J. Sci. Hiroshima Univ. Ser. A-I
32 (1968), 17-53

Direct Solution of Partial Difference Equations for a Rectangle

Hisayoshi Shintani

(Received February 20, 1968)

1. Introduction

In this paper, we are concerned with the direct solution of the systems of linear algebraic equations arising from the discretization of linear partial differential equations over a rectangle. Such a system is usually solved by means of the iterative methods, and the direct methods are rarely used because of storage capacity [11][1]. Among the direct methods, however, there are known the square root method [11], the hypermatrix method [9, 36], the tensor product method [18], the method of summary representation [32], the method of lines [12, 20, 25, 26, 27, 37, 46], and so on [13, 16, 23, 39, 40, 45].

Although the results stated in this paper are not all new, they are summarized in a somewhat unified form. The methods can easily be extended to the problems in higher dimensions and to the domains consisting of rectangles. Several examples to which the direct methods are applicable are presented.

2. Preliminaries

2.1 Tridiagonal matrices

Let x be a real number and let $U_r(x)$ and $V_r(x)$ be the solutions of the difference equation

$$y_{r+1}-x y_r+y_{r-1}=0 \qquad (r=0, 1, \cdots) \tag{2.1}$$

satisfying the initial conditions $y_{-1}=0$, $y_0=1$ and $y_{-1}=1$, $y_0=x/2$ respectively. Then, as is easily checked, we have the following

Lemma 1. *$U_r(x)$ and $V_r(x)$ are expressed as follows:*

$$U_r(x)=\begin{cases} \dfrac{\sinh(r+1)\omega}{\sinh\omega}, & 2\cosh\omega=x \qquad (x\geqq 2) \\[2ex] \dfrac{\sin(r+1)\theta}{\sin\theta}, & 2\cos\theta=x \qquad (|x|<2) \\[2ex] (-1)^r\dfrac{\sinh(r+1)\omega}{\sinh\omega}, & 2\cosh\omega=|x| \quad (x\leqq -2) \end{cases}$$

1) Numbers in square brackets refer to the references listed at the end of this paper.

$$V_r(x) = \begin{cases} \cosh(r+1)\omega, & 2\cosh\omega = x \quad (x \geqq 2) \\ \cos(r+1)\theta, & 2\cos\theta = x \quad (|x|<2) \\ (-1)^{r+1}\cosh(r+1)\omega, & 2\cosh\omega = |x| \quad (x \leqq -2). \end{cases}$$

The general solution of the equation (2.1) *is given by the formula*

$$y_r = C_1 U_{r-1}(x) + C_2 V_{r-1}(x), \tag{2.2}$$

where C_1 and C_2 are arbitrary constants.

We introduce the following $k\times k$ matrices ($k \geqq 3$):

$$I_k = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}, \quad J_k = \begin{pmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{pmatrix}, \quad K_k = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad Z_k = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix},$$

$$U_k = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & & 0 \end{pmatrix}, \quad V_k = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 0 & \\ & & \ddots & \ddots \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix}, \quad U_k^J = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ & & 0 \\ 0 & & & 1 \end{pmatrix}, \quad V_k^J = \begin{pmatrix} 0 & & & 0 \\ 0 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 0 & \ddots \\ & & & 1 & 0 \end{pmatrix}.$$

Let p, q, α and β be the real numbers such that

$$1+\alpha>0, \quad 1+\beta>0, \tag{2.3}$$

and put

$$L = L(k; p, q; \alpha, \beta) = J_k + pU_k + qU_k^J + \alpha V_k + \beta V_k^J$$

$$= \begin{pmatrix} p, & 1+\alpha & & & 0 \\ 1, & 0, & 1, & & \\ & \ddots & \ddots & \ddots & \\ & & 1, & 0, & 1 \\ 0 & & & 1+\beta, & q \end{pmatrix}$$

Then we have the following

LEMMA 2. *Under the condition* (2.3), *the eigenvalues of L are all real and distinct and they are the roots of the equation*

$$F_k(\lambda) = U_k(\lambda) - (p+q)U_{k-1}(\lambda) + (pq-\alpha-\beta)U_{k-2}(\lambda) + \tag{2.4}$$
$$+ (p\beta+q\alpha)U_{k-3}(\lambda) + \alpha\beta U_{k-4}(\lambda) = 0.$$

Let λ be an eigenvalue of L and put

$$x_j = U_{j-1}(\lambda) - pU_{j-2}(\lambda) - \alpha U_{j-3}(\lambda) \qquad (j=1, 2, \cdots, k), \tag{2.5}$$

$$x^T = (x_1, x_2, \cdots, x_k), \tag{2.6}$$

then x *is an eigenvector corresponding to* λ.

PROOF. Let $L=(l_{ij})$, then since L is a real tridiagonal matrix and $l_{i+1,\, i}l_{i,\, i+1}>0$ $(i=1, 2, \cdots, k-1)$, the eigenvalues of L are all real and distinct [44].

Let λ be an eigenvalue of L and x be an eigenvector corresponding to λ. Then there holds the relation $(L-\lambda I)x=\mathbf{0}$, namely

$$(p-\lambda)x_1+(1+\alpha)x_2=0, \tag{2.7}$$

$$x_{j-1}-\lambda x_j+x_{j+1}=0 \qquad (j=2, 3, \cdots, k-1), \tag{2.8}$$

$$(1+\beta)x_{k-1}+(q-\lambda)x_k=0. \tag{2.9}$$

By (2.2) x_j satisfying (2,8) can be written as follows:

$$x_j = C_1U_{j-1}(\lambda)+C_2V_{j-1}(\lambda) \tag{2.10}$$

and, by (2.7) and (2.9), constants C_1 and C_2 must satisfy the equations

$$\begin{aligned}(p-\lambda)x_1+(1+\alpha)x_2 &= px_1+\alpha x_2-x_0\\ &= (p+\alpha U_1(\lambda))C_1+(pV_0(\lambda)+\alpha V_1(\lambda)-1)C_2=0,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&(1+\beta)x_{k-1}+(q-\lambda)x_k = \beta x_{k-1}+qx_k-x_{k+1}\\ &= (\beta U_{k-2}(\lambda)+qU_{k-1}(\lambda)-U_k(\lambda))C_1+(\beta V_{k-2}(\lambda)+qV_{k-1}(\lambda)-V_k(\lambda))C_2=0.\end{aligned} \tag{2.12}$$

The necessary and sufficient condition for the equations (2.11) and (2.12) to have a non-trivial solution is that

$$\begin{aligned}&(pV_0+\alpha V_1-1)(\beta U_{k-2}+qU_{k-1}-U_k)-(pU_0+\alpha U_1)(\beta V_{k-2}+qV_{k-1}-V_k)\\ &= U_k-p(V_0U_k-U_0V_k)-qU_{k-1}+pq(V_0U_{k-1}-U_0V_{k-1})-\beta U_{k-2}-\\ &\quad -\alpha(V_1U_k-U_1V_k)+p\beta(V_0U_{k-2}-U_0V_{k-2})+q\alpha(V_1U_{k-1}-U_1V_{k-1})+\\ &\quad +\alpha\beta(V_1U_{k-2}-U_1V_{k-2})=0.\end{aligned} \tag{2.13}$$

Using Lemma 1 and addition theorems for trigonometric and hyperbolic functions, we can rewrite (2.13) as (2.4).

In the case where $p+\alpha U_1(\lambda)\neq 0$, from (2.11) we have

$$C_1 = C_2(1-pV_0(\lambda)-\alpha V_1(\lambda))/(p+\alpha U_1(\lambda))$$

and, if we put $C_2=p+\alpha U_1(\lambda)$, then it follows from (2.10) that

$$\begin{aligned}x_j &= U_{j-1}(\lambda)-p(V_0(\lambda)U_{j-1}(\lambda)-V_{j-1}(\lambda))-\alpha(V_1(\lambda)U_{j-1}(\lambda)-U_1(\lambda)V_{j-1}(\lambda))\\ &= U_{j-1}(\lambda)-pU_{j-2}(\lambda)-\alpha U_{j-3}(\lambda).\end{aligned}$$

In the case where $p+\alpha U_1(\lambda)=0$, from (2.3) it follows that

$$1-pV_0(\lambda)-\alpha V_1(\lambda)=1+\alpha(U_1(\lambda)V_0(\lambda)-V_1(\lambda))=1+\alpha>0,$$

so that $C_2=0$ from (2.11). If we put $C_1=1+\alpha$, then we have

$$x_j=C_1U_{j-1}(\lambda)=(1+\alpha)U_{j-1}(\lambda)$$
$$=U_{j-1}(\lambda)-pU_{j-2}(\lambda)-\alpha U_{j-3}(\lambda).$$

Thus the vector x given by (2.6) is an eigenvector corresponding to λ.

COROLLARY 1. *Let λ_j $(j=1, 2, \cdots, k)$ be the eigenvalues of L and put*

$$G(k; p, q; \alpha, \beta)=\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k),$$
$$R(k; p, q; \alpha, \beta)=(r_{ij}),$$
$$D(k; \alpha, \beta)=\operatorname{diag}(1/(1+\alpha), 1, 1, \cdots, 1, 1/(1+\beta)),$$

where

$$r_{ij}=c_j\tilde{r}_{ij},$$
$$\tilde{r}_{ij}=U_{i-1}(\lambda_j)-pU_{i-2}(\lambda_j)-\alpha U_{i-3}(\lambda_j),$$

$$c_j=1/\Big(\sum_{i=2}^{k-1}\tilde{r}_{ij}^2+\tilde{r}_{1j}^2/(1+\alpha)+\tilde{r}_{kj}^2/(1+\beta)\Big)^{1/2}. \tag{2.14}$$

Then it is valid that

$$L(k; p, q; \alpha, \beta)=R(k; p, q; \alpha, \beta)G(k; p, q; \alpha, \beta)R(k; p, q; \alpha, \beta)^{-1},$$
$$R(k; p, q; \alpha, \beta)^{-1}=R(k; p, q; \alpha, \beta)^T D(k; \alpha, \beta).$$

PROOF. Put

$$F=\operatorname{diag}(1/\sqrt{1+\alpha}, 1, 1, \cdots, 1, 1/\sqrt{1+\beta}).$$

Since $FLF^{-1}=S$ is a real symmetric matrix, there exists an orthogonal matrix T such that $S=TGT^{-1}$. If we put

$$\tilde{\boldsymbol{r}}_j^T=(\tilde{r}_{1j}, \tilde{r}_{2j}, \cdots, \tilde{r}_{kj}),$$

then $c\tilde{\boldsymbol{r}}_j$ $(c\neq 0)$ is an eigenvector of L corresponding to λ_j. Let R be the matrix

$$R=(c_1\tilde{\boldsymbol{r}}_1, c_2\tilde{\boldsymbol{r}}_2, \cdots, c_k\tilde{\boldsymbol{r}}_k),$$

then it follows that

$$LF^{-1}T=F^{-1}TG, \; LR=RG.$$

Hence we can choose c_j $(j=1, 2, \cdots, k)$ so that $FR=T$. Evidently such a c_j

is given by (2.14). Then it follows that

$$R^{-1}=T^{-1}F=T^{T}F=R^{T}F^{T}F=R^{T}F^{2}=R^{T}D.$$

From this corollary we directly obtain the following

COROLLARY 2. *Suppose that the matrix* $aI_k-L(k;\,p,\,q;\,\alpha,\,\beta)$ *is non-singular. Then*

$$\begin{aligned}&(aI_k-L(k;\,p,\,q;\,\alpha,\,\beta))^{-1}\\&\quad=R(k;\,p,\,q;\,\alpha,\,\beta)(aI_j-G(k;\,p,\,q;\,\alpha,\,\beta))^{-1}R(k;\,p,\,q;\,\alpha,\,\beta)^{-1}.\end{aligned}$$

Differentiating the formula (2.4) and using the relation (2.1), we have

COROLLARY 3. *The functions* $F_k(\lambda)$, $F_k'(\lambda)$ and $F_k''(\lambda)$ *satisfy the following recurrence formulas*:

$$\begin{aligned}&F_r(\lambda)=\lambda F_{r-1}(\lambda)-F_{r-2}(\lambda),\\&F_r'(\lambda)=\lambda F_{r-1}'(\lambda)-F_{r-2}'(\lambda)+F_{r-1}(\lambda)\qquad(r=3,\,4,\,\cdots,\,k),\\&F_r''(\lambda)=\lambda F_{r-1}''(\lambda)-F_{r-2}''(\lambda)+2F_{r-1}'(\lambda),\end{aligned}$$

where

$$\begin{aligned}&F_1(\lambda)=(1-\alpha\beta)\lambda-(p+q+p\beta+q\alpha),\\&F_2(\lambda)=\lambda^2-(p+q)\lambda+pq-\alpha-\beta-\alpha\beta-1,\\&F_1'(\lambda)=1-\alpha\beta,\quad F_2'(\lambda)=2\lambda-(p+q),\\&F_1''(\lambda)=0,\quad F_2''(\lambda)=2.\end{aligned}$$

Now put

$$L_1(k)=L(k;\,0,\,0;\,0,\,0),\quad L_2(k)=L(k;\,1,\,1;\,0,\,0),\quad L_3(k)=L(k;\,1,\,0;\,0,\,0),$$

$$L_4(k)=L(k;\,0,\,0;\,1,\,1),\quad L_5(k)=L(k;\,0,\,0;\,1,\,0),\quad L_6(k)=L_1(k)+Z_k,$$

$$G_i(k)=\mathrm{diag}(2\cos\theta_{i1},\,2\cos\theta_{i2},\,\cdots,\,2\cos\theta_{ik}),$$

where

$$\theta_{1j}=\frac{j\pi}{k+1},\quad \theta_{2j}=\frac{(j-1)\pi}{k},\quad \theta_{3j}=\frac{(2j-1)\pi}{2k+1},$$

$$\theta_{4j}=\frac{(j-1)\pi}{k-1},\quad \theta_{5j}=\frac{(2j-1)\pi}{2k},\quad \theta_{6j}=\frac{2(j-1)\pi}{k}.$$

Further put

$$R_1(k)=(\sin i\theta_{1j}),\quad R_2(k)=\Big(\sin\frac{(2i-1)}{2}\theta_{2j}\Big),$$

$$R_3(k)=\big(\sin(k+1-i)\theta_{3j}\big),\quad R_4(k)=\big(\cos(i-1)\theta_{4j}\big),$$

$$R_5(k)=\big(\cos(i-1)\theta_{5j}\big),\quad R_6(k)=(r_{ij}),$$

where

$$r_{i1}=1/\sqrt{2},\quad r_{ij}=\cos(i-1)\theta_{6j}\qquad(2\leqq j\leqq l-1),$$

$$r_{il}=\delta\cos(i-1)\theta_{6l},\quad r_{ij}=\sin(i-1)\theta_{6j}\qquad(l+1\leqq j\leqq k),$$

$$l=[k/2],\quad \delta=\begin{cases}1 & (k\text{: odd})\\ 1/\sqrt{2} & (k\text{: even}).\end{cases}$$

Then we have the following

THEOREM 1. *There holds the relation*

$$L_i(k)=R_i(k)G_i(k)R_i(k)^{-1}\qquad(i=1,\,2,\,\cdots,\,6),$$

and $R_i(k)^{-1}$ *are represented as follows*:

$$R_1(k)^{-1}=\frac{2}{k+1}R_1(k),\quad R_2(k)^{-1}=\frac{2}{k}R_2(k)^T,\quad R_3(k)^{-1}=\frac{4}{2k+1}R_3(k)^T,$$

$$R_4(k)^{-1}=\frac{2}{k-1}R_4(k)^TD_1,\quad R_5(k)^{-1}=\frac{2}{k}R_5(k)^TD_2,\quad R_6(k)^{-1}=\frac{2}{k}R_6(k)^T,$$

where

$$D_1=\operatorname{diag}(1/2,\,1,\,\cdots,\,1,\,1/2),\quad D_2=\operatorname{diag}(1/2,\,1,\,\cdots,\,1).$$

PROOF. The results for $i=1, 2, \cdots, 5$ follow directly from Corollary 1. The result for $i=6$ is obtained from the fact that $L_6(k)$ is a circulant matrix [19].

Now put

$$L_7(k;p,q)=L(k;p,q;0,0),\quad L_8(k;p,q)=L(k;p,q;1,1),$$

$$L_9(k;p)=L(k;p,0;0,0),\quad L_{10}(k;p)=L(k;p,0;1,0),$$

and let us define $G_i(k;p,q)$, $R_i(k;p,q)$ $(i=7, 8)$, $G_j(k;p)$ and $R_j(k;p)$ $(j=9, 10)$ likewise.

By Corollary 2 we can obtain the matrix $(aI_k-L)^{-1}$ in terms of the eigenvalues and eigenvectors of L. Without knowledge of eigenvalues, however, we can also write it explicitly by the following

LEMMA 3. *Under the condition* (2.3), *suppose that the matrix* $aI_k-L(k;p,q;\alpha,\beta)$ *is non-singular. Then it is valid that*

$$(aI_k - L(k; p, q; \alpha, \beta))^{-1} = (r_{ij}),$$

where

$$r_{i1} = \Delta^{-1}(U_{k-i} - qU_{k-i-1} - \beta U_{k-i-2}),$$

$$r_{ij} = \begin{cases} \Delta^{-1}(U_{i-1} - pU_{i-2} - \alpha U_{i-3})(U_{k-j} - qU_{k-j-1} - \beta U_{k-j-2}) & (j \geqq i), \\ \Delta^{-1}(U_{k-i} - qU_{k-i-1} - \beta U_{k-i-2})(U_{j-1} - pU_{j-2} - \alpha U_{j-3}) & (j < i), \end{cases}$$

$$(2 \leqq j \leqq k-1)$$

$$r_{ik} = \Delta^{-1}(U_{i-1} - pU_{i-2} - \alpha U_{i-3}),$$

$$\Delta = F_k(a), \quad U_j = U_j(a) \qquad (j = -2, -1, 0, \cdots).$$

PROOF. We consider the system of equations

$$(aI_k - L(k; p, q; \alpha, \beta))\boldsymbol{x} = \boldsymbol{f}, \tag{2.15}$$

where

$$\boldsymbol{x}^T = (x_1, x_2, \cdots, x_k), \quad \boldsymbol{f}^T = (f_1, f_2, \cdots, f_k).$$

From the first $k-1$ equations of (2.15) we obtain inductively

$$x_l = -\sum_{i=2}^{l-1} U_{l-1-i} f_i - U_{l-2} f_1/(1+\alpha) + x_1(U_{l-1} - pU_{l-2} - \alpha U_{l-3})/(1+\alpha). \tag{2.16}$$

$$(l = 1, 2, \cdots, k)$$

Substituting the expressions for x_{k-1} and x_k into the last equation of (2.15), we have

$$\Delta x_1/(1+\alpha) = f_k + \sum_{i=2}^{k-1}(U_{k-i} - qU_{k-i-1} - \beta U_{k-i-2})f_i + \\ + (U_{k-1} - qU_{k-2} - \beta U_{k-3})f_1/(1+\alpha). \tag{2.17}$$

Multiplying (2.16) by Δ and substituting (2.17) into it, we have

$$\Delta x_l = \big((U_{l-1} - pU_{l-2} - \alpha U_{l-3})(U_{k-1} - qU_{k-2} - \beta U_{k-3}) - \Delta U_{l-2}\big)f_1/(1+\alpha) + \\ + \sum_{i=2}^{l-1}\big((U_{k-i} - qU_{k-i-1} - \beta U_{k-i-2})(U_{l-1} - pU_{l-2} - \alpha U_{l-3}) - \Delta U_{l-1-i}\big)f_i + \\ + \sum_{i=l}^{k-1}(U_{k-i} - qU_{k-i-1} - \beta U_{k-i-2})(U_{l-1} - pU_{l-2} - \alpha U_{l-3})f_i + \\ + (U_{k-1} - pU_{k-2} - \alpha U_{k-3})f_k. \tag{2.18}$$

Using Lemma 1 and addition theorems for trigonometric and hyperbolic functions, we can rewrite (2.18) as follows:

$$\Delta x_l=(U_{l-1}-pU_{l-2}-\alpha U_{l-3})\Big(f_k+\sum_{i=l}^{k-1}(U_{k-i}-qU_{k-1-i}-\beta U_{k-2-i})f_i\Big)+$$

$$+(U_{k-l}-qU_{k-1-l}-\beta U_{k-2-l})\Big(\sum_{i=2}^{l-1}(U_{i-1}-pU_{i-2}-\alpha U_{i-3})f_i+f_1\Big).$$

This completes the proof of the lemma.

LEMMA 4. *Let W be a $k\times k$ non-singular matrix and let p and q be constants. Suppose that $(W-pU_k-qU_k^t)$ is non-singular. Then it is valid that*

$$(W-pU_k-qU_k^t)^{-1}=W^{-1}+W^{-1}ZW^{-1},$$

where

$$Z=\begin{pmatrix} p\Delta^{-1}(1-qw_{kk}), & 0, & \cdots, & 0, & pq\Delta^{-1}w_{1k} \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ pq\Delta^{-1}w_{k1}, & 0, & \cdots, & 0, & q\Delta^{-1}(1-pw_{11}) \end{pmatrix}$$

$$W^{-1}=(w_{ij}),\quad \Delta=(1-pw_{11})(1-qw_{kk})-pqw_{1k}w_{k1}.$$

PROOF. Consider the system of equations

$$(W-pU_k-qU_k^t)\boldsymbol{x}=\boldsymbol{f}.$$

Then we have

$$Ax=(I_k-pW^{-1}U_k-qW^{-1}U_k^t)\boldsymbol{x}=W^{-1}\boldsymbol{f}=\boldsymbol{g}, \tag{2.19}$$

where

$$A=\begin{pmatrix} 1-pw_{11}, & & 0 & -qw_{1k} \\ -pw_{21}, & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & 1, & -qw_{k-1k} \\ -pw_{k1} & 0 & & 1-qw_{kk} \end{pmatrix},\quad \boldsymbol{g}=\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ \vdots \\ g_k \end{pmatrix}.$$

From the equations

$$(1-pw_{11})x_1-qw_{1k}x_k=g_1,$$

$$-pw_{k1}x_1+(1-qw_{kk})x_k=g_k,$$

we have

$$\begin{aligned} x_1&=\Delta^{-1}(1-qw_{kk})g_1+\Delta^{-1}qw_{1k}g_k, \\ x_k&=\Delta^{-1}pw_{k1}g_1+\Delta^{-1}(1-pw_{11})g_k, \end{aligned} \tag{2.20}$$

and substituting these into the remaining equations of (2.19), we obtain

$$\begin{aligned} x_l &= g_l + pw_{l1}x_1 + qw_{lk}g_k \\ &= g_l + \big(pw_{l1}\Delta^{-1}(1-qw_{kk}) + qw_{lk}\Delta^{-1}pw_{k1}\big)g_1 + \\ &\quad + \big(pw_{l1}\Delta^{-1}qw_{1k} + qw_{lk}\Delta^{-1}(1-pw_{11})\big)g_k \qquad (l=2,\ 3,\ \cdots,\ k-1). \end{aligned}$$

Further (2.20) can be rewritten as follows:

$$\begin{aligned} x_1 &= g_1 + p\Delta^{-1}\big(w_{11}(1-qw_{kk}) + qw_{1k}w_{k1}\big)g_1 + \Delta^{-1}qw_{1k}g_1, \\ x_k &= g_k + p\Delta^{-1}w_{k1}g_1 + q\Delta^{-1}\big(w_{kk}(1-pw_{11}) + pw_{1k}w_{k1}\big)g_k. \end{aligned}$$

Hence it follows that $A^{-1} = I_k + W^{-1}Z$. Thus the lemma has been proved.

Let A be an $m\times m$ matrix and B be an $n\times n$ matrix. Then we define an $mn\times mn$ matrix $A\otimes B$ by

$$A\otimes B = (a_{ij}B).$$

For simplicity, in the sequel, the matrices A_k, $A_j(k)$ and $A_j(k;p,q)$ are written as A, A_j and $A_j(p,q)$ respectively when $k=n$ and they are written as A_m, $\hat{A}_j$ and $\hat{A}_j(p,q)$ respectively when $k=m$. Further we put

$$S = I_m\otimes R_1, \quad P = \hat{R}_1\otimes I. \tag{2.21}$$

The following lemma is an extension of Lemma 4 and it can be proved analogously.

LEMMA 5. *Let W be an $mn\times mn$ matrix and let p and q be constants. Suppose that the matrix $W-(pU_m+qU_m^J)\otimes I$ is non-singular. Then it is valid that*

$$(W-(pU_m+qU_m^J)\otimes I)^{-1} = W^{-1} + W^{-1}ZW^{-1},$$

where

$$Z = \begin{pmatrix} p\Delta_1^{-1}, 0, \cdots, 0, pq(I-pW_{11})^{-1}W_{1m}\Delta_m^{-1} \\ 0 \qquad\qquad\qquad 0 \\ \vdots \qquad\qquad\qquad \vdots \\ 0 \qquad\qquad\qquad 0 \\ pq(I-qW_{mm})^{-1}W_{m1}\Delta_1^{-1}, 0, \cdots, 0, q\Delta_m^{-1} \end{pmatrix},$$

$$\begin{aligned} W^{-1} &= (W_{ij}) \qquad (i,j=1,\ 2,\ \cdots,\ m), \\ \Delta_1 &= (I-pW_{11}) - pq(I-qW_{mm})^{-1}W_{m1}, \\ \Delta_m &= (I-qW_{mm}) - pq(I-pW_{11})^{-1}W_{1m}, \end{aligned}$$

and W_{ij}'s are $n\times n$ matrices.

Let p, q, α, β, γ and δ be real numbers and put

$$\begin{aligned} M(k;p,q;\alpha,\beta;\gamma,\delta) &= \gamma(K_k + pU_k + \alpha V_k) + \gamma\delta^2(K_k^T + qU_k^J + \beta V_k^J) \\ &= \begin{pmatrix} \gamma p, & \gamma\delta^2+\gamma\alpha & & & \\ \gamma, & 0, & \gamma\delta^2 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \gamma, & 0, & \gamma\delta^2 \\ 0 & & & \gamma+\gamma\delta^2\beta, & \gamma\delta^2 q \end{pmatrix} \end{aligned}$$

where it is assumed that

$$\text{(2.22)} \qquad \gamma \neq 0, \quad \delta > 0, \quad \delta \neq 1.$$

Then, as is easily seen, we have the following

LEMMA 6. *Let E_k be the matrix defined by*

$$E_k = \operatorname{diag}(1, \delta, \delta^2, \dots, \delta^{k-1}).$$

Then, under the condition (2.22), *it is valid that*

$$M(k; p, q; \alpha, \beta; \gamma, \delta) = E_k^{-1} \gamma \delta L(k; p\delta^{-1}, q\delta; \alpha\delta^{-2}, \beta\delta^2) E_k.$$

This lemma reduces the problem of finding Jordan's canonical form of M to that of finding the canonical form of $L(k; p\delta^{-1}, q\delta; \alpha\delta^{-2}, \beta\delta^2)$. As special cases of L, we consider two cases where $p=q=1$, $\alpha=\beta=0$ and $p=q=0$, $\alpha=\beta=1$. Put

$$L_{11}(k; \delta) = L(k; \delta^{-1}, \delta; 0, 0), \quad L_{12}(k; \delta) = L(k; 0, 0; \delta^{-2}, \delta^2),$$

$$G_{11}(k; \delta) = \operatorname{diag}\left(2\cos\frac{\pi}{k}, \dots, 2\cos\frac{(k-1)\pi}{k}, \delta + \delta^{-1}\right)$$

$$G_{12}(k; \delta) = \operatorname{diag}\left(2\cos\frac{\pi}{k-1}, \dots, 2\cos\frac{(k-2)\pi}{k-1}, \delta + \delta^{-1}, -(\delta + \delta^{-1})\right),$$

$$R_{11}(k; \delta) = (r_{ij}), \quad R_{12}(k; \delta) = (s_{ij}),$$

where

$$r_{ij} = \sqrt{2/k}\left(\delta \sin\frac{ij\pi}{k} - \sin\frac{(i-1)j\pi}{k}\right) \Big/ \left(1 + \delta^2 - 2\delta\cos\frac{j\pi}{k}\right)^{1/2} \quad (1 \leqq j \leqq k-1),$$

$$r_{ik} = \sqrt{(1-\delta^2)/(1-\delta^{2k})}\,\delta^{i-1},$$

$$s_{ij} = \left(\delta \sin\frac{ij\pi}{k-1} - \delta^{-1}\sin\frac{(i-2)j\pi}{k-1}\right) \Big/ \left(\left(\frac{k-1}{2} - \sin^2\frac{j\pi}{k-1}\right)(\delta - \delta^{-1})^2 + \right.$$

$$\left. + (\delta + \delta^{-1})^2 \sin^2\frac{j\pi}{k-1}\right)^{1/2} \qquad (1 \leqq j \leqq k-2),$$

$$s_{i\,k-1} = \sqrt{(1-\delta^4)/\big(2(\delta^2 - \delta^{2k})\big)}\,\delta^{i-1},$$

$$s_{ik} = \sqrt{(1-\delta^4)/\big(2(\delta^2 - \delta^{2k})\big)}\,(-\delta)^{i-1}.$$

Then by Corollary 1 we have the following

THEOREM 2. *Under the condition* (2.22), *it is valid that*

$$L_j(k; \delta) = R_j(k; \delta) G_j(k; \delta) R_j(k; \delta)^{-1} \qquad (j = 11, 12),$$

$$R_{11}(k; \delta)^{-1} = R_{11}(k; \delta)^T, \quad R_{12}(k; \delta)^{-1} = R_{12}(k; \delta)^T D_3,$$

where

$$D_3 = \operatorname{diag}(\delta^2/(1+\delta^2),\ 1,\ \cdots,\ 1,\ 1/(1+\delta^2)).$$

We consider further the matrices

$$L_{13}(k;p,q;\delta) = L(k;p\delta^{-1},q\delta;0,0),$$

$$L_{14}(k;p,q;\delta) = L(k;p\delta^{-1},q\delta;\delta^{-2},\delta^2),$$

and define $G_j(k;p,q;\delta)$ and $R_j(k;p,q;\delta)$ $(j=13, 14)$ likewise.

Since

$$aI_k - M(k;p,q;\alpha,\beta;\delta) = E_k^{-1}\gamma\delta((\gamma\delta)^{-1}aI_k - L(k;p\delta^{-1},q\delta;\alpha\delta^{-2},\beta\delta^2))E_k,$$

we can obtain $(aI_k - M)^{-1}$ by Lemma 3.

2.2 Quidiagonal matrices

Let A be the matrix defined by

$$A = aI - 2bJ + (J^2 - 2I), \tag{2.23}$$

where

$$J^2 - 2I = \begin{pmatrix} -1, & 0, & 1 & & & & \\ 0, & 0, & 0, & 1 & & 0 & \\ 1, & 0, & \ddots & \ddots & \ddots & & \\ & 1, & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1 \\ & 0 & & \ddots & \ddots & 0, & 0 \\ & & & & 1, & 0, & -1 \end{pmatrix}.$$

Then we have

$$A = R_1((a-2-b^2)I + (bI-J)^2)R_1^{-1},$$

Thus the matrices of the form (2.23) can be diagonalized easily.

Let $S_k(\lambda)$ and $T_k(\lambda)$ be the solutions of the difference equation

$$y_{k+2} - 2ay_{k+1} + dy_k - 2ay_{k-1} + y_{k-2} = 0 \qquad (k=0, 1, 2, \cdots) \tag{2.24}$$

satisfying the initial conditions

$$y_{-1} = y_0 = 0, \quad y_1 = 1, \quad y_2 = 0,$$

and

$$y_{-1} = y_0 = y_1 = 0, \quad y_2 = 1$$

respectively, where

$$d = a^2 + 2 - \lambda \qquad (a > 0).$$

Put

$$R_j(\lambda)=T_j(\lambda)S_{j-1}(\lambda)-T_{j-1}(\lambda)S_j(\lambda).$$

Then, as is easily checked, we have the following

LEMMA 7. *$R_j(j=0, 1, 2, \dots)$ are the solutions of the difference equation*

(2.25) $R_{j+3}-dR_{j+2}+(4a^2-1)R_{j+1}-(8a^2-2d)R_j+(4a^2-1)R_{j-1}-dR_{j-2}+R_{j-3}=0$

satisfying the initial condition

$$R_{-2}=1,\quad R_{-1}=R_0=R_1=0,\quad R_2=1,\quad R_3=d.$$

Moreover it is valid that

$$R_j(\lambda)=T_j(\lambda)^2-T_{j-1}(\lambda)T_{j+1}(\lambda).$$

Solving the characteristic equation of (2.25), we have the following

LEMMA 8. *$R_j(\lambda)$ can be expressed as follows:*
In the case where $\lambda>(a+2)^2$ or $\lambda<(a-2)^2$,

$$R_j(\lambda)=\frac{2}{(r-2)(s-2)}+\frac{1}{(r-2)(r-s)}(U_j(r)-U_{j-2}(r))-$$
$$-\frac{1}{(s-2)(r-s)}(U_j(s)-U_{j-2}(s)),$$

where

$$r=\frac{1}{2}(a^2-\lambda+\sqrt{(a^2-4-\lambda)^2-16\lambda}),\quad s=\frac{1}{2}(a^2-\lambda-\sqrt{(a^2-4-\lambda)^2-16\lambda}).$$

In the case where $(a-2)^2<\lambda<(a+2)^2$,

$$R_j(\lambda)=\frac{1}{4\lambda}(2-2U_{j-1}(r)U_{j-1}(s)+U_{j-2}(s)U_j(r)+U_j(s)U_{j-2}(r)),$$

where

$$r=a+\sqrt{\lambda},\quad s=a-\sqrt{\lambda}.$$

In the case where $\lambda=(a+2)^2$ or $\lambda=(a-2)^2$,

$$R_j(\lambda)=\begin{cases}\dfrac{2}{(r-2)^2}(1-U_{j-1}(r))+\dfrac{1}{r-2}jU_{j-1}(r) & (r\neq 2),\\[2mm] \dfrac{1}{12}j^2(j^2-1) & (r=2),\end{cases}$$

where

$$r=(a^2-\lambda)/2.$$

Solving the characteristic equation of (2.24) and using the initial conditions, we have the following

LEMMA 9. *$S_k(\lambda)$ and $T_k(\lambda)$ can be expressed explicitly as follows: In the case where $\lambda>0$,*

$$S_k(\lambda)=\frac{1}{2c}(\rho U_{k-1}(\mu)-\mu U_{k-1}(\rho)+U_{k-2}(\mu)-U_{k-2}(\rho)),$$

$$T_k(\lambda)=\frac{1}{2c}(U_{k-1}(\rho)-U_{k-1}(\mu)),$$

where

$$c=\sqrt{\lambda}\,,\quad \rho=a+c,\quad \mu=a-c.$$

In the case where $\lambda<0$,

$$S_k(\lambda)=\frac{1}{4c}(U_{k+1}(\mu)U_{k-3}(\rho)-U_{k-3}(\mu)U_{k+1}(\rho)+ \\ +U_{k-1}(\mu)U_{k-3}(\rho)-U_{k-3}(\mu)U_{k-1}(\rho)),$$

$$T_k(\lambda)=\frac{1}{4c}(U_{k-2}(\mu)U_k(\rho)-U_k(\mu)U_{k-2}(\rho)),$$

where

$$\mu=2\cos\theta=\frac{1}{2}((a+2)^2-\lambda-\sqrt{(a-2)^2-\lambda}),$$

$$\rho=2\cosh\sigma=\frac{1}{2}((a+2)^2-\lambda+\sqrt{(a-2)^2-\lambda}),$$

$$c=\cosh^2\sigma-\cos^2\theta.$$

In the case where $\lambda=0$,

$$S_k(\lambda)=\frac{1}{4-a^2}((2+2k)U_{k+1}(a)+(k-2)U_{k-1}(a)-(2+2k)U_{k-3}(a)),$$

$$T_k(\lambda)=\frac{1}{4-a^2}((k+1)U_{k-2}(a)-U_k(a)).$$

Put

$$N(a;p,q)=(a^2+2)I-2aJ+(J^2-2I)+(p+1)U+(q+1)U^J$$

$$=\begin{pmatrix} a^2+2+p, & -2a, & 1 & & & & \\ -2a, & a^2+2, & -2a, & 1 & & 0 & \\ 1, & -2a, & a^2+2, & -2a, & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1, & -2a, & a^2+2, & -2a, & 1 \\ & 0 & & 1, & -2a, & a^2+2, & -2a \\ & & & & 1, & -2a, & a^2+2+q \end{pmatrix}$$

Then there holds the following

LEMMA 10. *The eigenvalues of* $N(a; p, q)$ *are all real and they are the solutions of the equation*

$$H(\lambda)=R_{n+2}(\lambda)+(p+q)R_{n+1}(\lambda)+pqR_n(\lambda)=0. \tag{2.26}$$

Let λ *be an eigenvalue of* $N(a; p, q)$ and x_i $(i=1, 2, \cdots, n)$ *be the solution of the difference equation*

$$x_{r+2}-2ax_{r+1}+dx_r-2ax_{r-1}+x_{r-2}=0 \qquad (r=1, 2, \cdots) \tag{2.27}$$

satisfying the initial condition

$$x_{-1}=pT_{n+1}(\lambda), \quad x_0=0, \quad x_1=T_{n+1}(\lambda), \quad x_2=-S_{n+1}(\lambda)+pT_n(\lambda)$$

and put

$$\boldsymbol{x}^T=(x_1, x_2, \cdots, x_n).$$

Then $\boldsymbol{x}$ *is an eigenvector corresponding to* λ.

PROOF. Since $N(a; p, q)$ is a real symmetric matrix, its eigenvalues are all real. The equation $N(a; p, q)\boldsymbol{x}-\lambda\boldsymbol{x}=\boldsymbol{0}$ can be written as follows:

$$\begin{cases} (d+p)x_1-2ax_2+x_3=0, \\ -2ax_1+dx_2-2ax_3+x_4=0, \\ x_{i-2}-2ax_{i-1}+dx_i-2ax_{i+1}+x_{i+2}=0 \qquad (i=3, 4, \cdots, n-2), \\ x_{n-3}-2ax_{n-2}+dx_{n-1}-2ax_n=0, \\ x_{n-2}-2ax_{n-1}+(d+q)x_n=0, \end{cases} \tag{2.28}$$

where $d=a^2+2-\lambda$. Then we have inductively

$$x_j=\big(S_j(\lambda)-pT_{j-1}(\lambda)\big)x_1+T_j(\lambda)x_2 \qquad (j=1, 2, \cdots) \tag{2.29}$$

and the last two equations of the system (2.28) become as follows:

$$x_{n-3}-2ax_{n-2}+dx_{n-1}-2ax_n=-x_{n+1}=0, \tag{2.30}$$

$$x_{n-2}-2ax_{n-1}+(d+q)x_n=qx_n+2ax_{n+1}-x_{n+2}=0. \tag{2.31}$$

Subtracting (2.30) from (2.31) and substituting (2.29) into them, we have

$$\big(S_{n+1}(\lambda)-pT_n(\lambda)\big)x_1+T_{n+1}(\lambda)x_2=0, \tag{2.32}$$

$$\big(S_{n+2}(\lambda)-pT_{n+1}(\lambda)-q\big(S_n(\lambda)-pT_{n-1}(\lambda)\big)\big)x_1+\big(T_{n+2}(\lambda)-qT_n(\lambda)\big)x_2=0.$$

For these equations to have a non-trivial solution it is necessary and sufficient that

$$H(\lambda)= T_{n+2}(\lambda)S_{n+1}(\lambda)- T_{n+1}(\lambda)S_{n+2}(\lambda)+p\big(T_{n+1}(\lambda)^2- T_n(\lambda)T_{n+2}(\lambda)\big)+$$
$$+q\big(T_{n+1}(\lambda)S_n(\lambda)- T_n(\lambda)S_{n+1}(\lambda)\big)+pq\big(T_n(\lambda)^2- T_{n-1}(\lambda)T_{n+1}(\lambda)\big)=0.$$

Using Lemma 7, we can rewrite this equation in the form (2.26).

Evidently x_i ($i=1, 2, \cdots, n$) defined by (2.29) satisfy the equation (2.27). By (2.32) we set

$$x_1= T_{n+1}(\lambda),\quad x_2= - S_{n+1}(\lambda)+p\,T_n(\lambda).$$

Then we have from (2.29)

$$x_0=\big(S_0(\lambda)-p\,T_{-1}(\lambda)\big)T_{n+1}(\lambda)+ T_0(\lambda)\big(p\,T_n(\lambda)- S_{n+1}(\lambda)\big)=0,$$
$$x_{-1}=\big(S_{-1}(\lambda)-p\,T_{-2}(\lambda)\big)T_{n+1}(\lambda)+ T_{-1}(\lambda)\big(p\,T_n(\lambda)- S_{n+1}(\lambda)\big)=p\,T_{n+1}(\lambda).$$

This completes the proof of the lemma.

Now we consider the equation

(2.33) $$(N(a;p,q)-\lambda I)\boldsymbol{x}=\boldsymbol{f},$$

where λ is a real number that is not an eigenvalue of $N(a;p,q)$. Then we have the following

Lemma 11. *The solution of the equation* (2.33) *is given by the formula*

(2.34) $$x_r=\sum_{j=1}^{r-2} T_{r-j}(\lambda)f_j+\big(S_r(\lambda)-p\,T_{r-1}(\lambda)\big)x_1+ T_r(\lambda)x_2,$$
$$(r=1, 2, \cdots, n)$$

(2.35) $$x_1=H(\lambda)^{-1}\sum_{j=1}^{n} X_j f_{n+1-j},$$

(2.36) $$x_2=H(\lambda)^{-1}\sum_{j=1}^{n} Y_j f_{n+1-j},$$

where X_j and Y_j are the solutions of the equation (2.24) *satisfying the initial conditions*

$$X_{-1}=q\,T_{n+1}(\lambda),\quad X_0=0,\quad X_1= T_{n+2}(\lambda),\quad X_2=(d+q)T_n(\lambda)-2a\,T_{n-1}(\lambda)+ T_{n-2}(\lambda)$$

and

$$Y_{-1}= -q\,S_{n+1}(\lambda)+pq\,T_n(\lambda),\quad Y_0=0,\quad Y_1= - S_{n+1}(\lambda)+p\,T_n(\lambda),$$
$$Y_2=S_{n+2}(\lambda)-2a\,S_{n+1}(\lambda)-q\,S_n(\lambda)-p\big(T_{n+1}(\lambda)-2a\,T_n(\lambda)-q\,T_{n-1}(\lambda)\big)$$

respectively.

Proof. Form the system (2.33) we have inductively the formula (2.34), and from the last two equations of the system (2.33) it follows that

$$(S_{n+1}(\lambda)-pT_n(\lambda))x_1+T_{n+1}(\lambda)x_2=-\sum_{j=1}^{n}T_{n+1-j}(\lambda)f_j,$$

$$(S_{n+2}(\lambda)-qS_n(\lambda)-pT_{n+1}(\lambda)+pqT_{n-1}(\lambda))x_1+(T_{n+2}(\lambda)-qT_n(\lambda))x_2=$$

$$=-\sum_{j=1}^{n}(T_{n+2-j}(\lambda)-qT_{n-j}(\lambda))f_j.$$

Solving these equations, we have

$$H(\lambda)x_1=T_{n+1}(\lambda)\sum_{j=1}^{n}(T_{n+2-j}(\lambda)-qT_{n-j}(\lambda))f_j-$$

$$-(T_{n+2}(\lambda)-qT_n(\lambda))\sum_{j=1}^{n}T_{n+1-j}(\lambda)f_j,$$

$$H(\lambda)x_2=(S_{n+2}(\lambda)-qS_n(\lambda)-pT_{n+1}(\lambda)+pqT_{n-1}(\lambda))\sum_{j=1}^{n}T_{n+1-j}(\lambda)f_j-$$

$$-(S_{n+1}(\lambda)-pT_n(\lambda))\sum_{j=1}^{n}(T_{n+2-j}(\lambda)-qT_{n-j}(\lambda))f_j.$$

From these (2.35) and (2.36) are obtained.

3. Second order elliptic equations

3.1 Methods for the solution

The problem of solving approximately the second order elliptic equations is often reduced to that of solving the difference equations of the following form:

$$\text{(3.1)} \qquad M\boldsymbol{x}=\begin{pmatrix} A_1, & -C_1, & & & \\ -B_2, & A_2, & -C_2, & & \\ & \ddots & \ddots & \ddots & \\ & & -B_{m-1}, & A_{m-1}, & -C_{m-1} \\ & & & -B_m, & A_m \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_{m-1} \\ \boldsymbol{x}_m \end{pmatrix}=\begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_{m-1} \\ \boldsymbol{f}_m \end{pmatrix}=\boldsymbol{f},$$

Where A_i, B_i and C_i are $n\times n$ matrices. For convenience we consider that $B_1=C_m=0$.

Methods for solving the equation (3.1) are considered in the following three cases:

1°. Case where M is similar to a block-diagonal matrix. When M is expressed as

$$\text{(3.2)} \qquad M=E\operatorname{diag}(D_1, D_2, \cdots, D_m)E^{-1},$$

since

$$M^{-1}=E\operatorname{diag}(D_1^{-1}, D_2^{-1}, \cdots, D_m^{-1})E^{-1},$$

the problem is reduced to that of finding the matrices D_i^{-1} $(i=1, 2, \cdots, m)$.

2°. Case where M is decomposed as $M=W+N$ and W^{-1} is easily obtained. The equation (3.1) can be rewritten as follows:

$$(3.3) \qquad (I+W^{-1}N)\boldsymbol{x}=W^{-1}\boldsymbol{f}.$$

This decomposition is effective when the problem of solving (3.3) is reduced to that of solving the equations of the lower order.

3°. Case where all the block principal minor matrices

$$M_i=\begin{pmatrix} A_1, & -C_1, & & \\ -B_2, & A_2, & -C_2 & \\ & \ddots & \ddots & \\ & & -B_i & A_i \end{pmatrix} \qquad (i=1, 2, \cdots, m)$$

of M are non-singular. M can be decomposed into the form LU, where

$$L=\begin{pmatrix} I, & & & 0 \\ -B_2P_1^{-1}, & I & & \\ & \ddots & \ddots & \\ 0 & & -B_mP_{m-1}^{-1}, & I \end{pmatrix}, \quad U=\begin{pmatrix} P_1, & -C_1 & & 0 \\ & P_2, & -C_2 & \\ & & \ddots & \ddots \\ 0 & & & -C_{m-1} \\ & & & P_m \end{pmatrix},$$

$$P_1=A_1,$$

$$(3.4) \qquad P_k=A_k-B_kP_{k-1}^{-1}C_{k-1} \qquad (k=2, 3, \cdots, m).$$

Then since

$$U\boldsymbol{x}=L^{-1}\boldsymbol{f}=\boldsymbol{g}, \quad \boldsymbol{x}=U^{-1}\boldsymbol{g},$$

$\boldsymbol{x}_i$ $(i=1, 2, \cdots, m)$ can be obtained through the recurrence formulas

$$\boldsymbol{g}_1=\boldsymbol{f}_1, \quad \boldsymbol{g}_k=\boldsymbol{f}_k+B_kP_{k-1}^{-1}\boldsymbol{g}_{k-1} \qquad (k=2, 3, \cdots, m),$$

$$\boldsymbol{x}_m=P_m^{-1}\boldsymbol{g}_m, \quad \boldsymbol{x}_k=P_k^{-1}(\boldsymbol{g}_k+C_k\boldsymbol{x}_{k+1}) \qquad (k=m-1, m-2, \cdots, 1).$$

In the case where A_k, B_k, and C_k can be diagonalized by the same similarity transformation, namely where there exists a matrix F such that

$$A_k=F\hat{A}_kF^{-1}, \quad B_k=F\hat{B}_kF^{-1}, \quad C=F\hat{C}_kF^{-1} \qquad (k=1, 2, \cdots, m)$$

with diagonal matrices $\hat{A}_k$, $\hat{B}_k$ and $\hat{C}_k$, this method is easily applied. Since

$$(3.5) \qquad M=(I_m\otimes F)\begin{pmatrix} \hat{A}_1, & -\hat{C}_1, & & & 0 \\ -\hat{B}_2, & \hat{A}_2, & -\hat{C}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\hat{B}_{m-1}, & \hat{A}_{m-1}, & -\hat{C}_{m-1} \\ 0 & & & -\hat{B}_m, & \hat{A}_m \end{pmatrix}(I_m\otimes F)^{-1},$$

if we put $\boldsymbol{z}_i=F^{-1}\boldsymbol{x}_i$ and $\hat{\boldsymbol{f}}_i=F^{-1}\boldsymbol{f}_i$, then the system (3.1) can be rewritten as follows:

$$\hat{M}\boldsymbol{z} = \begin{pmatrix} \hat{A}_1, & -\hat{C}_1, & & \\ -\hat{B}_2, & \ddots & \ddots & \\ & \ddots & \ddots & -\hat{C}_{m-1} \\ & & -\hat{B}_m, & \hat{A}_m \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \vdots \\ \boldsymbol{z}_{m-1} \\ \boldsymbol{z}_m \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{f}}_1 \\ \hat{\boldsymbol{f}}_2 \\ \vdots \\ \hat{\boldsymbol{f}}_{m-1} \\ \hat{\boldsymbol{f}}_m \end{pmatrix} = \hat{\boldsymbol{f}}.$$

P_k and P_k^{-1} are easily obtained because $\hat{A}_k$, $\hat{B}_k$ and $\hat{C}_k$ are diagonal matrices.

In the particular case where

$$\hat{A}_i = A = \mathrm{diag}(a_1, a_2, \cdots, a_n) \qquad (a_j>0;\ i=1, 2, \cdots, m),$$

$$\hat{B}_i = B = \mathrm{diag}(b_1, b_2, \cdots, b_n) \qquad (b_j>0;\ i=2, 3, \cdots, m),$$

$$\hat{C}_i = C = \mathrm{diag}(c_1, c_2, \cdots, c_n) \qquad (c_j>0;\ i=1, 2, \cdots, m-1),$$

we investigate the stability of this numerical process.

THEOREM 3. *Suppose that*

(3.6) $\quad a_i \geqq \max(2\sqrt{b_i c_i},\ 2b_i c_i,\ 2b_i,\ 2c_i,\ b_i+c_i,\ 1+b_i c_i) \qquad (i=1, 2, \cdots, n).$

Then both the forward process

(3.7) $$\boldsymbol{g}_k = \hat{\boldsymbol{f}}_k + BP_{k-1}^{-1}\boldsymbol{g}_{k-1} \qquad (k=2, 3, \cdots, m)$$

and the backward process

(3.8) $$\boldsymbol{z}_k = P_k^{-1}C\boldsymbol{z}_{k+1} + P_k^{-1}\boldsymbol{g}_k \qquad (k=m, m-1, \cdots, 1)$$

are numerically stable.

PROOF. The vectors $\boldsymbol{g}_k$ and $\boldsymbol{z}_k$ are written explicitly in terms of $\hat{\boldsymbol{f}}_j$ and $\boldsymbol{g}_j$ as follows:

$$\boldsymbol{g}_k = \hat{\boldsymbol{f}}_k + \sum_{j=1}^{k-1}\Big(\sum_{l=j}^{k-1} BP_l^{-1}\Big)P_j^{-1}\hat{\boldsymbol{f}}_j \qquad (k=1, 2, \cdots, m),$$

$$\boldsymbol{z}_k = P_k^{-1}\boldsymbol{g}_k + \sum_{j=k+1}^{m}\Big(\sum_{l=k}^{j-1} P_l^{-1}C\Big)P_j^{-1}\boldsymbol{g}_j \quad (k=m, m-1, \cdots, 1).$$

Hence in order that the round-off errors incurred in the course of numerical computation may not grow, it is sufficient that the eigenvalues of P_l^{-1}, BP_l^{-1} and $P_l^{-1}C$ $(l=1, 2, \cdots, m)$ are all less than one in modulus.

Put $P_j = Q_{j-1}^{-1}Q_j$ $(j=1, 2, \cdots)$, where Q_j are diagonal matrices. Then, in view of (3.5), we have

$$Q_j = AQ_{j-1} - BCQ_{j-2} \qquad (j=2, 3, \cdots, m)$$

$$Q_0 = I, \quad Q_1 = A.$$

Since by (3.6) $a_i \geqq 2d_i = 2\sqrt{b_i c_i}$, Q_j can be written as follows:

$$Q_j = \operatorname{diag}\left(\cdots, \frac{d_i^j \sinh(j+1)\omega_i}{\sinh \omega_i}, \cdots\right),$$

where

$$e^{-\omega_i} = \frac{1}{2d_i}(a_i - \sqrt{a_i^2 - 4d_i^2}).$$

Hence we have

$$P_j^{-1} = \operatorname{diag}\left(\cdots, \frac{\sinh j\omega_i}{d_i \sinh(j+1)\omega_i}, \cdots\right).$$

On the other hand, since $\cosh(j+1)\omega/\sinh(j+1)\omega > 1 (\omega > 0)$, it follows that

$$\frac{\sinh j\omega}{\sinh(j+1)} = \cosh\omega - \frac{\cosh(j+1)\omega}{\sinh(j+1)\omega}\sinh\omega$$

$$< \cosh\omega - \sinh\omega = e^{-\omega}.$$

Hence we have only to show that

$$e^{-\omega_i}/d_i \leqq 1, \quad b_i e^{-\omega_i}/d_i \leqq 1, \quad c_i e^{-\omega_i}/d_i \leqq 1.$$

Since $b_i \leqq a_i - c_i$, it follows that

$$4b_i^2 \leqq 4a_i b_i - 4d_i^2, \quad (a_i - 2b_i)^2 \leqq a_i^2 - 4d_i^2,$$

and so

$$a_i - \sqrt{a_i^2 - 4d_i^2} \leqq 2b_i.$$

Similarly we obtain the result

$$a_i - \sqrt{a_i^2 - 4d_i^2} \leqq 2c_i.$$

From these follows that

$$b_i e^{-\omega_i}/d_i \leqq 1, \quad c_i e^{-\omega_i}/d_i \leqq 1.$$

Since $b_i c_i - a_i < -1$, it follows that

$$4b_i^2 c_i^2 - 4a_i b_i c_i \leqq -4d_i^2, \quad (a_i - 2d_i^2)^2 \leqq a_i^2 - 4d_i^2,$$

so that

$$a_i - \sqrt{a_i^2 - 4d_i^2} \leqq 2d_i^2.$$

This means that $e^{-\omega_i}/d_i \leqq 1$. Thus the theorem has been proved.

3.2 Examples

In the following examples, we are concerned with partial differential equations over a rectangle R with sides parallel to the x- and y-axes. We denote by UH and LH the upper and lower horizontal sides of R respectively and by RV and LV the right and left vertical sides respectively. Let h and h_1 be the mesh-sizes in the x- and y-directions respectively, and put

(3.9) $$\sigma = h/h_1, \quad b = \sigma^2, \quad a = 2(1+b).$$

Values of the unknown function $u_{ij} = u(x_i, y_j)$ are arranged in the following manner:

$$x_i^T = (u_{1i}, u_{2i}, \cdots, u_{ni}) \qquad (i=1, 2, \cdots, m).$$

Laplace's operator Δ is approximated by the following two formulas:

(I) Five point formula

$$-(\Delta u)_{ij} = h^{-2}Hu_{ij} + O(h^2)$$

$$= \frac{1}{h^2} \begin{array}{|c|c|c|} \hline & -b & \\ \hline -1 & a & -1 \\ \hline & -b & \\ \hline \end{array} \; u_{ij} + O(h^2).$$

(II) Hermitian difference formula

$$-\Omega(\Delta u)_{ij} = -\frac{1}{12} \begin{array}{|c|c|c|} \hline 1 & 10 & 1 \\ \hline 10 & 100 & 10 \\ \hline 1 & 10 & 1 \\ \hline \end{array} \; (\Delta u)_{ij}$$

$$= \frac{1}{h^2} \begin{array}{|c|c|c|} \hline -(1+b) & -(10b-2) & -(1+b) \\ \hline -(10-2b) & 10a & -(10-2b) \\ \hline -(1+b) & -(10b-2) & -(1+b) \\ \hline \end{array} \; u_{ij} + O(h^4).$$

3.2.1 Example 1

We consider the equation

$$-\Delta u + \lambda u = f(x, y) \qquad (\lambda \geqq 0).$$

(I) Case where five point formula is used.
The matrix M takes the form

$$M = I_m \otimes A - B \otimes I, \quad A = (a+\lambda h^2)I - bC,$$

where B is an $m\times m$ matrix, A and C are $n\times n$ matrices and, according to the boundary conditions imposed on LH and UH, C becomes as follows:

(a) when u is given on LH and UH, $C=L_1$.

(b) when u is periodic in the y-direction, $C=L_6$.

(c) when u is given on UH and u_y is given on LH,

(i) in the case where $u_y(x, y)$ is approximated by the forward difference $(u(x, y+h_1)-u(x, y))/h_1$ or by the backward difference $(u(x, y)-u(x, y-h_1))/h_1$, $C=L_3$.

(ii) in the case where $u_y(x, y)$ is approximated by the central difference $(u(x, y+h_1)-u(x, y-h_1))/(2h_1)$, $C=L_5$.

(d) when u_y is given on LH and UH, $C=L_2$ in the case (i) and $C=L_4$ in the case (ii).

(e) when $u_y+\sigma_1 u$ is given on LH and $u_y+\sigma_2 u$ is given on UH,

$$C=L_7(p, q),\quad p=1+h_1\sigma_1,\quad q=1+h_1\sigma_2 \qquad \text{in the case (i);}$$

$$C=L_8(p, q),\quad p=2h_1\sigma_1,\quad q=2h_1\sigma_2, \qquad \text{in the case (ii),}$$

where σ_1 and σ_2 are constants.

(f) when u is given on UH and $u_y+\sigma_1 u$ is given on LH,

$$C=L_9(p),\quad p=1+h_1\sigma_1 \qquad \text{in the case (i);}$$

$$C=L_{10}(p),\quad p=2h_1\sigma_1 \qquad \text{in the case (ii).}$$

If UH, LH, u_y, L_i, C, p, q, y, σ_1, and σ_2 are replaced with RV, LV, u_x, $\hat{L}_i$, B, r, s, x, σ_3 and σ_4 respectively, then B is determined similarly.

Thus we have the matrices

$$M_{ij}=I_m\otimes((a+\lambda h^2)I-bL_i)-\hat{L}_j\otimes I \qquad (i, j=1, 2, \cdots, 10). \tag{3.10}$$

Since

$$M_{ij}=(I_m\otimes R_i)(I_m\otimes((a+\lambda h^2)I-bG_i)-\hat{L}_j\otimes I)(I_m\otimes R_i)^{-1}, \tag{3.11}$$

matrices M_{ij} are of the form (3.5) except for the case $j=6$.

On the other hand, since

$$M_{ij}=(\hat{R}_j\otimes I)(I_m\otimes((a+\lambda h^2)I-bG_i)-\hat{G}_j\otimes I)(\hat{R}_j\otimes I)^{-1}, \tag{3.12}$$

matrices M_{ij} are of the form (3.2). Moreover, it follows that

$$\begin{aligned} M_{ij}&=(I_m\otimes R_i)(\hat{R}_j\otimes I)\Lambda_{ij}(\hat{R}_j\otimes I)^{-1}(I_m\otimes R_i)^{-1}, \\ \Lambda_{ij}&=I_m\otimes((a+\lambda h^2)I-bG_i)-\hat{G}_j\otimes I, \\ &=\mathrm{diag}(\Lambda_{ij}^{(1)}, \Lambda_{ij}^{(2)}, \cdots, \Lambda_{ij}^{(m)}), \\ \Lambda_{ij}^{(k)}&=\mathrm{diag}(\lambda_{ij1}^{(k)}, \lambda_{ij2}^{(k)}, \cdots, \lambda_{ijn}^{(k)}), \\ \lambda_{ijl}^{(k)}&=(a+\lambda h^2)-b\lambda_{il}-\mu_{jk}, \end{aligned} \tag{3.13}$$

where

$$G_i = \mathrm{diag}(\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}),$$

$$\hat{G}_j = \mathrm{diag}(\mu_{j1}, \mu_{j2}, \cdots, \mu_{jm}).$$

In particular, we have

$$\lambda_{ijl}^{(k)} = \lambda h^2 + 4b\sin^2\frac{\theta_{il}}{2} + 4\sin^2\frac{\theta_{jk}}{2} \qquad (i, j=1, 2, \cdots, 6). \tag{3.14}$$

When M_{ij} are non-singular, evidently their inverse matrices are given by the formula

$$M_{ij}^{-1} = (I_m \otimes R_i)(\hat{R}_j \otimes I)\Lambda_{ij}^{-1}(\hat{R}_j \otimes I)^{-1}(\hat{R}_j \otimes I)^{-1}.$$

Since

$$M_{i7} = M_{i1} - (rU_m + sU_m^J)\otimes I, \quad M_{i8} = M_{i4} - (rU_m + sU_m^J)\otimes I,$$
$$M_{i9} = M_{i1} - rU_m \otimes I, \quad M_{i10} = M_{i5} - rU_m \otimes I,$$

matrices M_{i7}^{-1}, M_{i8}^{-1}, M_{i9}^{-1} and M_{i10}^{-1} can also be obtained by Lemma 5. In addition, since

$$\begin{aligned} M_{ij} &= I_m \otimes ((a+\lambda h^2)I - bL_i(p, q)) - \hat{L}_j \otimes I \\ &= (\hat{R}_j \otimes I)\Omega_{ij}(\hat{R}_j \otimes I)^{-1} \qquad (i=7, 8, 9; j=1, 4, 5), \\ \Omega_{ij} &= I_m \otimes ((a+\lambda h^2)I - bL_i(p, q)) - \hat{G}_j \otimes I \\ &= \mathrm{diag}(\Omega_{ij}^{(1)}, \Omega_{ij}^{(2)}, \cdots, \Omega_{ij}^{(m)}), \\ \Omega_{ij}^{(k)} &= (a + \lambda h^2 - \mu_{jk})I - bL_i(p, q), \end{aligned} \tag{3.15}$$

$\Omega_{ij}^{(k)-1}$ are obtained by Lemma 3. Hence M_{i1}^{-1}, M_{i4}^{-1} and M_{i5}^{-1} can be obtained without knowledge of the eigenvalues of $L_i(p, q)$.

(II) Case where Hermitian difference formula is used. Put

$$a_1 = 10a + \frac{25}{3}\lambda h^2, \quad a_2 = 10b - 2 - \frac{5}{6}\lambda h^2,$$

$$b_1 = 10 - 2b - \frac{5}{6}\lambda h^2, \quad b_2 = 1 + b - \frac{1}{12}\lambda h^2,$$

$$A = a_1 I - a_2 J, \quad B = b_1 I + b_2 J.$$

Then we have the formula

$-b_2$	$-a_2$	$-b_2$
$-b_1$	a_1	$-b_1$
$-b_2$	$-a_2$	$-b_2$

$u_{ij} = h^2 \Omega f_{ij} + O(h^6).$

The partial derivatives u_x and u_y are to be approximated by the central difference. Then we have the following results:

1°. when u is given on the whole boundary,

$$M_1 = I_m \otimes A - \hat{L}_1 \otimes B.$$

2°. when u is periodic in both directions,

$$M_2 = I_m \otimes (a_1 I - a_2 L_6) - \hat{L}_6 \otimes (b_1 I + b_2 J).$$

3°. when u is given on LH and UH and u is periodic in the x-direction,

$$M_3 = I_m \otimes A - \hat{L}_6 \otimes B.$$

4°. when u is given on LH and UH and u_x is given on LV and RV,

$$M_4 = I_m \otimes A - \hat{L}_4 \otimes B.$$

5°. when u_y is given on LH and UH and u_x is given on RV and LV,

$$M_5 = I_m \otimes (a_1 I - a_2 L_4) - \hat{L}_4 \otimes (b_1 I + b_2 L_4).$$

6°. when u is given on LH and UH, $u_x + \sigma_3 u$ is given on LV and $u_x + \sigma_4 u$ is given on LV,

$$M_6 = I_m \otimes A - \hat{L}_8(p, q) \otimes B, \quad p = 2h\sigma_3, \quad q = 2h\sigma_4.$$

7°. when u is given on LH, UH and RV and $u_x + \sigma_3 u$ is given on LV,

$$M_7 = I_m \otimes A - \hat{L}_{10}(p) \otimes B, \quad p = 2h\sigma_3.$$

Since

$$I_m \otimes A - \hat{L}_i \otimes B = (I_m \otimes R_1)\big(I_m \otimes (a_1 I - a_2 G) - \hat{L}_i \otimes (b_1 I + b_2 G_1)\big)(I_m \otimes R_1)^{-1},$$
$$(i = 1, 3, 4, 6, 7)$$

$$M_2 = (I_m \otimes R_6)\big(I_m \otimes (a_1 I - a_2 G_6) - \hat{L}_6 \otimes (b_1 I + b_2 G_6)\big)(I_m \otimes R_6)^{-1},$$

$$M_5 = (I_m \otimes R_4)\big(I_m \otimes (a_1 I - a_2 G_4) - \hat{L}_4 (b_1 I + b_2 G_4)\big)(I_m \otimes R_4)^{-1},$$

each block of M_i can be diagonalized.

3.2.2 Example 2

We consider the equation

$$-\Delta u + d u_x + e u_y + g u = f(x, y).$$

(I) Case where five point formula is used.
We assume first that

$$d = d(x), \quad e = \text{const.}, \quad g = g(x),$$

The mesh-size h is to be chosen small so that

$$\gamma=\left(1-\frac{h}{2}e\right)>0,\quad \gamma\delta^2=\left(1+\frac{h}{2}e\right)>0\qquad(\delta>0).$$

Then A_k, B_k and C_k become as follows:

$$A_1=\left(a+h^2g_1-r\left(1+\frac{h}{2}d_1\right)\right)I-M(p,q;\alpha,\beta;\gamma,\delta),$$

$$A_i=(a+h^2g_i)I-M(p,q;\alpha,\beta;\gamma,\delta)\qquad(i=2,3,\cdots,m-1),$$

$$A_m=\left(a+h^2g_m-s\left(1-\frac{h}{2}d_m\right)\right)I-M(p,q;\alpha,\beta;\gamma,\delta),$$

$$C_1=\left(1-\frac{h}{2}d_1+w\left(1+\frac{h}{2}d_1\right)\right)I,$$

$$C_i=\left(1-\frac{h}{2}d_i\right)I,\quad B_i=\left(1+\frac{h}{2}d_i\right)I\qquad(i=2,3,\cdots,m-1),$$

$$B_m=\left(1+\frac{h}{2}d_m+z\left(1-\frac{h}{2}d_m\right)\right)I.$$

The values of p, q, α and β are determined according to the boundary conditions as follows:

in the case (a), $p=q=\alpha=\beta=0$;
in the case (c) (i), $p=1,\quad q=\alpha=\beta=0$;
in the case (c) (ii), $p=0,\quad q=0,\quad \alpha=1,\quad \beta=0$;
in the case (d) (i), $p=q=1,\quad \alpha=\beta=0$;
in the case (d) (ii), $p=q=0,\quad \alpha=\beta=1$;
in the case (e) (i), $p=1+h_1\sigma_1,\quad q=1+h_1\sigma_2,\quad \alpha=\beta=0$;
in the case (e) (ii), $p=2h_1\sigma_1,\quad q=2h_1\sigma_2,\quad \alpha=\beta=1$;
in the case (f) (i), $p=1+h_1\sigma_1,\quad q=0,\quad \alpha=\beta=0$;
in the case (f) (ii), $p=2h_1\sigma_1,\quad q=0,\quad \alpha=1,\quad \beta=0$.

If LH, UH, u_y, p, q, α, β, σ_1, σ_2 and h_1 are replaced with LV, RV, u_x, r, s, w, z, σ_3, σ_4 and h respectively, then the values of r, s, w, and z are determined similarly.

In each case it is readily seen by Lemma 6 that A_i $(i=1,2,\cdots,m)$ can be diagonalized by the same similarity transformation.

By interchanging the roles of x and y, the case where

$$d=\text{const.},\quad e=e(y),\quad g=g(y)$$

can be treated analogously.

Next we are concerned with the case where d, e and g are constants.

We choose h small so that

$$\mu=\left(1+\frac{h}{2}d\right)>0,\quad \mu\rho^2=\left(1-\frac{h}{2}d\right)>0 \qquad (\rho>0),$$

and put

$$F=\mathrm{diag}(1,\rho,\rho^2,\cdots,\rho^{m-1}).$$

Then it is valid that

$$M=I_m\otimes((a+h^2g)I-M(p,q;\alpha,\beta;\gamma,\delta))-\hat{M}(r,s;w,z;\mu,\rho)\otimes I.$$

Since

$$M=(I_m\otimes E)^{-1}(F\otimes I)^{-1}\Omega(F\otimes I)(I_m\otimes E),$$

$$\Omega=I_m\otimes((a+h^2g)I-\gamma\delta L(p\delta^{-1},q\delta;\alpha\delta^{-2},\beta\delta^2))-$$

$$-\mu\rho\hat{L}(\gamma\rho^{-1},s\rho;w\rho^{-2},z\rho^2)\otimes I,$$

M can be reduced to the form (3.5).

(II) Case where Hermitian difference formula is used.

We assume that

$$d=d(x),\quad e=0,\quad g=g(x),$$

and put

$$a_i=10a-2h\delta_x d_i+8h^2g_i+h^2\delta_x^2 g_i+2h^2d_i^2-\frac{1}{2}h^3d_i\delta_x g_i,$$

$$b_i=(5-b)(2+hd_i)-h\delta_x d_i+\frac{h}{2}\delta_x^2 d_i+h^2(d_i^2-g_i)+$$

$$+\frac{h^2}{2}\delta_x g_i-\frac{h^2}{4}d_i\delta_x d_i-\frac{h^3}{2}d_i g_i,$$

$$c_i=(5-b)(2-hd_i)-h\delta_x d_i-\frac{h}{2}\delta_x^2 d_i+h^2(d_i^2-g_i)-$$

$$-\frac{h^2}{2}\delta_x g_i+\frac{h^2}{4}d_i\delta_x d_i+\frac{h^3}{2}d_i g_i,$$

$$\alpha_i=10b-h^2g_i,\quad \beta_i=(1+b)\left(1+\frac{h}{2}d_i\right),\quad \gamma_i=(1+b)\left(1-\frac{h}{2}d_i\right),$$

where

$$\delta_x f_{ij}=f_{i+1j}-f_{i-1j},\quad \delta_x^2 f_{ij}=f_{i+1j}-2f_{ij}+f_{i-1j},$$

$$\delta_y f_{ij}=f_{ij+1}-f_{ij-1},\quad \delta_y^2 f_{ij}=f_{ij+1}-2f_{ij}+f_{ij-1}.$$

Then we have the formula

$-\beta_i$	$-\alpha_i$	$-\gamma_i$
$-b_i$	a_i	$-c_i$
$-\beta_i$	$-\alpha_i$	$-\gamma_i$

$$u_{ij}=h^2\Big(8f_{ij}+f_{ij+1}+f_{ij-1}+\Big(1+\frac{h}{2}d_i\Big)f_{i-1j}+ +\Big(1-\frac{h}{2}d_i\Big)f_{i+1j}\Big)+O(h^6),$$

and the following results are obtained:

1°. when u is given on the whole bounary,

$$A_i=a_iI-\alpha_iJ \quad (i=1, 2, \cdots, m),$$

$$B_i=b_iI+\beta_iJ \quad (i=2, 3, \cdots, m),$$

$$C_i=c_iI+\gamma_iJ \quad (i=1, 2, \cdots, m-1).$$

2°. when u is given on LH and UH and u_x is given on LV and RV,

$$A_i=a_iI-\alpha_iJ \qquad (i=1, 2, \cdots, m),$$

$$C_1=(b_1+c_1)I+(\beta_1+\gamma_1)J,$$

$$C_i=c_iI+\gamma_iJ, \quad B_i=b_iI+\beta_iJ \quad (i=2, 3, \cdots, m-1),$$

$$B_m=(b_m+c_m)I+(\beta_m+\gamma_m)J.$$

3°. when u_y is given on LH and UH and u_x is given on LV and RV,

$$A_i=a_iI-\alpha_iL_4 \qquad (i=1, 2, \cdots, m),$$

$$C_1=(b_1+c_1)I+(\beta_1+\gamma_1)L_4,$$

$$C_i=c_iI+\gamma_iL_4, \quad B_i=b_iI+\beta_iL_4 \qquad (i=2, 3, \cdots, m-1),$$

$$B_m=(b_m+c_m)I+(\beta_m+\gamma_m)L_4.$$

In each case M can be reduced to the form (3.5).

3.2.3 Example 3

We consider the axially symmetric problem

$$\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}=f(r, z) \qquad (0<r<1, 0<z<1),$$

where $u(r, 0)$ is given and u is regular at $r=0$. Let $h=1/n$ be the mesh-size in the r-direction and $h_1=h/\sigma$ be the mesh-size in the z-direction. 'If we use five point formula, then M becomes as follows:

$$M_{ij}=I_m\otimes A_i-B_j\otimes bI,$$

where A_i and B_j are determined according to the boundary conditions in the following manner:

(i) when $u(1, z)$ is given,

$$A_1 = C + \left(-1 + \frac{1}{2(n-1)}\right) V^J.$$

(ii) when $u_r(1, z)$ is given,

$$A_2 = C - 2V^J.$$

(iii) when $u_r(1, z) + \sigma_1 u(1, z)$ is given,

$$A_3 = C - 2V^J + 2h\sigma_1\left(1 + \frac{1}{2(n-1)}\right) U^J.$$

1°. when $u\big(r, (m+1)h_1\big)$ is given,

$$B_1 = J_m.$$

2°. when $u_z(r, mh_1)$ is given,

$$B_2 = J_m + V_m^J.$$

3°. when $u_z(r, mh_1) + \sigma_2 u(r, mh_1)$ is given,

$$B_3 = J_m + V_m^J - 2h\sigma_2 U_m^J,$$

where

$$C = \begin{pmatrix} 2(2+b), & -4, & & & \\ -\frac{1}{2}, & 2(1+b), & -\frac{3}{2} & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -1+\frac{1}{2(n-2)}, & 2(1+b), & -1-\frac{1}{2(n-2)} \\ & & & 0, & 2(1+b) \end{pmatrix}.$$

Let

$$B_j = S_j A_j S_j^{-1}, \quad A_j = \operatorname{diag}(\lambda_{j1}, \lambda_{j2}, \cdots, \lambda_{jm}).$$

Then since

$$M_{ij} = (S_j \otimes I)\Omega_{ij}(S_j \otimes I)^{-1},$$

$$\Omega_{ij} = I_m \otimes A_i - A_j \otimes bI$$

$$= \operatorname{diag}(\Omega_{ij}^{(1)}, \Omega_{ij}^{(2)}, \cdots, \Omega_{ij}^{(m)}),$$

$$\Omega_{ij}^{(k)} = A_i - b\lambda_{jk} I,$$

matrices M_{ij} are of the form (3.2).

3.2.4 Example 4

We consider Poisson's equation in polar coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=f(r,\theta) \qquad (0<r<1).$$

In the case where $u(1, \theta)$ is given, we have

$$M=\begin{pmatrix} B_1, & \left(1+\frac{1}{2}\right)I & & & \\ \left(1-\frac{1}{4}\right)I, & B_2, & \left(1+\frac{1}{4}\right)I & & \\ & \ddots & \ddots & \ddots & \\ & & \left(1-\frac{1}{2(m-1)}\right)I, & B_{m-1}, & \left(1+\frac{1}{2(m-1)}\right)I \\ & & & \left(1-\frac{1}{2m}\right)I, & B_m \end{pmatrix},$$

where

$$B_p=-2\left(1+\frac{1}{(p\delta\theta)^2}\right)I+\frac{1}{(p\delta\theta)^2}J.$$

In the case where $u_r(1, \theta)$ is given, we have

$$M=\begin{pmatrix} B_1, & \left(1+\frac{1}{2}\right)I & & & \\ \left(1-\frac{1}{4}\right)I, & B_2, & \left(1+\frac{1}{4}\right)I & & \\ & \ddots & \ddots & \ddots & \\ & & \left(1-\frac{1}{2(m-1)}\right)I, & B_{m-1}, & \left(1+\frac{1}{2(m-1)}\right)I \\ & & & 2I, & B_m \end{pmatrix}.$$

Since

$$B_p=R_1\left(-2\left(1+\frac{1}{(p\delta\theta)^2}\right)I+\frac{1}{(p\delta\theta)^2}G_1\right)R_1^{-1},$$

each block of M can be diagonalized.

4. Fourth order elliptic equations

The problem of solving approximately the fourth order elliptic equations is often reduced to that of solving the system of equations of the following form:

$$(4.1)\quad N\boldsymbol{x}=\begin{pmatrix} A_1, & -C_1, & E_1 & & & & & \\ -B_2, & A_2, & -C_2, & E_2 & & & 0 & \\ D_3, & -B_3, & A_3, & -C_3, & E_3 & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & D_{m-2}, & -B_{m-2}, & A_{m-2}, & -C_{m-2}, & E_{m-2} & \\ 0 & & & D_{m-1}, & -B_{m-1}, & A_{m-1}, & -C_{m-1} & \\ & & & & D_m, & -B_m, & A_m & \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \vdots \\ \boldsymbol{x}_{m-2} \\ \boldsymbol{x}_{m-1} \\ \boldsymbol{x}_m \end{pmatrix}=\begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \\ \vdots \\ \boldsymbol{f}_{m-2} \\ \boldsymbol{f}_{m-1} \\ \boldsymbol{f}_m \end{pmatrix}=\boldsymbol{f}.$$

In the case where all the block principal minor matrices of N are non-singular, N can be decomposed into the form LU, where

$$L=\begin{pmatrix} I & & & 0 \\ -L_2, & I & & \\ M_3, & -L_3, & I & \\ 0 & \ddots & \ddots & \ddots \\ & M_m, & -L_m, & I \end{pmatrix},\quad U=\begin{pmatrix} P, & -U_1, & E_1 & 0 \\ & \ddots & \ddots & \ddots \\ & & \ddots & E_{m-2} \\ 0 & P_{m-1}, & & -U_{m-1} \\ & & & P_m \end{pmatrix},$$

$$M_i=D_iP_{i-2}^{-1},$$

$$L_i=(B_i-M_iU_{i-2})P_{i-1}^{-1},$$

$$U_i=C_i-L_iE_{i-1},\qquad (i=1,2,\cdots,m).$$

$$P_i=A_i-L_iU_{i-1}-M_iE_{i-2}$$

This method is easily applied when A_i, B_i, C_i, D_i and E_i can be diagonalized by the same similarity transformation.

Example We consider the equation

$$\Delta\Delta u+2\alpha\Delta u+\beta u=f(x,\ y),$$

where α and β are constants and u is given on the entire boundary. Put

$$A=(a^2+2b^2+2+2\alpha h^2a+\beta h^4)I-2(a+\alpha h^2)bJ+b^2(J^2-2I),$$

$$B=(a+\alpha h^2)I-bJ.$$

We consider the following three cases:

(i) when u_{xx} is given on LV and RV and u_{yy} is given on LH and UH,

$$N_1=I_m\otimes A-2J_m\otimes B+(J_m^2-2I_m)\otimes I.$$

Since

$$N_1=S(I_m\otimes((2+(\beta-\alpha^2)h^4)I+((a+\alpha h^2)I-bG_1)^2)+$$
$$+J_m\otimes 2((a+\alpha h^2)I-bG_1)+(J_m^2-2I_m)\otimes I)S^{-1},$$

each block of N_1 can be diagonalized. In this case, moreover, it is valid that

$$N_1=SP\Lambda P^{-1}S^{-1},$$

$$\Lambda = I_m \otimes (\beta-\alpha^2)h^4 I + (I_m \otimes ((a+\alpha h^2)I - bG_1) - \hat{G}_1 \otimes I)^2$$
$$= \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_m),$$
$$\Lambda_k = (\beta-\alpha^2)h^4 + \left((a+\alpha h^2)I - bG_1 - 2\cos\frac{k\pi}{m+1} I\right)^2$$
$$= \mathrm{diag}(\lambda_{k1}, \lambda_{k2}, \cdots \lambda_{kn}),$$
$$\lambda_{kj} = (\beta-\alpha^2)h^4 + \left(\alpha h^2 + 4b\sin^2\frac{j\pi}{2(n+1)} + 4\sin^2\frac{k\pi}{2(m+1)}\right)^2.$$

(ii) when u_{yy} is given on LH and UH and u_x is given on LV and RV,

$$N_2 = N_1 + 2(U_m + U_m^J) \otimes I.$$

In this case each block of N_2 can be diagonalized and since

$$N_2 = S(P\Lambda P^{-1} + 2(U_m + U_m^J) \otimes I)S^{-1},$$

N_2^{-1} can also be obtained by Lemma 5.

(iii) when u_y is given on LH and UH and u_x is given on LV and RV,

$$N_3 = N_4 + 2(U_m + U_m^J) \otimes I,$$

where

$$N_4 = I_m \otimes (A + 2b^2(U + U^J)) - 2J_m \otimes B + (J_m^2 - 2I_m) \otimes I.$$

In this case it is valid that

$$N_4 = P\Omega P^{-1},$$
$$\Omega = I_m \otimes (A - 2I + 2b^2(U + U^J)) - 2\hat{G}_1 \otimes B + \hat{G}_1^2 \otimes I$$
$$= \mathrm{diag}(\Omega_1, \Omega_2, \cdots, \Omega_m),$$
$$\Omega_k = \left(\left(2b + \alpha h^2 + 4\sin^2\frac{k\pi}{2(m+1)}\right)I - bJ\right)^2 + (\beta-\alpha^2)h^4 I + 2b^2(U + U^J).$$

Ω_k^{-1} can be obtained either by Lemma 4 or by LU-decomposition, so that N_4^{-1} can be obtained easily and then N_3^{-1} can be computed by Lemma 5. Lemma 10 and Lemma 11 can also be applied.

5. Parabolic equations

5.1 One-dimensional second order parabolic equation

Let us consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1),$$

where $u(x, 0)$ is given. Let h be the mesh-size in the x-direction and h_1 be the mesh-size in the t-direction and put $r=h_1/h^2$.

In the case where $u(0, t)=u(1, t)=0$ $(t>0)$, using Crank-Nicolson's formula, we have [41]

$$B_1\boldsymbol{u}_{l+1}=(4I-B_1)\boldsymbol{u}_l \qquad (l=0, 1, \cdots),$$

where

$$B_1=2I-rJ.$$

In the case where the boundary conditions are given by

$$\frac{\partial u}{\partial x}(0, t)=k_1(u-v_1), \quad \frac{\partial u}{\partial x}(1, t)=-k_2(u-v_2) \qquad (t>0),$$

with constants k_1, k_2, v_1 and v_2, we have

$$B_2\boldsymbol{u}_{l+1}=(4I-B_2)\boldsymbol{u}_l+\boldsymbol{f}_l \qquad (l=0, 1, \cdots),$$

where

$$B_2=2(1+r)I-rL(-2hk_1, -2hk_2; 1, 1).$$

Both cases can be treated easily.

5.2 Two-dimensional second order parabolic equation

We consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, \qquad (0<x,\ y<1,\ 0<t\leqq T)$$

with the initial conditions

$$u(x, y, 0)=f(x, y),$$

and boundary conditions

$$u(0, y, t)=u(1, y, t)=0 \qquad (0\leqq y\leqq 1,\ 0\leqq t\leqq T)$$

$$u(x, 0, t)=u(x, 1, t)=0 \qquad (0\leqq x\leqq 1,\ 0\leqq t\leqq T).$$

Put

$$u_{i,j,k}=u(ih, jh_1, kl), \quad \omega=h^2/l, \quad \gamma=\omega+\alpha+\beta, \qquad 0\leqq\alpha,\ \beta\leqq 1.$$

Using the formula [38]

$$\begin{aligned} u_{i,j,k+1}=\frac{1}{2\gamma}(&\alpha u_{i+1,j,k+1}+\alpha u_{i-1,j,k+1}+\beta u_{i,j+1,k+1}+\\ &+\beta u_{i,j-1,k+1}+(2-\alpha)u_{i+1,j,k}+(2-\beta)u_{i,j+1,k}+\\ &+(2-\alpha)u_{i-1,j,k}+(2-\beta)u_{i,j-1,k}-2(4-\gamma)u_{i,j,k}),\end{aligned}$$

we have

$$Au_{p+1}=Bu_p \qquad (p=0,1,\cdots),$$

where

$$A=I\otimes D-J\otimes\frac{\beta}{2}I,\quad B=A+I\otimes T+J\otimes I,$$

$$D=\gamma I-\frac{\alpha}{2}J,\quad T=-4I+J.$$

$A^{-1}B$ is easily obtained.

5.3 Fourth order parabolic equation

Let us consider the equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=0 \qquad (0\leqq x\leqq 1,\ t>0),$$

with the initial conditions

$$u(x,0)=g_0(x),\quad \frac{\partial u}{\partial t}(x,0)=g_1(x) \qquad (0\leqq x\leqq 1),$$

and boundary conditions

$$u(0,t)=f_0(t),\quad u(1,t)=f_1(t)$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=p_0(t),\quad \frac{\partial^2 u}{\partial x^2}(1,t)=p_1(t).$$

Put

$$\Phi=\frac{\partial u}{\partial t},\quad \Psi=\frac{\partial^2 u}{\partial x^2},\quad \Omega=\begin{pmatrix}\Phi\\ \Psi\end{pmatrix},\quad C=\begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix}.$$

Then the given equation can be rewritten as follows [10]:

$$\frac{\partial \Omega}{\partial t}=C\frac{\partial^2\Omega}{\partial x^2}.$$

Let h be the mesh-size in the x-direction and h_1 be the mesh-size in the t-direction and put $r=h_1/h^2$.

When Crank-Nicolson method is used, we have

$$A\Omega_{p+1}=B\Omega_p+f_p \qquad (p=0,1,\cdots),$$

where

$$A=I_m\otimes A_1+J_m\otimes A_2,\quad B=I_m\otimes B_1+J_m\otimes B_2,$$

$$A_1 = I_2 + rC, \quad A_2 = -\frac{r}{2}C, \quad B_1 = I_2 - rC, \quad B_2 = -A_2.$$

In this case it is valid that

$$A = (\hat{R}_1 \otimes I_2) D (\hat{R}_1 \otimes I_2)^{-1}, \quad B = (\hat{R}_1 \otimes I_2) F (\hat{R}_1 \otimes I_2)^{-1},$$

$$D = I_m \otimes A_1 + \hat{G}_1 \otimes A_2 = \mathrm{diag}(D_1, D_2, \cdots, D_m),$$

$$F = I_m \otimes B_1 + \hat{G}_1 \otimes B_2 = \mathrm{diag}(F_1, F_2, \cdots, F_m),$$

$$D_j = A_1 + 2\cos\frac{j\pi}{m+1} A_2 = I_2 + 2r\sin^2\frac{j\pi}{2(m+1)} C,$$

$$F_j = B_1 + 2\cos\frac{j\pi}{m+1} B_2 = I_2 - 2r\sin^2\frac{j\pi}{2(m+1)} C.$$

When Douglas' high order correct method [8] is used, we have

$$A_1 = 10 I_2 + 12rC, \quad A_2 = I_2 - 6rC,$$

$$B_1 = 10 I_2 - 12rC, \quad B_2 = I_2 + 6rC.$$

In this case it is valid that

$$D_j = \left(8 + 4\cos^2\frac{j\pi}{2(m+1)}\right) I_2 + 24r\sin^2\frac{j\pi}{2(m+1)} C,$$

$$F_j = \left(8 + 4\cos^2\frac{j\pi}{2(m+1)}\right) I_2 - 24r\sin^2\frac{j\pi}{2(m+1)} C.$$

Since

$$(I_2 + \sigma C)^{-1} = \frac{1}{1+\sigma^2}(I_2 - \sigma C),$$

$A^{-1}B$ can be obtained easily.

5.4 Periodic parabolic problem

We consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1)$$

with the boundary condition [42]

$$u(0, t) = f(t), \quad u(1, t) = g(t), \quad u(x, 0) = u(x, T),$$

where

$$f(t+T) = f(t), \; g(t+T) = g(t) \qquad (t \geq 0).$$

Put

$$l = T/\text{m}, \quad h = 1/(n+1), \quad \sigma = 1/h$$

and let Q_m be an $m \times m$ matrix defined by

$$Q_m = \begin{pmatrix} 0 & & & & 1 \\ 1, & 0 & & & \\ & 1, & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ 0 & & & 1, & 0 \end{pmatrix}$$

Then, according as explicit formula or implicit formula is used, the problem is reduced to the solution of the following systems of equations:

$$(5.1) \qquad (I_m \otimes I - Q_m \otimes M)\boldsymbol{x} = \boldsymbol{f}$$

or

$$(5.2) \qquad (I_m \otimes N - Q_m \otimes I)\boldsymbol{x} = \boldsymbol{g},$$

where

$$M = (1-2\sigma)I + \sigma J \qquad (\sigma \leqq 1/2), \qquad N = (1+2\sigma)I - \sigma J.$$

Since

$$M = R_1 D R_1^{-1}, \quad N = R_1 E^{-1} R_1^{-1},$$

where

$$D = (1-2\sigma)I + \sigma G_1, \quad E^{-1} = (1+2\sigma)I - \sigma G_1,$$

we can write (5.1) and (5.2) as follows:

$$(5.3) \qquad S(I_m \otimes I - Q_m \otimes D)S^{-1}\boldsymbol{x} = \boldsymbol{f},$$

$$(5.4) \qquad S(I_m \otimes I - Q_m \otimes E)(I_m \otimes E^{-1})S^{-1}\boldsymbol{x} = \boldsymbol{g}.$$

Then, for (5.3), it is valid that

$$(I_m \otimes I + Q_m \otimes D + \cdots + Q_m^{m-1} \otimes D^{m-1})S^{-1}\boldsymbol{f} =$$
$$= (I_m \otimes I - Q_m^m \otimes D^m)S^{-1}\boldsymbol{x} = I_m \otimes (I - D^m)S^{-1}\boldsymbol{x},$$

because $Q_m^m = I_m$, and it follows that

$$\boldsymbol{x} = S(I_m \otimes (I - D^m)^{-1})(I_m \otimes I + Q_m \otimes D + \cdots + Q_m^{m-1} \otimes D^{m-1})S^{-1}\boldsymbol{f}.$$

Similarly for (5.4) we have

$$\boldsymbol{x} = S(I_m \otimes E)(I_m \otimes (I - E^m)^{-1})(I_m \otimes I + Q_m \otimes E + \cdots + Q_m^{m-1} \otimes E^{m-1})S^{-1}\boldsymbol{g}.$$

5.5 Three level difference scheme

Let us consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (0<x<1),$$

where boundary values are given. If Mitchell-Pearce nine point formula [24] is used, we have

$$\boldsymbol{A}\boldsymbol{u}_{k+1}=\boldsymbol{B}\boldsymbol{u}_k+\boldsymbol{C}\boldsymbol{u}_{k-1}+\boldsymbol{f}_{k+1} \qquad (k=1,\ 2,\ \cdots),$$

where

$$A=aI+bJ, \quad B=cI+dJ, \quad C=eI+fJ$$

$$a=4p^4+5p^3-\frac{1}{10}p^2-\frac{23}{84}p-\frac{313}{12600},$$

$$b=-2p^4+\frac{1}{2}p^3+\frac{1}{20}p^2-\frac{11}{840}p+\frac{13}{25200},$$

$$c=-16p^4+p^2-\frac{313}{6300},$$

$$d=8p^4-\frac{1}{2}p^2+\frac{13}{12600},$$

$$e=-4p^4+5p^3+\frac{1}{10}p^2-\frac{23}{84}p+\frac{313}{12600},$$

$$f=2p^4+\frac{1}{2}p^3-\frac{1}{20}p^2-\frac{11}{840}p-\frac{13}{25200} \qquad (p\leqq\sqrt{5}/10).$$

Matrices A, B, and C can easily be diagonalized.

References

[1] Asplund, S. O.: *Finite boundary value problems solved by Green's matrix*, Math. Scand., **7** (1959), 49-56.

[2] Bublik, B. N. and G. N. Polozhii: *Eigenvalues of the basic biharmonic problem for a rectangle*, Dopovidi Akad. Nauk Ukrain, RSR, (1967), 3-6.

[3] Buledza, A. V.: *On an approximate method of solving boundary problems for biharmonic equations*, Dokl. Akad. Nauk SSSR, **170** (1966), 495-496.

[4] Buledza, A. V.: *On difference methods for solution of boundary-value problems for differential equations of elliptic type*, Differential'nye Uravnenija, **1** (1965), 687-691.

[5] Conte, S. D. and T. T. Dames: *An alternating direction method for solving the biharmonic equation*, Math. Tables Aids Comput., 12 (1958), 198-205.

[6] Cornock, A. F.: *The numerical solution of Poisson's and bi-harmonic equations by matrices*, Proc. Cambridge Philos. Soc., **50** (1954), 524-535.

[7] Didenko, V. I. and I. N. Lyashenko: *On the numerical solution of boundary value problems for elliptic*

differential equations uith constant coefficients, Dopovidi Akad. Nauk Ukrain. RSR, (1964), 1273-1276.

[8] Douglas, J: *The solution of the diffusion equation by a high order correct difference equation*, J. Math. Phys., **35** (1956), 145-151.

[9] Egérvary, E.: *Über eine Methode zur numerischen Lösung der Poissonschen Differentialgleichung für beliebig Gebiete*, Acta Math. Acad. Sci. Hungar., **11** (1960), 341-361.

]10] Fairweather, C. and A. R. Gourlay: *Some stable difference approximation to a fourth-order parabolic partial differential equation*, Math. Comp., **21** (1967), 1-11.

[11] Forsythe, G. E. and W. R. Wasow: *Finite difference methods for partial differential equations*. John Wiley and Sons, 1960.

[12] Hicks, J. S. and J. Wei: *Numerical solution of parabolic partial differential equations with two-point boundary conditions by use of the method of lines*, J. Assoc. Comput. Mach., **14** (1967), 549-562.

[13] Hockney, R. W.: *A fast direct solution of Poisson's equation using Fourier analysis*, J. Assoc. Comput. Mach., **12** (1965), 93-113.

[14] Keast, P. and A. R. Mitchell: *On the instability of the Crank Nicholson formula under derivative boundary conditions*, Computer J., **9** (1966), 110-114.

[15] Keast, P. and A. R. Mitchell: *Finite difference solution of the third boundary problem in elliptic and parabolic equations*, Numer. Math., **10** (1967), 67-75.

[16] Lyashenko, I. N.: *Remarks on the matrix driving through method*, Dopovidi Akad. Nauk Ukrain. RSR, (1964), 458-461.

[17] Lyashenko, I. N.: *On eigenvalues and eigenfunctions of finite difference Dirichlet's and Neumann's problems for Helmholtz's equation*, Dopovidi Akad. Nauk Ukrain. RSR, (1967), 415-419.

[18] Lynch, R. E., J. R. Rice and D. H. Thomas: *Direct solution of partial difference equations by tensor product methods*, Numer. Math., **6** (1964), 185-199.

[19] Marcus, M.: *Basic theorems in matrix theory*. Nat. Bur. Standards Appl. Math. Ser., **57** (1960).

[20] Martynovic, T. D. and M. Korduba: *Solution of the spatial axially symmetric Dirichlet problem by the method of straight lines with higher accuracy of approximation*, Ž. Vyčisl. Mat. i Mat. Fiz., 4 (1966) Suppl., 214-218.

[21] Meister, B. and W. Prager: *On the construction of symmetric difference operators for square and cubic lattices*, Z. Angew. Math. Phys., **16** (1965), 403-410.

[22] Merzrath, E.: *Direct solution of partial difference equations*, Numer. Math., **9** (1967), 431-436.

[23] Misnik, A. F.: *On a technique of realizing the sweep method for solving Poisson's equations*, Ž. Vyčisl. Mat. i Mat. Fiz., **5** (1965), 136-139.

[24] Mitchell, A. R. and R. P. Pearce: *High accuracy difference formulae for the numerical solution of the heat conduction equation*, Computer J., **5** (1962), 142-146.

[25] Nemchinov, S. V.: *The solution of boundary problems for partial differential equations of the second order of the elliptic type using the network nethod*, Ž. Vyčisl. Mat. i Mat. Fiz., **2** (1962), 418-436.

[26] Nemchinov, S. V. and S. L. Libov: *A more accurate straight line method for the solution of boundary problems for the Helmholtz equation on a net of points in a rectangle*, Ž. Vyčisl. Mat. i Mat. Fiz., **4** (1964), 771-773.

[27] Omarov, E. O.: *Approximate solution by the straight line method of a partial differential equation of the first elliptic type*, Ž. Vyčisl. Mat. i Mat. Fiz., **4** (1964), 585-592.

[28] Osborne, M. R.: *Direct methods for the solution of finite-difference approximations to separable partial differential equations*, Computer J., **8** (1965), 150-156.

[29] Osborne, M. R.: *A note on the numerical solution of a periodic parabolic problem*, Computer J., **7** (1964), 163-168.

[30] Parker, I. B. and J. Crank: *Persistent discretization errors in partial differential equations of parabolic type*, Computer J., **7** (1964), 155-158.

[31] Parter, S. V.: *Some computational results on "two-line" iterative methods for the biharmonic difference equation*, J. Assoc. Comput. Mach., **8** (1961), 359-362.

[32] Polozhii, G. N.: *The method of summary representation for numerical solution of problems of mathematical physics*. Pergamon Press, 1965.

[33] Polozhii, G. N. and I. N. Lyashenko: *Eigenvalues of finite-difference Laplace operator for domains consisting of rectangles*, Dopovidi Akad. Nauk Ukrain. RSR, (1967), 31-35.

[34] Prikhodko, E. M.: *Application of the method of summary representation to the numerical solution of biharmonic equations*, Dopovidi Akad. Nauk Ukrain. RSR, (1964), 856-860.

[35] Richtmyer, R. D.: *Difference methods for initial value problems*, Interscience, 1957.

[36] Rozsa, P. and I. Tóth: *Über die direkte numerische Lösung von partiellen Differentialgleichungen mit Anwendung der Hypermatrizen*, Apl. Mat., **10** (1965), 289-292.

[37] Sarmin, E. N,: *Application of straight line method to the solution of boundary value problem for some non-self-adjoint two-dimensional elliptic equations of the second order*, Ž. Vyčisl. Mat. i Mat. Fiz., **5** (1965), 945-949.

[38] Saul'yev, V. K.: *Integration of equations of parabolic type by the method of nets*. Pergamon Press, 1964.

[39] Schechter, S.: *Quasi-tridiagonal matrices and type-insensitive difference equations*, Quart. Appl. Math., **18** (1960-1961), 258-295.

[40] Shintani,H.: *Direct methods for the numerical solution of partial difference equations for a rectangle*, J. Sci. Hiroshima Univ. Ser. A-I, **30** (1966), 75-90.

[41] Smith, G. D.: *Numerical solution of partial differential equations*. Oxford, 1965.

[42] Tee, G. J.: *An application of p-cyclic matrices for solving periodic parabolic problems*, Numer. Math., **6** (1964), 142-159.

[43] Tee, G. J.: *A novel finite-difference approximation to the biharmonic operator*, Computer J., **6** (1963), 177-192.

[44] Varga, R. S.: *Matrix Iterative Analysis*. Prentice-Hall, 1962.

[45] von Holdt, R. E.: *Inversion of a triple-diagonal compound matrices*, J. Assoc. Comput. Mach., **9** (1962), 71-83.

[46] Wadsworth, M. and A. Wragg: *On matrix methods for the solution of partial differential equations*, Proc. Cambridge Philos. Soc., **61** (1965), 129-132.

Department of Mathematics
Faculty of Science
Hiroshima University