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On Atomistic Lattices with the Covering Property

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In the investigations of lattices in geometries, a matroid lattice is defined as an upper continuous atomistic lattice with the covering property. (See [5] and [3]. A matroid lattice is called a geometric lattice in [2].) On the other hand, there exist atomistic lattices of another type with the covering property, for instance, the lattice of all closed subspaces of a normed space or a Hilbert space. Such a lattice L has the following property:

(*) Both L and its dual are atomistic and have the covering property.

In this paper, a lattice with the property (*) is called a DAC-lattice. Since it can be proved that the dual of a matroid lattice is atomistic, the defference between a matroid lattice and a DAC-lattice is that the former is upper continuous and the latter has the dual covering property.

In the literature, the properties of matroid lattices are well investigated. The main purpose of this paper is to investigate the properties of DAC-lattices compared with those of matroid lattices.

An important common property is that the modular relation is symmetric (see §2). Other common properties appear in the arguments on the perspectivity of atoms and on irreducible decomposability (see §3 and §4).

An important difference between them is that the atoms of a DAC-lattice form a projective space but those of a non-modular matroid lattice do not. For this reason, in the theory of matroid lattices parallelism is very important but in a DAC-lattice there exists only trivial parallelism, and we have an embedding theorem of DAC-lattices into projective lattice (see §5).

In the last section of this paper, we give some examples of DAC-lattices: the lattices of closed subspaces of some vector spaces, and discuss representation theorems.

§1. DAC-lattices and matroid lattices

Definition. (i) Let a and b be elements of a lattice L. We say that b *covers* a and write $a \lessdot b$ if $a < b$ and there does not exist $c \in L$ with $a < c < b$.

(ii) Let L be a lattice with 0. An element p of L is called an *atom* (or a *point*) if $0 \lessdot p$. L is called *atomic* if every non-zero element of L contains an atom. L is called *atomistic* if every element of L is the join of some set (which may be empty) of atoms. It is easy to show that L is atomistic if and only if L is relatively atomic, that is, $a < b$ implies $a < a \cup p \leqq b$ for some atom p.

(iii) Let L be a lattice with 1. An element h of L is called a *dual-atom* (or a *hyperplane*) if $h \lessdot 1$. L is called *dual-atomic* (resp. *dual-atomistic*) if the dual L^* of L is atomic (resp. atomistic).

(iv) Let a and b be elements of a lattice L. We say that (a, b) is a *modular pair* and write (a, b)M if

$$(c \cup a) \cap b = c \cup (a \cap b) \qquad \text{for every} \quad c \leqq b.$$

We say that (a, b) is a *dual-modular pair* and write $(a, b)M^*$ if

$$(a \cup b) \cap c = a \cup (b \cap c) \qquad \text{for every} \quad c \geqq a.$$

Evidently, $(a, b)M^*$ holds in L if and only if $(b, a)M$ holds in L^*. L is called *M-symmetric* if $(a, b)M$ implies $(b, a)M$ in L.

(v) The following property of a lattice L with 0 is called the *covering property.*

(C) If p is an atom and $p \not\leqq a$ then $a \lessdot a \cup p$.

When L is atomistic, (C) is equivalent to each of the following statements (see [8], Lemma 1).

(C_1) If p is an atom then $(p, a)M$ holds for every a.

(C_2) If p and q are atoms and $p \not\leqq a$ then $p \leqq a \cup q$ implies $q \leqq a \cup p$. (Exchange property)

(C_3) If $a \cap b \lessdot a$ then $b \lessdot a \cup b$.

It follows from (C_1) that if L is M-symmetric then it has the covering property. For brevity, an atomistic lattice with the covering property is called an *AC-lattice.*

(vi) An upper continuous AC-lattice is called a *matroid lattice*. An upper continuous atomistic modular lattice, that is, a modular matroid lattice is called a *projective lattice* (see [3]).

(vii) A lattice L with 0 and 1 is called a *DAC-lattice* if both L and its dual L^* are AC-lattices. It follows from (C_3) that L is a DAC-lattice if and only if L satisfies the following two conditions:

(D_1) L is atomistic and dual-atomistic,

(D_2) $a \cap b \lessdot a$ is equivalent to $b \lessdot a \cup b$.

If L is a DAC-lattice then so is L^* evidently. (In [10], McLaughlin called a complete DAC-lattice a C-lattice.)

LEMMA 1.1. *If an AC-lattice L is relatively complemented then L is dual-atomistic.*

PROOF. Let $a < b$ in L. It suffices to show the existence of a dual-atom

h such that $a \leqq h$ and $b \nleqq h$. Since L is relatively atomic, there exists an atom p such that $p \leqq a$ and $p \nleqq b$. Since the sublattice $L(a, 1)$ is complemented, there exists an element h such that

$$(a \cup p) \cap h = a \quad \text{and} \quad (a \cup p) \cup h = 1.$$

Since $p \nleqq a$ and $a \leqq h$, we have

$$h \cap p = h \cap (a \cup p) \cap p = a \cap p = 0 \quad \text{and} \quad h \cup p = h \cup a \cup p = 1.$$

Hence, by the covering property, we have $h \lessdot h \cup p = 1$, that is, h is a dual-atom. Since $p \nleqq h$ we have $b \nleqq h$. This completes the proof.

Lemma 1.2. *Let L be a matroid lattice.*
(i) *L is relatively complemented.*
(ii) *L is dual-atomistic.*

Proof. (i) follows from [5], Lemma 9. (ii) follows from (i) and Lemma 1.1.

Remark. Both a matroid lattice and a DAC-lattice are dual-atomistic AC-lattices. The difference between them is that the former is upper continuous and the latter has the dual covering property.

Theorem 1.1. (i) *Any complemented modular atomic lattice is a DAC-lattice.*
(ii) *A projective lattice is an upper continuous DAC-lattice and conversely.*

Proof. (i) Let L be a complemented modular atomic lattice. Since a complemented modular lattice is relatively complemented, L is relatively atomic and hence is atomistic. Since L and L^* are modular, they have the covering property. It follows from Lemma 1.1 that L is dual-atomistic. Thus L is a DAC-lattice.

(ii) A projective lattice is complemented by Lemma 1.2 (i) and hence it is a DAC-lattice by (i). Conversely, if L is an upper continuous DAC-lattice then L is a matroid lattice satisfying (D_2). Hence L is modular by [8], Corollary of Theorem 1.

Remark. The above theorem gives some examples of modular DAC-lattices. Here we shall give those of non-modular DAC-lattices.

(i) An orthocomplemented AC-lattice is a DAC-lattice since it is self-dual. This lattice is closely related to the quantum theory. In [11], Piron shows that a system of propositions forms a complete orthocomplemented (moreover orthomodular) AC-lattice.

(ii) The set of all closed subspaces of a normed space forms a DAC-lattice which we shall discuss in §6.

§2. Modularity in AC-lattices

DEFINITION. (i) In a lattice L with 0, an element is called *finite* if it is the join of a finite set (which may be empty) of atoms. The set of all finite elements of L is denoted by $F(L)$.

(ii) An element a of a lattice L is called *modular* if $(b, a)M$ holds for every $b \in L$. A lattice L with 0 is called *finite-modular* if every finite element of L is modular.

LEMMA 2.1. *If L is an AC-lattice then $F(L)$ is an ideal of L.*

PROOF. [4], Theorem 4.1. (Cf. [8], Lemma 3.)

LEMMA 2.2. (i) *Let L be an AC-lattice. The following five statements are equivalent.*

(α) *L is finite-modular.*

(β) *If l is the join of two different atoms then l is modular.*

(γ) *If l is the join of two different atoms then $a \lessdot a \cup l$ implies $a \cap l \lessdot l$.*

(δ) *If p and q are atoms and $p \leqq q \cup a (a \neq 0)$, then there exists an atom r such that $p \leqq q \cup r$ and $r \leqq a$.*

(ε) *If p is an atom, if a or b is finite and if $p \leqq a \cup b (a \neq 0, b \neq 0)$, then there exist two atoms q and r such that $p \leqq q \cup r$, $q \leqq a$ and $r \leqq b$.*

(ii) *In a finite-modular AC-lattice, if a is a finite element then $(a, b)M$, $(b, a)M$, $(a, b)M^*$ and $(b, a)M^*$ hold for every b.*

PROOF. This lemma is a consequence of [8], Theorem 1.

LEMMA 2.3. *Let L be a finite-modular AC-lattice.*

(i) *$(a, b)M^*$ is equivalent to the following: if p is an atom and $p \leqq a \cup b$ then there exist atoms q and r such that $p \leqq q \cup r$, $q \leqq a$ and $r \leqq b$.*

(ii) *The dual of L is M-symmetric.*

PROOF. [8], Lemma 4.

THEOREM 2.1. (i) *Any DAC-lattice is finite-modular. If a finite-modular AC-lattice is dual-atomistic then it is a DAC-lattice.*

(ii) *Any DAC-lattice is M-symmetric.*

PROOF. (i) if L is a DAC-lattice, then it follows from (D_2) that L satisfies (γ) of Lemma 2.2, whence L is finite-modular. Next, if L is a dual-atomistic finite-modular AC-lattice, then by Lemma 2.3 L^* is M-symmetric, whence L^* has the covering property. Thus L is a DAC-lattice.

Statement (ii) is a consequence of (i) and Lemma 2.3. (Cf. [8], Lemma 5.)

REMARK. Any matroid lattice is also M-symmetric. (This can be proved using the upper continuity. Cf. [13].) But, except for projective lattices, matroid lattices are not finite-modular. Because if L is a matroid lattice, then by the upper continuity, (ε) of Lemma 2.2 implies the mcdularity (cf. [8], Corollary of Theorem 1).

Next we give a theorem for a DAC-lattice L, which does not hold when L is a non-modular matroid lattice.

Definition. In a lattice L with 1, an element is called *dual-finite* if it is finite in the dual L^*. In a lattice L with 0 and 1, the set of all finite elements and all dual-finite elements is denoted by $\check{F}(L)$. That is,

$$\check{F}(L) = F(L) \cup F(L^*).$$

Theorem 2.2. *Let L be a DAC-lattice.*

(i) *If $a \in \check{F}(L)$ then $(a, b)M$, $(b, a)M$, $(a, b)M^*$ and $(b, a)M^*$ hold for every $b \in L$.*

(ii) *$\check{F}(L)$ is a DAC-sublattice of L which is complemented and modular.*

Proof. (i) If $a \in F(L)$, then it follows from Theorem 2.1 and Lemma 2.2 (ii) that the four modular relations for a and b hold. If $a \in F(L^*)$, then since L^* is a DAC-lattice the four modular relations for a and b hold in L^*, whence they hold in L also.

(ii) Let $a, b \in \check{F}(L)$. We shall show $a \cap b \in \check{F}(L)$. Since $F(L)$ is an ideal by Lemma 2.1, $a \cap b \in F(L)$ when either a or b is finite. When both a and b are dual-finite, $a \cap b$ is dual-finite by the dual statement of Lemma 2.1. Thus we have $a \cap b \in \check{F}(L)$. Similarly we have $a \cup b \in \check{F}(L)$. Hence $\check{F}(L)$ is a sublattice of L. $\check{F}(L)$ is modular by (i). Since $\check{F}(L)$ contains all atoms and all dual-atoms of L, $\check{F}(L)$ is atomistic and dual-atomistic. Hence $\check{F}(L)$ is a modular DAC-lattice. Finally we shall prove that it is complemented. By the duality, it suffices to show that if $a \in F(L^*)$ then a has a complement which belongs to $F(L)$. When $a \in F(L^*)$, by the dual covering property there exists a finite chain

$$a = a_0 \lessdot a_1 \lessdot \cdots \lessdot a_n = 1,$$

and hence there exist atoms $p_1, \cdots, p_n$ such that

$${}_ip \cap a_{i-1} = 0, \quad p_i \cup a_{i-1} = a_i \qquad (i = 1, \cdots, n).$$

For each i with $2 \leqq i \leqq n$, since $(p_i, a_{i-1})M$ by (C_1) and $p_1 \cup \cdots \cup p_{i-1} \leqq a_{i-1}$, we have

$$\begin{aligned} a \cap (p_1 \cup \cdots \cup p_i) &= a \cap (p_1 \cup \cdots \cup p_{i-1} \cup p_i) \cap a_{i-1} \\ &= a \cap \{(p_1 \cup \cdots \cup p_{i-1}) \cup (p_i \cap a_{i-1})\} = a \cap (p_1 \cup \cdots \cup p_{i-1}). \end{aligned}$$

Hence $a \cap (p_1 \cup \cdots \cup p_n) = a \cap (p_1 \cup \cdots \cup p_{n-1}) = \cdots = a \cap p_1 = 0$. It is evident that

$a\cup(p_1\cup\cdots\cup p_n)=1$. Therefore the element $p_1\cup\cdots\cup p_n$ of $F(L)$ is a complement of a.

§3. Perspectivity of atoms in AC-lattices

DEFINITION. Let a and b be elements of a lattice L with 0. If there exists $x\in L$ such that

$$a\cup x=b\cup x \quad \text{and} \quad a\cap x=b\cap x=0,$$

then we say that a and b are *perspective* and write $a\sim_x b$ (or simply $a\sim b$). It is evident that $a\sim_0 a$.

We write $a\bigtriangledown b$ if $(x\cup a)\cap b=x\cap b$ for every $x\in L$. It is easily seen that if $a\sim b$ and $a\bigtriangledown b$ then $a=b=0$ and that if $a\bigtriangledown b$, $a_1\leqq a$ and $b_1\leqq b$ then $a_1\bigtriangledown b_1$.

LEMMA 3.1. *Let a lattice L with 0 have the covering property, and let p and q be atoms of L.*

(i) *If $q\leqq p\cup x$ and $q\cap x=0$ (p is subperspective to q) then $p\sim_x q$.*

(ii) *If $p\neq q$ and if $p\cup q$ contains a third atom r then $p\sim_r q$.*

(iii) *$p\sim q$ holds if and only if $p\bigtriangledown q$ does not hold.*

PROOF. (i) Let $q\leqq p\cup x$ and $q\cap x=0$. If $p\leqq x$ then $q\leqq p\cup x=x$ which contradicts $q\cap x=0$. Hence $p\cap x=0$. Since $x\lessdot p\cup x$ by the covering property and since $x<q\cup x\leqq p\cup x$, we have $q\cup x=p\cup x$. Therefore $p\sim_x q$.

(ii) If $p\neq q$ and if r is an atom such that $r\leqq p\cup q$, $r\neq p$ and $r\neq q$, then by the covering property we have $p\cup r=p\cup q$ and $q\cup r=p\cup q$. Hence $p\sim_r q$.

(iii) If $p\sim q$ holds then $p\bigtriangledown q$ does not hold since $p\neq 0$. Conversely, if $p\bigtriangledown q$ does not hold then there exists $x\in L$ such that $(x\cup p)\cap q>x\cap q$. Then $q\leqq p\cup x$ and $q\cap x=0$; because, if $q\nleqq p\cup x$ then $(x\cup p)\cap q=0=x\cap q$, a contradiction, and if $q\leqq x$ then $(x\cup p)\cap q=q=x\cap q$, a contradiction. Hence we have $p\sim_x q$ by (i).

LEMMA 3.2. *Let L be an AC-lattice. L is finite-modular if and only if L has the following property:*

(*) *If p and q are different atoms of L such that $p\sim_x q$ then $p\cup q$ contains a third atom r with $r\leqq x$.*

PROOF. (i) Assume that L is finite-modular. If $p\neq q$ and $p\sim_x q$, then since $p\leqq q\cup x$ it follows from (δ) of Lemma 2.2 that there exists an atom r such that $p\leqq q\cup r$ and $r\leqq x$. By (C_2) we have $r\leqq p\cup q$. Moreover $r\neq p$ since $p\cap r\leqq p\cap x=0$; and similarly $r\neq q$.

(ii) Assume that L has the property (*). We shall show that (δ) of Lemma 2.2 holds. Let $p\leqq q\cup a$. When $p=q$, any atom r with $r\leqq a$ satisfies (γ). When $p\leqq a$, $r=p$ satisfies (δ). When $p\neq q$ and $p\nleqq a$, since $p\sim_a q$ by Lemma 3.1 (i), it follows from (*) that there exists an atom r such that $r\neq q$, $r\leqq p\cup q$ and $r\leqq a$. By (C_2) we have $p\leqq q\cup r$. Therefore L is finite-modular by Lemma 2.2.

THEOREM 3.1. *Let L be a finite-modular AC-lattice (in particular a DAC-lattice).*

(i) *Two different atoms p and q of L are perspective if and only if $p\cup q$ contains a third atom.*

(ii) *If p, q and r are atoms of L then $p\sim q$ and $q\sim r$ imply $p\sim r$. (Transitivity of perspectivity)*

PROOF. (i) If $p\sim q$ then it follows from Lemma 3.2 that $p\cup q$ contains a third atom. The converse follows from Lemma 3.1 (ii).

(ii) Let $p\sim q$ and $q\sim r$. We shall prove $p\sim r$. We may assume $p\neq q$, $q\neq r$ and $p\neq r$. It follows from (i) that there exist atoms s and t such that

$$s\leqq p\cup q \quad\text{and}\quad t\leqq q\cup r \quad\text{with}\quad s\neq p,\ s\neq q,\ t\neq q \quad\text{and}\quad t\neq r.$$

By the covering property we have

$$p\cup s=q\cup s=p\cup q \quad\text{and}\quad q\cup t=r\cup t=q\cup r.$$

Since $t\leqq q\cup r\leqq s\cup p\cup r$, by ($\delta$) of Lemma 2.2 there exists an atom u such that $t\leqq s\cup u$ and $u\leqq p\cup r$. When $u\neq p$ and $u\neq r$, we have $p\sim_u r$ by Lemma 3.1 (ii). When $u=p$, we have

$$r\leqq q\cup t\leqq q\cup s\cup u=p\cup q.$$

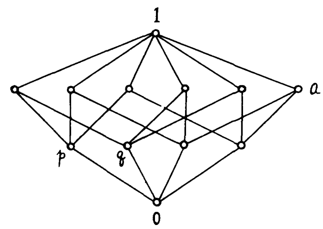
Hence $p\sim_q r$ by Lemma 3.1 (i). When $u=r$, we have

$$q\leqq r\cup t\leqq r\cup s\cup u=r\cup s.$$

By (C_2) we have $s\leqq q\cup r$ and then $p\leqq q\cup s\leqq q\cup r$. Hence $p\sim_q r$ by Lemma 3.1 (i). This completes the proof.

REMARK. (i) When L is a non-modular matroid lattice, the statement (*) of Lemma 3.2 does not hold, since L is not finite-modular.

(ii) If L is the lattice of all affine subspaces of a vector space E, then L is a non-modular matroid lattice, but for any different atoms p and q (points of E) the element $p\cup q$ is a line in E and contains infinitely many atoms. Hence in this lattice statement (i) of Theorem 3.1 holds. On the contrary, the following figure shows a non-modular matroid lattice where statement (i) of Theorem 3.1 does not hold; because in this lattice $p\sim_a q$ but $p\cup q$ does not contain a third atom.



(iii) It was proved in [14], Theorem 1 that in any matroid lattice statement (ii) of Theorem 3.1 holds.

LEMMA 3.3. *Let p and q_α ($\alpha \in I$) be atoms of a DAC-lattice L, and assume that $\bigvee(q_\alpha;\, \alpha \in I)$ exists. If for some x*

$$p \leqq \bigvee(q_\alpha;\, \alpha \in I) \cup x \quad \text{and} \quad p \cap x = 0$$

(in particular if $p \leqq \bigvee(q_\alpha;\, \alpha \in I)$), then p is perspective to q_α for some α.

PROOF. Since $p \not\leqq x$ and L is dual-atomistic, there exists a dual-atom h such that $h \geqq x$ and $h \not\geqq p$. Put

$$a = \bigvee(q_\alpha;\, \alpha \in I) \quad \text{and} \quad b = h \cap (a \cup x).$$

We have $h \not\geqq a \cup x$ since $h \not\geqq p \leqq a \cup x$. Hence $b \lessdot a \cup x$ by the dual-covering property. If $q_\alpha \leqq b$ for every $\alpha \in I$, then $a \leqq b \leqq h$ and then $h \geqq a \cup x$, a contradiction. Hence there exists q_α such that $q_\alpha \not\leqq b$. Since $p \cap b \leqq p \cap h = 0$ and $b \lessdot a \cup x$, we have $p \cup b = a \cup x \geqq q_\alpha$. Hence $p \sim_b q_\alpha$ by Lemma 3.1 (i).

THEOREM 3.2. *For elements a and b of a DAC-lattice L, the following three statements are equivalent.*

(α) $a \bigtriangledown b$.

(β) *There do not exist atoms p and q such that*

$$p \sim q, \quad p \leqq a \quad \text{and} \quad q \leqq b.$$

(γ) *There do not exist non-zero elements a_1 and b_1 such that*

$$a_1 \sim b_1, \quad a_1 \leqq a \quad \text{and} \quad b_1 \leqq b.$$

PROOF. (α)$\Rightarrow$(γ). If $0 \neq a_1 \leqq a$, $0 \neq b_1 \leqq b$ and $a_1 \sim b_1$, then $a_1 \bigtriangledown b_1$ does not hold, whence $a \bigtriangledown b$ does not hold. (γ)$\Rightarrow$(β). This is trivial. (β)$\Rightarrow$(α). If $a \bigtriangledown b$ does not hold, then there exists x such that $(a \cup x) \cap b > x \cap b$. Since L is relatively atomic, there exists an atom q such that

$$q \leqq (a \cup x) \cap b \quad \text{and} \quad q \not\leqq x \cap b.$$

Since $q \leqq a \cup x$ and $q \cap x = q \cap b \cap x = 0$, it follows from Lemma 3.3 that there exists an atom $p \leqq a$ such that $p \sim q$. Hence (β) does not hold. This completes the proof.

We get from this proof the following

Corollary. *In a DAC-lattice, if $a \sim b$ and q is an atom with $q \leqq b$, then there exists an atom p such that $p \sim q$ and $p \leqq a$.*

Remark. We can prove that Lemma 3.3 holds when L is a matroid lattice. Because by the upper continuity of L, if $p \leqq \bigvee(q_\alpha; \alpha \in I) \cup x$ then there exist $\alpha_1, \cdots, \alpha_n \in I$ such that $p \leqq q_{\alpha_1} \cup \cdots \cup q_{\alpha_n} \cup x$ (cf. [7], Lemmas 3.2 and 3.3). Therefore Theorem 3.2 also holds when L is a matroid lattice.

§4. Irreducible decompositions and central covers in complete atomistic lattices

We have seen in §3 that both matroid lattices and complete DAC-lattices are complete atomistic lattices with the following property:

(P) If $p \leqq \bigvee(q_\alpha; \alpha \in I) \cup x$ and $p \cap x = 0$ (p and q_α are atoms), then p is perspective to q_α for some α.

In this section, we shall give the irreducible decomposition of such a lattice.

Lemma 4.1. *An element z of a complete atomistic lattice L with the property* (P) *is a central element if and only if*

$$(*) \qquad p \sim q \leqq z \quad \text{implies} \quad p \leqq z,$$

where p and q are atoms of L.

Proof. It is evident that (*) holds if z is central. To prove the converse, we remark that z is central if z has a complement z' satisfying the following two conditions (see [6], Kap. I, Proof of Satz 3.3, $(\beta) \Rightarrow (\alpha)$):

$$(1) \qquad z \bigtriangledown z' \quad \text{and} \quad z' \bigtriangledown z,$$

$$(2) \qquad a = (a \cap z) \cup (a \cap z') \qquad \text{for every} \quad a \in L.$$

Let z satisfy (*) and put $z' = \bigvee(p; p \not\leqq z)$. Evidently $z \cup z' = 1$. If there exist atoms p and q such that $p \sim q$, $p \leqq z$ and $q \leqq z'$, then by (P) there exists an atom r such that $q \sim r$ and $r \not\leqq z$. From (*), $r \sim q \sim p \leqq z$ implies $r \leqq z$, a contradiction. Hence it follows from the proof $(\beta) \Rightarrow (\alpha)$ of Theorem 3.2 that (1) holds, and then z' is a complement of z. Next, since for any atom p either $p \leqq z$ or $p \leqq z'$, we have

$$a=\bigvee(p;\,p\leqq a)\leqq(a\cap z)\cup(a\cap z')\leqq a.$$

Hence (2) holds. This completes the proof.

LEMMA 4.2. *If a complete atomistic lattice L has property* (P), *then L is a Z-lattice* (*see* [9]), *that is,*

(1) *the center $Z(L)$ of L is a complete sublattice of L, and*

(2) *if $z_\alpha \in Z(L)$ for every $\alpha \in I$ then*

$$\bigvee(z_\alpha;\,\alpha \in I)\cap a=\bigvee(z_\alpha\cap a;\,\alpha\in I).$$

In particular, any complete DAC-lattice or any matroid lattice is a Z-lattice.

PROOF. It follows from [9], Corollary 3 that (2) is implied by (1). We shall prove (1). Let $z_\alpha \in Z(L)$. If $p\sim q\leqq\bigwedge_\alpha z_\alpha$, then by Lemma 4.1 we have $p\leqq z_\alpha$ for every α and then $p\leqq\bigwedge_\alpha z_\alpha$. Hence $\bigwedge_\alpha z_\alpha \in Z(L)$ by Lemma 4.1. Next, if $p\sim q\leqq\bigvee_\alpha z_\alpha$, then by (P) there exists an atom r such that $q\sim r$ and $r\leqq z_\alpha$ for some $\alpha\in I$. Then $q\leqq z_\alpha$ and $p\leqq z_\alpha\leqq\bigvee_\alpha z_\alpha$. Hence $\bigvee_\alpha z_\alpha\in Z(L)$.

DEFINITION. For any element a of a Z-lattice, there exists the least central element z with $a\leqq z$. We denote it by $e(a)$.

LEMMA 4.3. *If a complete atomistic lattice L is a Z-lattice, then L is the direct sum of irreducible sublattices $L(0,z_\alpha)$ of L. For every element $a\in L$,*

$$a=\bigvee(a_\alpha;\,\alpha\in I)\quad \textit{with}\quad a_\alpha\in L(0,\,z_\alpha)$$

and the elements a_α are uniquely determined.

PROOF. We shall show that the center $Z(L)$ is atomistic. It is easily seen that if p is an atom of L then $e(p)$ is an atom of $Z(L)$. For every $z\in Z(L)$ we have

$$\bigvee\big(e(p);\,p\leqq z\big)=e\big(\bigvee(p;\,p\leqq z)\big)=e(z)=z.$$

Hence $Z(L)$ is atomistic.

Let $\{z_\alpha;\,\alpha\in I\}$ be the set of all atoms of $Z(L)$. We have $z_\alpha\cap z_\beta=0$ if $\alpha\neq\beta$, and $\bigvee(z_\alpha;\,\alpha\in I)=1$. Hence, by [9], Theorem 2, L is a direct sum of $L(0,z_\alpha)$ and the expression given above for $a\in L$ is unique. $L(0,z_\alpha)$ is irreducible since z_α is an atom of $Z(L)$.

DEFINITION. For atoms p and q in a lattice L with 0, we write $p\approx q$ if there exist atoms r_i $(i=0,1,\cdots,n)$ such that

$$r_0=p,\quad r_n=q\quad \text{and}\quad r_{i-1}\sim r_i\qquad (i=1,\cdots,n).$$

We write $p\lessapprox a$ (p is an atom) if there exists an atom q such that $p\approx q\leqq a$.

Lemma 4.4. *Let L be a complete atomistic lattice with property* (P). *Let p and q be atoms of L and let $a \in I$.*

(i) $p \leqq e(a)$ *if and only if* $p \lessapprox a$.

(ii) $e(a) = \bigvee(p; p \lessapprox a)$.

(iii) $e(p) = e(q)$ *if and only if* $p \approx q$.

Proof. Put $z = \bigvee(p; p \lessapprox a)$. If q is an atom with $q \leqq z$, then by (P) there exists p such that $q \sim p$ and $p \lessapprox a$. Hence $q \lessapprox a$. Thus

$$(1) \qquad q \leqq z \quad \text{if and only if} \quad q \lessapprox a.$$

It follows from (1) that $p \sim q \leqq z$ implies $p \leqq z$. Hence $z \in Z(L)$ by Lemma 4.1. Since $a = \bigvee(p; p \leqq a) \leqq z$, we have $e(a) \leqq z$. If $p \lessapprox a$, then since $p \lessapprox e(a) \in Z(L)$ we have $p \leqq e(a)$. Therefore $e(a) = z$. This completes the proof of (i) and (ii).

If $e(p) = e(q)$, then by (i) we have $p \lessapprox q$, whence $p \approx q$. Conversely, if $p \approx q$ then it is evident that $e(p) = e(q)$.

Theorem 4.1. *Let L be a complete atomistic lattice with property* (P).

(i) *L is a direct sum of irreducible sublattices $L(0, z_\alpha)$ of L. For every element $a \in L$,*

$$a = \bigvee(a_\alpha; \alpha \in I) \quad \textit{with} \quad a_\alpha \in L(0, z_\alpha)$$

and the elements a_α are uniquely determined.

(ii) *Two atoms $p, q \in L$ are contained in the same $L(0, z_\alpha)$ if and only if $p \approx q$.*

Proof. (i) follows from Lemmas 4.2 and 4.3, and (ii) follows from Lemma 4.4 (iii).

Corollary 1. *If L is a complete DAC-lattice of a matroid lattice, then (i) of Theorem 4.1 holds and two atoms $p, q \in L$ are contained in the same $L(0, z_\alpha)$ if and only if p and q are perspective.*

Proof. This follows from Theorem 4.1 since $p \approx q$ is equivalent to $p \sim q$ by Theorem 3.1 (ii).

Remark. The result of this corollary was proved in [10], Theorem 2 and [14], Theorem 2 on a complete DAC-lattice and on a matroid lattice respectively. These theorems are unified by Theorem 4.1.

Corollary 2. *Let L be a complete DAC-lattice or a matroid lattice. L is irreducible if and only if all atoms are perspective.*

Theorem 4.2. *Let L be a complete DAC-lattice* (*or a matroid lattice*). *For two elements a and b of L, $a \bigtriangledown b$ holds if and only if $e(a) \frown e(b) = 0$.*

PROOF. (i) If $e(a)\cap e(b)\neq 0$, then taking an atom p with $p\leqq e(a)\cap e(b)$, we have $p\lesssim a$ and $p\lesssim b$ by Lemma 4.4 (i). Then it follows from Theorem 3.1 (ii) and Theorem 3.2 that $a\bigtriangledown b$ does not hold.

(ii) If $a\bigtriangledown b$ does not hold, then it follows from Theorem 3.2 that there exist atoms p and q such that $p\sim q$, $p\leqq a$ and $q\leqq b$. Then $e(a)\cap e(b)\geqq e(p)\cap e(q)=e(p)\neq 0$.

Finally we add the following result.

LEMMA 4.5. *A complete DAC-lattice L is $\bigtriangledown$-continuous, that is, if $a_\delta\bigtriangledown b$ and $a_\delta\uparrow a$ then $a\bigtriangledown b$.*

PROOF. Let $a_\delta\bigtriangledown b$ and $a_\delta\uparrow a$. Assume that $a\bigtriangledown b$ does not hold. It follows from Theorem 3.2 that there exist atoms p and q such that $p\sim q$, $p\leqq a$ and $q\leqq b$. Since

$$p\leqq a=\bigcup_\delta a_\delta=\bigcup_\delta\left(\bigcup(r;\, r\leqq a_\delta)\right),$$

by property (P) there exists an atom r such that $r\sim p$ and $r\leqq a_\delta$ for some δ. Then it follows from Theorem 3.1 (ii) and Theorem 3.2 that $a_\delta\bigtriangledown b$ does not hold, a contradiction.

REMARK. The $\bigtriangledown$-continuity of a matroid lattice L follows directly from the upper continuity of L.

§5. Finite-modular AC-lattices and projective lattices

First we shall construct a complete finite-modular AC-lattice from a projective lattice.

THEOREM 5.1. *Let Λ be a projective lattice with the lattice operations $\vee$ and $\wedge$. If a subset L of Λ satisfies the following three conditions:*

(1) $0, 1\in L$,

(2) *if p is an atom of Λ and $a\in L$ then $a\vee p\in L$,*

(3) *if $a_\alpha\in L$ for every α then $\wedge_\alpha a_\alpha\in L$,*

then the subset L with the same order as Λ is a complete finite-modular AC-lattice.

PROOF. (i) It follows from (3) that, in L, the meet $\bigcap_\alpha a_\alpha$ of elements a_α exists and is equal to $\wedge_\alpha a_\alpha$. Hence the join $\bigcup_\alpha a_\alpha$ also exists (in fact, it is $\wedge(x\in L;\, a_\alpha\leqq x$ for every α)). Thus L is a complete lattice.

(ii) It follows from (1) and (2) that if p is an atom of Λ then $p\in L$ and hence p is an atom of L. Since Λ is atomistic, for every $a\in L$ we have

$$a=\vee(p;\, p\leqq a)\leqq\bigcup(p;\, p\leqq a)\leqq a.$$

Hence L is atomistic. Moreover, any atom p of L is an atom of Λ. Because, since Λ is atomistic there exists an atom q of Λ with $q\leqq p$, and since $q\in L$, we have $q=p$.

(iii) Let $a \in L$ and p be an atom with $p \nleq a$. Since Λ has the covering property, we have $a \lessdot a \cup p$ in Λ. Since $a \vee p = a \cup p$ by (2), we have $a \lessdot a \cup p$ in L. Thus L has the covering property. Moreover, since Λ is modular, it has the property (δ) of Lemma 2.2. Hence by (2) L has the same property and hence L is finite-modular.

Note that this theorem also holds when Λ is not a projective lattice but only a complete finite-modular AC-lattice.

Next, we shall construct a projective lattice from a finite-modular AC-lattice.

Definition. A *projective space* Ω is a set of points with a system of subsets, called *lines*, satisfying the following two conditions:

(PG 1) Every line contains at least two points. Two different points p and q are in one and only one line, which is denoted by $\overline{pq}$.

(PG 2) Let p, q and r be different points which are not contained in one line. If s and t are different points such that $s \in \overline{pq}$ and $t \in \overline{qr}$ then there exists a point u such that $u \in \overline{pr}$ and $u \in \overline{st}$.

A subset S of Ω is called a linear set if $p, q \in S$ implies $\overline{pq} \subset S$. The set $L(\Omega)$ of all linear set of Ω forms a projective lattice, ordered by set-inclusion, where the meet $\wedge_\alpha S_\alpha$ is the intersection of linear sets S_α and the join $S_1 \vee S_2$ of two linear sets is equal to the set $\{r \in \Omega;\ r \in \overline{pq},\ p \in S_1,\ q \in S_2\}$. (Cf. [6], Kap. III, §3.)

Lemma 5.1. *Let L be an atomic lattice and assume that $(p, q \cup r)M$ and $(p \cup q, r \cup s)M$ hold for all atoms p, q, r and s of L. Then the set $\Omega(L)$ of all atoms of L forms a projective space where $\overline{pq} = \{r \in \Omega(L);\ r \leqq p \cup q\}$. Moreover for every $a \in L$ the set $\omega(a) = \{p \in \Omega(L);\ p \leqq a\}$ is a linear set of Ω.*

Proof. First we shall show that

(*) if p, q and r are atoms such that $r \leqq p \cup q$ and $p \neq r$ then $q \leqq p \cup r$ and $p \cup q = p \cup r$.

Since $(q, p \cup r)M$ holds, we have

$$p \cup \{q \cap (p \cup r)\} = (p \cup q) \cap (p \cup r) = p \cup r > p.$$

Hence $q \cap (p \cup r) \neq 0$. Then $q \leqq p \cup r$ and $p \cup q = p \cup r$.

Next we shall show that

(**) if p, q, r and s are atoms such that $p \leqq q \cup r \cup s$ then there exists an atom t such that $p \leqq q \cup t$ and $t \leqq r \cup s$.

When $p=q$, $t=r$ may be used, and when $q \leqq r \cup s$, $t=p$ may be used. When $p \neq q$ and $q \nleq r \cup s$, it follows from $(r \cup s, p \cup q)M$ that

$$q \cup \{(r \cup s) \cap (p \cup q)\} = (q \cup r \cup s) \cap (p \cup q) \geqq p.$$

Hence $(r\cup s)\cap(p\cup q)\neq 0$, and since L is atomic there exists an atom t such that $t\leqq(r\cup s)\cap(p\cup q)$. Since $q\nleqq r\cup s$, we have $t\neq q$. Hence it follows from (*) that $p\leqq q\cup t$. Thus (**) has been proved.

In the set $\Omega(L)$ of atoms, let the set $\overline{pq}=\{r\in\Omega(L);\, r\leqq p\cup q\}$ be a line for every pair of different atoms p and q. We shall show that (PG 1) holds. Let p and q are different atoms contained in a line $\overline{rs}$ $(r\neq s)$. When $r=p$, we have $q\leqq p\cup s$. Hence it follows from (*) that $p\cup s=p\cup q$, which implies $\overline{pq}=\overline{rs}$. When $r\neq p$, $p\leqq r\cup s$ implies $r\cup p=r\cup s$ by (*). Hence $q\leqq r\cup p$, and by (*) again we have $p\cup q=r\cup p$. Hence $\overline{pq}=\overline{rs}$. Thus (PG 1) holds.

Next we shall show that (PG 2) holds. Let $s\in\overline{pq}$, $t\in\overline{qr}$ and $s\neq t$. When $s=p$, $u=p$ may be used. When $s\neq p$, by (*) we have $q\leqq p\cup s$, whence $t\leqq q\cup r \leqq s\cup p\cup r$. It follows from (**) that there exists an atom u such that $t\leqq s\cup u$ and $u\leqq p\cup r$. Then $u\in\overline{pr}$, and by (*) $u\in\overline{st}$. Thus we have proved that $\Omega(L)$ is a projective space. It is obvious that $\omega(a)$ is a linear set.

THEOREM 5.2. *Let L be a finite-modular AC-lattice. The set $\Omega(L)$ of all atoms of L forms a projective space and there exists a one-to-one, order-preserving mapping ω of L into the projective lattice $L(\Omega(L))$ of linear sets of $\Omega(L)$, with the following four properties*:

(1) $\omega(0)=0$, *and if L has* 1 *then* $\omega(1)=1$.

(2) $\omega(a\cap b)=\omega(a)\wedge\omega(b)$ *for every* $a,\, b\in L$. *Moreover* $\omega(\bigcap_\alpha a_\alpha)=\bigwedge_\alpha\omega(a_\alpha)$ *if* $\bigcap_\alpha a_\alpha$ *exists.*

(3) $\omega(a\cup b)\geqq\omega(a)\vee\omega(b)$ *for every* $a,\, b\in L$, *and equality holds if and only if* $(a, b)M^*$ *holds in* L.

(4) *If* $a\in F(L)$ *then* $\omega(a\cup b)=\omega(a)\vee\omega(b)$.

If L is a complete finite-modular AC-lattice induced from a projective lattice Λ by the method of Theorem 5.1, *then $L(\Omega(L))$ is isomorphic to Λ.*

PROOF. It follows from Lemma 5.1 that $\Omega(L)$ forms a projective space and that $\omega(a)\in L(\Omega(L))$ for every $a\in L$. ω is evidently order-preserving and is one-to-one since L is atomistic. Statement (1) evidently holds. (2) holdes since $\bigwedge_\alpha\omega(a_\alpha)$ is the intersection of linear sets $\omega(a_\alpha)$. Next it is evident that

$$(*)\qquad \omega(a\cup b)\geqq\omega(a)\vee\omega(b).$$

Since
$$\omega(a)\vee\omega(b)=\{p\in\Omega(L);\, p\in\overline{qr},\ q\in\omega(a),\ r\in\omega(b)\}$$
$$=\{p\in\Omega(L);\, p\leqq q\cup r,\ q\leqq a,\ r\leqq b\},$$

it follows from Lemma 2.3 (i) that equality holds in (*) if and only if $(a, b)M^*$ holds. Thus (3) has been proved. (4) follows from (3) and Lemma 2.2 (ii).

If L is a complete finite-modular AC-lattice induced from a projective lattice Λ then $\Omega(L)=\Omega(\Lambda)$. It follows from [6], Kap. III, Satz 3.2 that Λ is isomorphic to $L(\Omega(L))$.

REMARK. If L is a DAC-lattice then property (4) may be replaced by the following property:

(4′) If $a \in \check{F}(L)$ then $\omega(a \cup b) = \omega(a) \vee \omega(b)$.

This theorem is a generalization of [11], Théorème XVIII.

§6. The lattice of closed subspaces

Let E be a vector space over a field K, where K is not necessarily commutative. It is easy to see that the set $L(E)$ of all subspaces of E forms an irreducible projective lattice, ordered by set-inclusion, where the meet of subspaces is equal to their intersection (see [2], p. 367).

Let E (resp. F) be a left (resp. right) vector space over a field K. Assume that there exists a bilinear mapping f of $E \times F$ onto K; precisely f satisfies the following two conditions for all $\xi, \eta \in E$, $\xi', \eta' \in F$ and $\lambda, \mu \in K$.

(1) $f(\lambda\xi + \mu\eta, \xi') = \lambda f(\xi, \xi') + \mu f(\eta, \xi')$.
(2) $f(\xi, \xi'\lambda + \eta'\mu) = f(\xi, \xi')\lambda + f(\xi, \eta')\mu$.

For any subset A of E we put

$$A^0 = \{\xi' \in F; f(\xi, \xi') = 0 \quad \text{for every} \quad \xi \in A\}$$

and for any subset B of F we put

$$B^0 = \{\xi \in E; f(\xi, \xi') = 0 \quad \text{for every} \quad \xi' \in B\}.$$

By (1) and (2), A^0 and B^0 are subspaces of F and E respectively, and we have $A \subset A^{00}$ and $B \subset B^{00}$. A subspace A of E is called *F-closed* if $A^{00} = A$, and a subspace B of F is called *E-closed* if $B^{00} = B$. E is the largest F-closed subspace and F^0 is the smallest one. F is the largest E-closed subspace and E^0 is the smallest one.

REMARK. In the above definitions, if the field K has an involutive anti-automorphism $\lambda \to \lambda'$ then the space F may be a left vector space replacing the element $\xi'\lambda$ by $\lambda'\xi'$. Then the mapping f is sesquilinear.

LEMMA 6.1. (1) *If $A_\alpha (\alpha \in I)$ are F-closed subspaces of E, then the intersection of A_α is also F-closed.*

(ii) *If A is an F-closed subspace and $\alpha \notin A$ then the sum $A + K\alpha$ is also F-closed.* ($A_1 + A_2 = \{\xi + \eta; \xi \in A_1, \eta \in A_2\}$)

The corresponding statements for E-closed subspaces also hold.

PROOF. It is easy to prove (i). We shall prove (ii). Let $\xi \in (A + K\alpha)^{00}$. Since $\alpha \notin A = A^{00}$, there exists $\alpha' \in A^0$ such that $f(\alpha, \alpha') \neq 0$. We may assume $f(\alpha, \alpha') = 1$, replacing α' by $\alpha' f(\alpha, \alpha')^{-1}$. Then for any $\xi' \in A^0$ we have

$$\xi' - \alpha' f(\alpha, \xi') \in (A + K\alpha)^0,$$

since $\xi' - \alpha' f(\alpha, \xi') \in A^0$ and since

$$f(\alpha, \xi' - \alpha' f(\alpha, \xi')) = f(\alpha, \xi') - f(\alpha, \alpha') f(\alpha, \xi') = 0.$$

Hence

$$\begin{aligned} 0 &= f(\xi, \xi' - \alpha' f(\alpha, \xi')) = f(\xi, \xi') - f(\xi, \alpha') f(\alpha, \xi') \\ &= f(\xi - f(\xi, \alpha')\alpha, \xi'). \end{aligned}$$

This means $\xi - f(\xi, a')\alpha \in A^{00} = A$, whence $\xi \in A + K\alpha$. Therefore $A + K\alpha$ is F-closed.

THEOREM 6.1. *If E (resp. F) is a left (resp. right) vector space over a field K and if there exists a bilinear mapping f of $E \times F$ onto K, then both the set $L_F(E)$ of all F-closed subspaces of E and the set $L_E(F)$ of all E-closed subspaces of F form irreducible complete DAC-lattices, ordered by set-inclusion.*

PROOF. The set $\Lambda = \{A \in L(E); F^0 \subset A\}$ is a projective sublattice of $L(E)$, where every atom has the form $F^0 + K\alpha$ ($\alpha \notin F^0$). The set $L_F(E)$ is a subset of Λ and, putting $L = L_F(E)$, it follows from Lemma 6.1 that L satisfies the conditions of Theorem 5.1. Hence $L_F(E)$ is a complete AC-lattice and $L_E(F)$ is also. Since $L_F(E)$ and $L_E(F)$ are dual-isomorphic by the mapping $A \to A^0$, they are DAC-lattices. Next, for two different atoms $F^0 + K\alpha$ and $F^0 + K\beta$ of $L_F(E)$ there exists a third atom $F^0 + K(\alpha + \beta)$ included in $F^0 + K\alpha + K\beta$. Hence $F^0 + K\alpha$ and $F^0 + K\beta$ are perspective. It follows from Corollary 2 of Theorem 4.1 that $L_F(E)$ is irreducible.

REMARK. McLaughlin [10] shows that if L is an irreducible complete DAC-lattice whose length is at least 4 then there exists a pair of vector spaces E and F with a bilinear mapping f such that L is isomorphic to $L_F(E)$. Note that this representation theorem may be implied from the well-known representation theorem of projective lattices and Theorem 5.2.

COROLLARY 1. *The set $L_C(E)$ of all closed subspaces of a locally convex space E (in particular a normed space) forms an irreducible complete DAC-lattice.*

PROOF. Let E^* be the set of all continuous linear forms on E and let $f(\xi, \xi')$ be the value of $\xi' \in E^*$ at $\xi \in E$. Then it follows from the theorem that $L_{E^*}(E)$ is an irreducible complete DAC-lattice. By the Hahn-Banach theorem, a subspace of E is closed if and only if it is E^*-closed. Hence $L_C(E)$ coincides with $L_{E^*}(E)$.

COROLLARY 2. *If a field K has an involutive anti-automorphism and if a left vector space E over K has a sesquilinear mapping $f: E \times E \to K$ which satisfies the following condition:*

$$(*) \qquad f(\xi, \xi) = 0 \quad implies \quad \xi = 0$$

(for instance if E is a prehilbert space), then $L_E(E)$ forms an irreducible complete orthocomplemented AC-lattice.

PROOF. It follows from (*) that the mapping $A \rightarrow A^0$ is an orthocomplementation in $L_E(E)$. Hence this corollary is a consequence of the theorem.

Finally we add two remarks on the theorems of Piron [11]. Théorème XXI of [11] can be reformed as follows without changing the proof.

THEOREM 6.2. *If L is an irreducible complete orthocomplemented AC-lattice whose length is at least 4, then there exists a field K with an involutive anti-automorphism and exists a left vector space E over K with a sesquilinear mapping f: $E \times E \rightarrow K$ such that L is isomorphic to $L_E(E)$.*

Théorème XXII of [11] can be reformed as follows:

THEOREM 6.3. *When E is a prehilbert space, the lattice $L_E(E)$ is orthomodular if and only if E is complete.*

The proof of this theorem is given in [1].

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