all* $i=1, \cdots, r$.

(ii) $0<\mu_{\boldsymbol{\alpha}}<r\kappa$ *(Partially confounded case) if and only if* $r\geq 2$ *and there exist a* B_i *such that* $\boldsymbol{\alpha} \in \mathfrak{P}(B_i')$. *In this case, the confounding coefficient is* $1/r$.

(iii) $\mu_{\boldsymbol{\alpha}}=r\kappa$ *(Totally confounded case) if and only if* $r=1$ *and* $\boldsymbol{\alpha} \in \mathfrak{P}(B_1')$.

PROOF. To evaluate $\mu_{\boldsymbol{\alpha}}$ we first evaluate the sum of $\lambda_{\boldsymbol{\beta}}$ which appears in (7.8) using the definition of $\lambda_{\boldsymbol{\beta}}$, i.e.,

$$\sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} \lambda_{\boldsymbol{\beta}} = \sum_{i=1}^{r} \sum_{\boldsymbol{u} \in \mathrm{EG}(l,s)} n_{\boldsymbol{x},i\boldsymbol{u}} \sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} n_{\boldsymbol{x}+\boldsymbol{\beta},i\boldsymbol{u}}. \tag{7.9}$$

Since $n_{\boldsymbol{x},i\boldsymbol{u}_i}$ can assume the value 1 when and only when $B_i\boldsymbol{x}=\boldsymbol{u}_i$ and zero otherwise for any fixed B_i and $\boldsymbol{x}$, we can simplify (7.9) as

$$\sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} \lambda_{\boldsymbol{\beta}} = \sum_{i=1}^{r} \sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} n_{\boldsymbol{x}+\boldsymbol{\beta},i\boldsymbol{u}_i} \tag{7.10}$$

where $B_i\boldsymbol{x}=\boldsymbol{u}_i$. To evaluate the sum of $\lambda_{\boldsymbol{\beta}}$ over $\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}$, since $n_{\boldsymbol{x}+\boldsymbol{\beta},i\boldsymbol{u}_i}$ can assume either 1 or 0, it is sufficient to evaluate the number of points $\boldsymbol{\beta}$ which satisfy $F\boldsymbol{\beta}=\mathbf{0}$, $\boldsymbol{\alpha}'\boldsymbol{\beta}=0$ and $B_i\boldsymbol{\beta}=\mathbf{0}$ simultaneously.

Thus we have

$$\sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} n_{\boldsymbol{x}+\boldsymbol{\beta},i\boldsymbol{u}_i} = \begin{cases} \dfrac{s^{k-p-l-1}-1}{s-1} & \text{if} \quad \boldsymbol{\alpha} \notin \mathfrak{P}(B_i') \\[2ex] \dfrac{s^{k-p-l}-1}{s-1} & \text{if} \quad \boldsymbol{\alpha} \in \mathfrak{P}(B_i'). \end{cases} \tag{7.11}$$

It should be noted that the following two cases can happen:

(i) $\boldsymbol{\alpha}$ does not belong to any $\mathfrak{P}(B_i')$ and (ii) $\boldsymbol{\alpha}$ belongs to one and only one $\mathfrak{P}(B_i')$. Thus we have

$$\sum_{\boldsymbol{\beta} \in \mathfrak{P}_{\boldsymbol{\alpha}}^{k-2-p}} \lambda_{\boldsymbol{\beta}} = \begin{cases} \dfrac{r(\kappa-s)}{s(s-1)} & \text{if} \quad \boldsymbol{\alpha} \notin \mathfrak{P}(B_i') \quad \text{for all} \quad i=1, \cdots, r, \\[2ex] \dfrac{1}{s}\left(\dfrac{r(\kappa-s)}{s-1}+\kappa\right) & \text{if there exists a } B_i \text{ such that } \boldsymbol{\alpha} \in \mathfrak{P}(B_i'). \end{cases}$$

Substituting these to the formula (7.8) we have

$$\mu_{\boldsymbol{\alpha}} = \begin{cases} 0 & \text{if} \quad \boldsymbol{\alpha} \notin \mathfrak{P}(B_i') \quad \text{for all} \quad i=1, \cdots, r \\ \kappa & \text{if there exists a } B_i \text{ such that } \boldsymbol{\alpha} \in \mathfrak{P}(B_i'). \end{cases}$$

The remaining part of the theorem is the immediate consequence of the theorem described in [4].

Theorem 5 shows that, when $r=1$, each component sum of squares is either orthogonal to the block space or totally confounded with the block space, and that, when $r\geq 2$, each component sum of squares is either orthogonal to the block space or partially confounded with the block space. The confounding coefficient of the latter case is r^{-1}.

Lemma 3 shows that

$$\sigma\left(A^{\#}_{\boldsymbol{\alpha}} = A^{\#}_{\xi\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} \,|\, \boldsymbol{\alpha}_1 \in \mathfrak{P}(\tilde{F}'),\ \ \boldsymbol{\alpha}_2 \in \mathfrak{P}(F')\right) = s^{-p} B^{\#}_{\boldsymbol{\alpha}_1}$$

holds for every $A^{\#}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \notin \bar{\mathfrak{P}}(F')$. For such idempotent, Theorem 5 shows that if $\boldsymbol{\alpha}_1 \in \mathfrak{P}(B'_i)$ for a certain i, the sum of squares corresponding to $B^{\#}_{\boldsymbol{\alpha}_1}$ is either confounded with the block factors when $r=1$ or partially confounded with block factors with confounding coefficient $1/r$ when $r\geqq 2$. The family of $s-1$ independent contrasts corresponding to such an $A^{\#}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} \notin \mathfrak{P}(F')$ and $\boldsymbol{\alpha} \in \mathfrak{P}(F'; B'_i))$ can be called the pseudo-block factors. $A^{\#}_{\boldsymbol{\alpha}}$ can be called pseudo-block idempotent.

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