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On the Distributional Boundary Values of Vector-Valued Holomorphic Functions

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The representations of distributions as distributional boundary values of holomorphic functions have been discussed by many authors. As for a distribution on a unit circle, G. Köthe [10] developed the theory of holomorphic representations through the Cauchy integral along the unit circle of the given distribution. On the other hand, for a distribution on the real axis, H. G. Tillmann [25] constructed its holomorphic representation by making use of a technique similar to the method of G. Mittag-Leffler. There the representation is not unique and we may add an entire function to obtain another representing function. Recently, in his article [12] motivated by the works of M. Sato [16, 17], A. Martineau has developed the theory of the distributional boundary values of holomorphic functions, and reduced the problem of representing a distribution to solving a non-homogeneous Cauchy-Riemann equation. The solution is a distribution which is holomorphic except on the line where the original distribution is given.

For a vector-valued distribution on the real axis, H. G. Tillmann [26] has shown that the Cauchy integral along the real axis gives rise to a holomorphic representation if the distribution happens to be of almost compact support, while the method of G. Mittag-Leffler just referred to cannot be applied to an arbitrary vector-valued distribution. We are naturally led to the question whether it is possible to give holomorphic representations of an arbitrary vector-valued distribution on the real axis. We can show that the answer is negative. This is because the space of entire functions has no topological supplement in the space of the solutions of the Cauchy-Riemann equations mentioned above. On the contrary, any vector-valued distribution on a unit circle is holomorphically represented by the Cauchy integral.

The main purpose of the present paper is to develop a general theory on holomorphic representations of the vector-valued distributions so that we may be able to answer the question raised above even in a Riemann surface M. Here a vector-valued distribution is given on a real analytic 1-dimensional oriented closed submanifold Γ which need not be connected. The problem will be reduced to solving a non-homogeneous Cauchy-Riemann equation in accordance with the idea of A. Martineau. Naturally his method requires a modification in its treatment of the subject.

Special attention will be paid to the cases where the original distribution is given on the real axis or on a unit circle.

Throughout this paper E will stand for a quasi-complete locally convex Hausdorff topological vector space.

The presentation of the material is arranged as follows: Section 1 is devoted to some preliminary discussions needed for our later purpose. In particular, the well-known fact that a distribution is locally a finite order derivative of a continuous function will be proved by modifying a technique due to A. Pietsch [14] so that we may obtain a clear-cut method of representing a bounded set of distributions in a similar form. In Section 2 the notions of the value of a distribution introduced by S. Łojasiewicz [11] will be extended to an E-valued distribution. Section 3 deals with the extension of the method of A. Martineau to E-valued distributions. As an analogue of Proposition 4 in A. Martineau [12, p. 204] it is shown that if an E-valued distribution $\vec{f}(x)$ is given on an open interval of the real axis, then any holomorphic representation is obtained as a solution $\vec{g}(z)$ of the differential equation $\frac{\partial g}{\partial \bar{z}}=\frac{i}{2}(f\otimes\delta_y)$. In Section 4 the distributional boundary value of an E-valued holomorphic function is investigated from a viewpoint of the Carleman's extension principle. The main result of this section is the theorem showing that if $\vec{h}_1(z)$ and $\vec{h}_2(z)$ are E-valued holomorphic functions on the upper and the lower half planes respectively and if, for each $\vec{e}'\in E'$, $<(\vec{h}_1(x+i\varepsilon)-\vec{h}_2(x-i\varepsilon)), \vec{e}'>$ converges to a distribution as $\varepsilon\rightarrow+0$, then $\vec{h}_1$, $\vec{h}_2$ have the distributional boundary values on the real axis R.

Section 5 is devoted to a general theory on holomorphic representations of E-valued distributions. Let M and Γ be the same as described before. The main result of this section is as follows. If E is taken arbitrary, the representation is only possible according to the cases;

(a) when M is open, Γ is compact,

(b) when M is compact, the E-valued distribution given on Γ satisfies certain conditions.

On the other hand, when M is open, the holomorphic representation is always possible if E is an (**F**)-space.

Section 6 is concerned with the distributional boundary value of holomorphic functions on a unit open disc D. We introduce the space of slowly increasing holomorphic functions on D, of which the boundary behaviors are investigated. Applying a theorem of N. A. Davydov, we can construct a boundary distribution which has no value at any point. We show that any two boundary distributions have the multiplicative product in a sense described in [21]. The result will be generalized by a conformal map to the boundary distributions on the same side of an analytic arc. In the final section we shall study the $\mathscr{S}'$-boundary value of holomorphic function on a half plane belonging to the class H^+. Improvements are given for the results of E. J. Beltrami-M. R. Wohlers [1] related to the Hilbert transform pairs.

1. Preliminaries

Let Ω be a non-empty open subset of an N-dimensional Euclidean space R^N. If $x=(x_1, \cdots, x_N)$, $y=(y_1, \cdots, y_N) \in R^N$ and λ is a real number, we write $x+y=(x_1+y_1, \cdots, x_N+y_N)$, $\lambda x=(\lambda x_1, \cdots, \lambda x_N)$ and $|x|=(\sum_{j=1}^{N}|x_j|^2)^{\frac{1}{2}}$. If p is an N-tuple $(p_1, \cdots, p_N)$ of non-negative integers, the sum $\sum_{j=1}^{N} p_j$ will be denoted by $|p|$ and with $D_x=(D_1, \cdots, D_N)$, $D_j=\frac{\partial}{\partial x_j}$, we put $D_x^p=D_1^{p_1}\cdots D_N^{p_N}$. In addition, we shall use the notation $\boldsymbol{D}^m=D_1^m \cdots D_N^m$ for an integer $m\geqq 0$.

A distribution $T \in \mathcal{D}'(\Omega)$ is locally a distributional derivative of a continuous function F. A. Pietsch [14] has devised an interesting method of constructing such a function F. In our treatment we shall need an analogue for a vector-valued distribution. In the following we shall modify his method of construction so that it may be convenient for our later discussions.

Let $Q=\prod_{j=1}^{N}(a_j, b_j)\subset\subset\Omega$. T is said to be of order at most m on $\bar{Q}$ if there exists a constant C such that

$$|<T, \phi>|\leqq C\sup_{|p|\leqq m}|D^p\phi|$$

for any $\phi \in \mathcal{D}_{\bar{Q}}$. Then, $\mathcal{D}_{\bar{Q}}$ being dense in $\mathcal{D}_{\bar{Q}}^m$, T will be uniquely extended to a continuous linear form on $\mathcal{D}_{\bar{Q}}^m$.

Proposition 1. *Given $Q\subset\subset\Omega$, there exists a function $G(x, y)=\prod_{j=1}^{N}G_j(x_j, y_j)$ defined on R^{2N} such that each $G_j \in \mathcal{D}^m_{[a_j,b_j]\times[a_j,b_j]}$ and such that for any $T \in \mathcal{D}'(\Omega)$ with order at most m on $\bar{Q}$ we can write $T=\boldsymbol{D}^{2m+2}F$ on Q, where $F(x)=<T_y, G(x, y)> \in \mathcal{D}_{\bar{Q}}^m$.*

Proof. For each j, $1\leqq j\leqq N$, we consider the ordinary differential equation

$$y^{(2m+2)}(t)=\delta_\tau \tag{1}$$

with boundary conditions

$$\begin{cases} y(a_j)=y'(a_j)=\cdots=y^{(m)}(a_j)=0, \\ y(b_j)=y'(b_j)=\cdots=y^{(m)}(b_j)=0, \end{cases} \tag{2}$$

where δ_τ, $a_j<\tau<b_j$, is the Dirac measure concentrated at τ. The general solution of (1) is given by

$$y=\frac{(t-\tau)_+^{2m+1}}{(2m+1)!}+c_0+c_1t+\cdots+c_{2m+1}t^{2m+1}$$

with constants $c_0, c_1, \cdots, c_{2m+1}$, which should be chosen to satisfy the boundary conditions (2). Then $c_0, c_1, \cdots, c_{2m+1}$ will be uniquely determined and will be polynomials of τ. Put

$$G_j(x_j, y_j)=\begin{cases}\dfrac{(x_j-y_j)_+^{2m+1}}{(2m+1)!}+c_0(y_j)+\cdots+c_{2m+1}(y_j)x_j^{2m+1} \\ \qquad\qquad \text{for} \quad (x_j, y_j)\in[a_j, b_j]\times[a_j, b_j], \\ 0 \qquad\qquad \text{otherwise.}\end{cases}$$

Clearly $G_j\in\mathcal{D}^m_{[a_j,b_j]\times[a_j,b_j]}$ and if we restrict the domain of G to $[a_j, b_j]\times[a_j, b_j]$, G_j will be of class C^{2m} and satisfy $G_j(x_j, y_j)=G_j(y_j, x_j)$ [4, p. 193]. Let $G(x, y)=\prod_{j=1}^{N}G_j(x_j, y_j)$, and put

$$F(x)=<T_y, G(x, y)>.$$

Then we have for any $\phi\in\mathcal{D}_{\bar{Q}}$

$$\begin{aligned}<\boldsymbol{D}^{2m+2}F, \phi> &= <F, \boldsymbol{D}^{2m+2}\phi> \\ &= <<T_y, G(x, y)>, \boldsymbol{D}^{2m+2}\phi(x)> \\ &= <T_y, \int G(x, y)\boldsymbol{D}^{2m+2}\phi(x)dx> = <T, \phi>,\end{aligned}$$

which completes the proof.

Let $\mathfrak{B}$ be a bounded subset of $\mathcal{D}'(\Omega)$. Since $\mathfrak{B}$ is equicontinuous, there exist for any $Q\subset\subset\Omega$ a constant $C=C(Q)$ and a non-negative integer $m=m(Q)$ such that

$$|<T, \phi>|\leqq C\sup_{|p|\leqq m}|D^p\phi|$$

for any $\phi\in\mathcal{D}_{\bar{Q}}$ and $T\in\mathfrak{B}$. Every distribution $T\in\mathfrak{B}$ can be written on Q

$$T=\boldsymbol{D}^{2m+2}F_T, \quad F_T(x)=<T_y, G(x, y)>\in\mathcal{D}^m_{\bar{Q}},$$

where $G(x, y)$ is the function considered in Proposition 1.

The set $\{G(x, \cdot)\}_{x\in\bar{Q}}$ forms a compact subset of $\mathcal{D}^m_{\bar{Q}}$ since the map $\bar{Q}\ni x\to G(x, \cdot)\in\mathcal{D}^m_{\bar{Q}}$ is continuous. From this and the fact that the strong and weak topologies on any equicontinuous subset of $\mathcal{D}'_{\bar{Q}}$ coincide, it follows from the Banach-Steinhaus theorem that a directed set $\{T_\lambda\}$ of $\mathfrak{B}$ converges to 0 in $\mathcal{D}'(\Omega)$ if and only if for any $Q\subset\subset\Omega$, $\{F_{T_\lambda}\}$ converges uniformly to 0.

Let L and M be any two locally convex Hausdorff topological vector spaces. The ε-product $L\varepsilon M$ is the linear space of bilinear forms on $L'_c\times M'_c$ hypocontinuous with respect to the equicontinuous subsets of L', M', which is equipped with the topology of uniform convergence on the products of equicontinuous subsets of L', M' [18, p. 18]. Let $\mathcal{L}_\varepsilon(L'_c; M)$ be the space of continuous linear maps of L'_c into M with the topology of uniform convergence on the equicontinuous subsets of L'. There exist the algebraic and topological isomorphisms among $L\varepsilon M$, $\mathcal{L}_\varepsilon(L'_c; M)$ and $\mathcal{L}_\varepsilon(M'_c; L)$ [18, p. 34]. Hence we shall identify $L\varepsilon M$ with $\mathcal{L}_\varepsilon(L'_c; M)$ or with $\mathcal{L}_\varepsilon(M'_c; L)$. The map χ: $L'_c \times (L\varepsilon M) \ni (\vec{x}', \vec{u}) \to \vec{u}(\vec{x}') \in M$ is hypocontinuous with respect to any equicontinuous subset of L'_c. This implies that if $\mathcal{L}$ is a relatively compact subset of $L\varepsilon M$ and A' is an equicontinuous subset of L', then the set $\{\vec{u}(\vec{x}'): \vec{u} \in \mathcal{L}, \vec{x}' \in A'\}$ is a relatively compact subset of M.

Let Ω be a non-empty open subset of R^N and E a locally convex Hausdorff topological vector space. We write $\mathcal{D}'_\Omega(E)$ instead of $\mathcal{D}'(\Omega)\varepsilon E$, a space of E-valued distributions. We shall assume that E is quasi-complete unless otherwise stated.

Let $\mathcal{L}$ be a relatively compact subset of $\mathcal{D}'_\Omega(E)$ and V° the polar of a 0-neighbourhood V of E. Then the set $\{< \vec{T}, \vec{e}' > \in \mathcal{D}'(\Omega): \vec{T} \in \mathcal{L}, \vec{e}' \in V^\circ\}$ is relatively compact and a fortiori bounded in $\mathcal{D}'(\Omega)$. From the preceding discussions we have immediately

Proposition 2. *Let $\mathcal{L}$ be a relatively compact subset of $\mathcal{D}'_\Omega(E)$. Given $Q \subset\subset \Omega$, we can write with the notations used before*

$$< \vec{T}, \vec{e}' > = \boldsymbol{D}^k F_{\vec{T},\vec{e}'}, \quad F_{\vec{T},\vec{e}'}(x) = << \vec{T}, \vec{e}' >, G(x, y)>_y,$$

where $\vec{T} \in \mathcal{L}$, $\vec{e}' \in V^\circ$ and the map χ: $\mathcal{L} \times V^\circ \ni (\vec{T}, \vec{e}') \to F_{\vec{T},\vec{e}'} \in C_{\bar{Q}}$ is continuous in the topology induced on $\mathcal{L} \times V^\circ$ by the product $\mathcal{D}'_\Omega(E) \times E'_c$.

In accordance with S. Łojasiewicz [11, p. 18] we shall show

Proposition 3. *We assume that $N=2$. Let $Q = I \times J \subset\subset \Omega$ with $I=(a, b)$, $J=(c, d)$ and let $\vec{X} \in \mathcal{D}'_\Omega(E)$. If $D_x^p D_y^q \vec{X} = 0$ on Q with positive integers p, q, then $\vec{X}$ may be written in the form*

$$\vec{X} = \sum_{\mu=1}^{p-1} x^\mu \vec{S}_\mu(y) + \sum_{\nu=1}^{q-1} y^\nu \vec{T}_\nu(x)$$

on Q with distributions $\vec{S}_\mu(y)$ and $\vec{T}_\nu(x)$ independent of x and y respectively.

Proof. Since the functions $1, x, \cdots, x^{p-1}$ are linearly independent in $\mathcal{D}'(I)$, we have the functions $a_\mu(x) \in \mathcal{D}(I)$ such that

$$< a_\mu(x), x^\nu > = -\delta_{\mu\nu}; \quad \mu, \nu = 0, 1, \cdots, p-1.$$

Let ϕ be any element of $\mathcal{D}(I)$. If we put

$$\psi(x) = \phi(x) + \sum_{\mu=0}^{p-1} < x^\mu, \phi(x) > a_\mu(x),$$

then $\chi(x) = \int_{-\infty}^x \frac{(x-\tau)^{p-1}}{(p-1)!} \psi(\tau) d\tau$ belongs to $\mathcal{D}(I)$ and $\psi = D_x^p \chi$. Similarly we have $b_\nu(y) \in \mathcal{D}(J)$ satisfying

$$< y^\mu, b_\nu(y)> = -\delta_{\mu\nu}, \qquad \mu, \nu=0, 1, \cdots, q-1.$$

For any $\xi(y) \in \mathcal{D}(J)$, the function

$$\eta(y)=\xi(y)+\sum_{\nu=0}^{q-1}< y^\nu, \xi(y)> b_\nu(y)$$

can be written as $\eta(y)=D_y^q\zeta(y)$ with $\zeta(y) \in \mathcal{D}(J)$. Since $\vec{X}$ has the property $D_x^pD_y^q\vec{X}=0$ on Q, we have

$$\begin{aligned}<\vec{X}, \psi(x)\otimes\eta(y)> &= <\vec{X}, D_x^p\chi(x)\otimes D_y^q\zeta(y)> \\ &=(-1)^{p+q}< D_x^pD_y^q\vec{X}, \chi\otimes\zeta>=0\end{aligned}$$

and therefore

$$\begin{aligned}-<\vec{X}, \phi(x)\otimes\xi(y)> =&\sum_{\mu=0}^{p-1}< x^\mu, \phi><\vec{X}, a_\mu(x)\otimes\xi(y)> \\ &+\sum_{\nu=0}^{q-1}< y^\nu, \xi><\vec{X}, \phi(x)\otimes b_\nu(y)> \\ &+\sum_{\mu=0}^{p-1}\sum_{\nu=0}^{q-1}< x^\mu, \phi>< y^\nu, \xi><\vec{X}, a_\mu(x)\otimes b_\nu(y)>,\end{aligned}$$

where

$$\begin{aligned}< x^\mu, \phi><\vec{X}, a_\mu(x)\otimes\xi(y)> &= < x^\mu, \phi><<\vec{X}_{t,y}, a_\mu(t)>, \xi(y)> \\ &= < x^\mu<\vec{X}_{t,y}, a_\mu(t)>, \phi\otimes\xi>,\end{aligned}$$

$$< y^\nu, \xi><\vec{X}, \phi(x)\otimes b_\nu(y)> = <<\vec{X}_{x,\tau}, b_\nu(\tau)> y^\nu, \phi\otimes\xi>,$$

$$< x^\mu, \phi>< y^\nu, \xi><\vec{X}, a_\mu(x)\otimes b_\nu(y)> = <<\vec{X}_{t,\tau}, a_\mu(t)\otimes b_\nu(\tau)> x^\mu y^\nu, \phi\otimes\xi>.$$

We have therefore the relation

$$\begin{aligned}-\vec{X}=&\sum_{\mu=0}^{p-1}<\vec{X}_{t,y}, a_\mu(t)> x^\mu+\sum_{\nu=0}^{q-1}<\vec{X}_{x,\tau}, b_\nu(\tau)> y^\nu \\ &+\sum_{\mu=0}^{p-1}\sum_{\nu=0}^{q-1}<\vec{X}_{t,\tau}, a_\mu(t)\otimes b_\nu(\tau)> x^\mu y^\nu.\end{aligned}$$

Putting

$$\vec{S}_\mu(y)=<\vec{X}_{t,y}, -a_\mu(t)>,$$

$$\vec{T}_\nu(x)=<\vec{X}_{x,\tau}, -b_\nu(\tau)>+\sum_{\mu=0}^{p-1}<\vec{X}_{t,\tau}, -a_\mu(t)\otimes b_\nu(\tau)> x^\mu,$$

then $\vec{S}_\mu(y)$, $\vec{T}_\nu(x)$ are independent of x and y respectively and we have $\vec{X}=\sum_{\mu=0}^{p-1}x^\mu\vec{S}_\mu(y)+\sum_{\nu=0}^{q-1}y^\nu\vec{T}_\nu(x)$ on Q, completing the proof.

2. Some remarks on fixation of vector-valued distributions

The notions of the value of a distribution at a point of R^N and of the section of a distribution will be extended to the vector-valued distributions.

Let $\vec{T}$ be an E-valued distribution defined on a neighbourhood U of $x_0 \in R^N$. $\vec{T}$ is said to have the value $\vec{T}(x_0)=\vec{e} \in E$ at $x=x_0$, if $\vec{T}(x_0+\lambda x)$, λ being a positive real number, converges to $\vec{e}$ in $\mathcal{D}'(E)$ as $\lambda \to +0$, namely, for any $\phi \in \mathcal{D}$, $< \vec{T}(x_0+\lambda x),\ \phi(x)> = < \vec{T}(x),\ \frac{1}{\lambda^N}\phi\left(\frac{x-x_0}{\lambda}\right)>$ converges to $\vec{e}\int \phi(x)dx$ as $\lambda \to +0$. If $\vec{T}$ is given on $U \backslash \{x_0\}$, $\lim_{x\to x_0} \vec{T}$ will be defined similarly.

With the aid of Proposition 2 we can show after the proof of Proposition 1 in S. Łojasiewicz [11, p. 17] the following

Proposition 4. *Let U be a neighbourhood of $x_0 \in R^N$ and $\vec{T}$ an E-valued distribution defined on $U \backslash \{x_0\}$. Then $\lim_{x\to x_0} \vec{T}=\vec{e}$ if and only if for any equicontinuous subset V° of E' and any neighbourhood $P \subset\subset U$ of x_0, there exists a continuous function $F_{\vec{e}'} \in C(R^N)$, $\vec{e}' \in E'_{V^\circ}$, such that for a non-negative integer k we have*

$$< \vec{T}, \vec{e}'> = <\vec{e}, \vec{e}'> + \boldsymbol{D}^k F_{\vec{e}'}$$

on $P \backslash \{x_0\}$, where $F_{\vec{e}'}=o(|x-x_0|^{kN})$ uniformly on V° as $|x-x_0| \to 0$.

Let us consider a restricted δ-sequence $\{\rho_n\}$, that is, a sequence of non-negative function $\rho_n \in \mathcal{D}(R^N)$ with the following properties:

(i) $\operatorname{supp} \rho_n$ converges to $\{0\}$ as $n \to \infty$;

(ii) $\int \rho_n(x)dx$ converges to 1 as $n \to \infty$;

(iii) $\int |x|^{|k|} |D^k \rho_n(x)|\, dx \leqq M_k$ (M_k being independent of n),

where the integral is extended over the whole N-dimensional space [22, p. 91]. Along the same line as in the proof of Proposition 3 in R. Shiraishi [22, p. 93] we can show

Proposition 5. *Let $\vec{T}$ be an E-valued distribution defined on a neighbourhood of $x_0 \in R^N$. $\vec{T}$ has the value $\vec{T}(x_0)=\vec{e} \in E$ if and only if*

$$\lim_{n\to\infty} < \vec{T},\ \rho_n(x-x_0)> = \vec{e}$$

for every restricted δ-sequence $\{\rho_n\}$.

By proceeding in the same way as in the proof of Lemma 3 in our previous paper [8, p. 156] we can show

Proposition 6. *Let $\vec{T}$ be an E-valued distribution defined on a neighbour-*

hood of 0 *in* R^N. *If the values* $\frac{\partial \vec{T}}{\partial x_j}(0)=\vec{e}_j \in E$ *exist for* $j=1, 2, \cdots, N$, *then the same is also true of* $\vec{T}$.

Let Ω be a non-empty open subset of $R^N=R_x^{N_1}\times R_y^{N_2}$ and $\vec{T}\in \mathcal{D}'_{\Omega}(E)$. For $x_0 \in R_x^{N_1}$ we put

$$\Omega_{x_0}=\{y\in R_y^{N_2}: (x_0, y)\in \Omega\}$$

and assume that Ω_{x_0} is not empty. If $\lim_{\lambda\to+0} T(x_0+\lambda x, y)=1_x\otimes \vec{S}(y)$ then we shall say that $x=x_0$ can be fixed in $\vec{T}(x, y)$ and that $S\in \mathcal{D}'_{\Omega_{x_0}}(E)$ is the section of $\vec{T}$ for $x=x_0$ which will be denoted by $\vec{T}(x_0, y)$. If $\vec{T}$ is defined on $\Omega\backslash\Omega_{x_0}$, one can define $\lim_{x\to x_0}\vec{T}(x, y)$ in obvious fashion.

The following Propositions 4′ and 5′ are the analogues of Propositions 4 and 5 and will be proved in a similar way.

PROPOSITION 4′. *Let* $\vec{T}$ *be an E-valued distribution defined on* $\Omega\backslash\Omega_{x_0}$. *Then* $\lim_{x\to x_0}\vec{T}(x, y)=\vec{S}(y)$ *if and only if for any equicontinuous subset* V° *of* E' *and any non-empty open subset* $G\subset\subset \Omega_{x_0}$, *there exist non-negative integers* k, l, *a neighbourhood* Δ *of* x_0 *and a continuous function* $F_{\overleftarrow{e}'}(x, y)\in C(R^N)$, $\overleftarrow{e}'\in E'_{V^\circ}$, *such that*

$$<\vec{T}, \overleftarrow{e}'> = <1_x\otimes \vec{S}(y), \overleftarrow{e}'> + D_x^k D_y^l F_{\overleftarrow{e}'}$$

on $(\Delta\backslash\{x_0\})\times G$, *where* $F_{\overleftarrow{e}'}=o(|x-x_0|^{kN})$ *uniformly on* G *and* V° *as* $|x-x_0|\to 0$.

PROPOSITION 5′. $\vec{T}$ *has the section* $\vec{T}(x_0, y)=\vec{S}(y)\in \mathcal{D}'_{\Omega_{x_0}}(E)$ *for* $x=x_0$ *if and only if*

$$\lim_{n\to\infty}<\vec{T}, \rho_n(x-x_0)>=1_x\otimes\vec{S}(y)$$

for every restricted δ*-sequence* $\{\rho_n\}$.

We assume that $N=2$. Let $\Omega_+=(a, b)\times(0, c)$ and $I=(a, b)$.

DEFINITION 1. *Let* $\vec{T}\in\mathcal{D}'_{\Omega_+}(E)$. *If there exists a distribution* $\vec{S}\in\mathcal{D}'_I(E)$ *such that*

$$\lim_{\lambda\to+0}\vec{T}(x, \lambda y)=\vec{S}(x)\otimes Y(y)$$

in $\mathcal{D}'_{I\times R_+}(E)$, Y *denoting the Heaviside function, then* $\vec{S}(x)$ *will be called the limit of* $\vec{T}$ *as* $y\to+0$ *and denoted by* $\lim_{y\to+0}\vec{T}(x, y)$. *This means that*

$$\lim_{\lambda\to+0}<\vec{T}, \frac{1}{\lambda}\phi(x)\psi\left(\frac{y}{\lambda}\right)> = <\vec{S}, \phi>\int_0^\infty \psi(y)dy$$

for any $\phi\in\mathcal{D}(I)$ *and* $\psi\in\mathcal{D}(R_+)$.

In the same way as in the proof of Proposition 4 we have

PROPOSITION 7. *Let* $\vec{T} \in \mathcal{D}'_{\Omega_+}(E)$. *Then* $\lim_{y\to+0} \vec{T}(x, y)=\vec{S}(x)$ *if and only if for any open intervals* $J_1 \subset\subset I$ *and* $J_2=(0, c_1)$ *with* $0<c_1<c$ *and any equicontinuous subset* V° *of* E' *there exists a function* $F_{\overleftarrow{e}'} \in C(R^2)$, $\overleftarrow{e}' \in E'_{V^\circ}$, *such that for some positive integer* k *we can write*

$$<\vec{T}, \overleftarrow{e}'> = <\vec{S}(x)\otimes Y(y), \overleftarrow{e}'> + D_x^k D_y^k F_{\overleftarrow{e}'}$$

on $J_1\times J_2$, *where* $F_{\overleftarrow{e}'}=o(y^k)$ *uniformly on* J_1 *and* V° *as* $y\to+0$.

PROPOSITION 8. *Let* $\vec{T} \in \mathcal{D}'_{\Omega_+}(E)$. *If* $\lim_{y\to+0} \dfrac{\partial \vec{T}}{\partial y} = \vec{S}(x)$ *exists, then* $\lim_{y\to+0} \vec{T}(x, y)$ *also exists.*

PROOF. Let V° be any equicontinuous subset of E'. Take J_1 and J_2 as in Proposition 7. Then there exist a positive integer k and a function $F_{\overleftarrow{e}'} \in C(R^2)$, $\overleftarrow{e}' \in E'_{V^\circ}$, for which

$$<\frac{\partial \vec{T}}{\partial y}, \overleftarrow{e}'> = <\vec{S}(x)\otimes Y(y), \overleftarrow{e}'> + D_x^k D_y^k F_{\overleftarrow{e}'}$$

on $J_1\times J_2$, where $F_{\overleftarrow{e}'}=o(y^k)$ uniformly on J_1 and V° as $y\to+0$. We have for any $\phi \in \mathcal{D}(J_1)$ and $\overleftarrow{e}' \in V^\circ$

$$\frac{\partial}{\partial y} <<\vec{T}, \phi>, \overleftarrow{e}'> = <<\frac{\partial \vec{T}}{\partial y}, \phi>, \overleftarrow{e}'>$$

$$= <<\vec{S}, \phi>, \overleftarrow{e}'> + (-1)^k D_y^k \int F_{\overleftarrow{e}'}(x, y)\phi^{(k)}(x)dx$$

and therefore

$$<<\vec{T}, \phi>, \overleftarrow{e}'> = <<\vec{S}, \phi>, \overleftarrow{e}'> y + (-1)^k D_y^{k-1} \int F_{\overleftarrow{e}'}(x, y)\phi^{(k)}(x)dx + c_{\phi,\overleftarrow{e}'},$$

where $c_{\phi,\overleftarrow{e}'}$ is a constant depending on ϕ and $\overleftarrow{e}'$.

This implies that for any given $\varepsilon>0$ and $\psi \in \mathcal{D}(J_2)$ we can choose $\delta>0$ so that

$$|<<\vec{T}(x, \lambda y)-\vec{T}(x, \lambda' y), \phi\otimes\psi>, \overleftarrow{e}'>| < \varepsilon$$

for any λ and λ' with $0<\lambda$, $\lambda'<\delta$. Since $\mathcal{D}'_{\Omega}(E)$ is quasi-complete, we can conclude that the limit $\lim_{y\to+0} \vec{T}(x, y)$ exists, completing the proof.

3. Boundary values of vector-valued holomorphic functions

Let $\boldsymbol{C}$ be a complex plane and z a complex variable with $z=x+iy$.

For any non-empty open subset $\Omega \subset \boldsymbol{C}$ we denote by $\mathscr{H}(\Omega)$ the space of holomorphic functions on Ω with the compact convergence topology. $\mathscr{H}_\Omega(E) = \mathscr{H}(\Omega)\varepsilon E$ will stand for the space of E-valued holomorphic functions defined on Ω.

Let $\Omega_+ = I \times (0, b)$ with $I = (-a, a)$ and $b > 0$. In his paper [12, p. 200], A. Martineau has shown some equivalent conditions for the existence of the distributional boundary value of $h \in \mathscr{H}(\Omega_+)$. There the distributional boundary value problem is reduced to solving a differential equation [12, p. 204]. In accordance with his idea we shall first study the analogues for vector-valued holomorphic functions.

Proposition 9. *Let $\vec{h} \in \mathscr{H}_{\Omega_+}(E)$. Then the following conditions are equivalent:*

(*a*) *$\vec{h}$ can be extended over $I = (-a, a)$ as an E-valued distribution.*

(*b*) *$\vec{h}$ is a slowly increasing function near each point of I, namely, for any $J \subset\subset I$ and any continuous semi-norm p of E, there exists a positive number k_0 such that*

$$\sup_{x \in J} p(y^{k_0} \vec{h}(x+iy))$$

is bounded for sufficiently small y.

(*c*) *The distributional limit $\lim_{\varepsilon \to +0} \vec{h}(x+i\varepsilon)$ exists.*

(*d*) *The distributional limit $\lim_{y \to +0} \vec{h}(x+iy)$ exists.*

Proof. $(a) \to (b)$. Let V be an absolutely convex neighbourhood of 0 in E such that p coincides with the gauge of V. Suppose $\vec{h}$ can be extended over $I = (-a, a)$ to an E-valued distribution which we shall denote by the same symbol $\vec{h}$ for the sake of simplicity. Let Ω be the domain of the extended $\vec{h}$. Choose positive numbers a_1, b_1 so that $\omega = (-a_1, a_1) \times (-b_1, b_1) \subset\subset \Omega$ and $J \subset\subset (-a_1, a_1)$. Then, by Proposition 2, there exists a continuous function $g_{\vec{e}'}(z)$ defined on the plane $\boldsymbol{C}$ such that for some positive integer k we can write

$$<\vec{h}(z), \vec{e}'> = D_x^k D_y^k g_{\vec{e}'}(z) \qquad \text{for any} \quad \vec{e}' \in V^\circ$$

on ω. The map $V^\circ \ni \vec{e}' \to g_{\vec{e}'} \in C(\bar{\omega})$ is continuous, where V° has the topology induced by E'_c.

Let $\vec{r}(z)$ be an E-valued holomorphic function on Ω_+ such that

$$i^k \frac{d^{2k}}{dz^{2k}} \vec{r}(z) = \vec{h}(z).$$

If we put $r_{\vec{e}'}(z) = <\vec{r}(z), \vec{e}'>$, then

$$D_x^k D_y^k (g_{\vec{e}'} - r_{\vec{e}'}) = 0 \qquad \text{for any } \vec{e}' \in V^\circ$$

on $\Omega_{+}\cap\omega$. Consequently we can find by Proposition 3 continuous functions $S_{\mu,\overleftarrow{e}'}(y)$, $T_{\nu,\overleftarrow{e}'}(x)$ on $(0, b_1)$ and on $(-a_1, a_1)$ respectively with the properties:

$$\text{(i)}\qquad g_{\overleftarrow{e}'}(z)-r_{\overleftarrow{e}'}(z)=\sum_{\mu=0}^{k-1}x^{\mu}S_{\mu,\overleftarrow{e}'}(y)+\sum_{\nu=0}^{k-1}y^{\nu}T_{\nu,\overleftarrow{e}'}(x)\qquad \text{on } \Omega_{+}\cap\omega,$$

$$\text{(ii)}\qquad S_{\mu,\overleftarrow{e}'}(y)=<g_{\overleftarrow{e}'}(z)-r_{\overleftarrow{e}'}(z),\ -a_{\mu}(x)>,\ \mu=0, 1, \cdots, k-1,$$

$$\begin{aligned}\text{(iii)}\qquad T_{\nu,\overleftarrow{e}'}(x)=&<g_{\overleftarrow{e}'}(z)-r_{\overleftarrow{e}'}(z),\ -b_{\nu}(y)>\\ &+\sum_{\mu=0}^{k-1}<g_{\overleftarrow{e}'}(z)-r_{\overleftarrow{e}'}(z),\ -a_{\mu}(x)\otimes b_{\nu}(y)>x^{\mu},\\ &\qquad\qquad \nu=0, 1, \cdots, k-1,\end{aligned}$$

where a_{μ}, b_{ν} have been chosen as in Proposition 3.

(ii) implies that the map $\overleftarrow{e}'\to S_{\mu,\overleftarrow{e}'}(y)\in C((0, b_1])$ is continuous where $C((0, b_1])$ has the compact convergence topology. (iii) implies that each $T_{\nu,\overleftarrow{e}'}(x)$ is a continuous function on $[-a_1, a_1]$ and the map $\overleftarrow{e}'\to T_{\nu,\overleftarrow{e}'}\in C([-a_1, a_1])$ is continuous.

If we put

$$l_{\overleftarrow{e}'}(z)=g_{\overleftarrow{e}'}(z)+\sum_{\mu=0}^{k-1}x^{\mu}<g_{\overleftarrow{e}'}(z),\ a_{\mu}(x)>-\sum_{\nu=0}^{k-1}y^{\nu}T_{\nu,\overleftarrow{e}'}(x),$$

it is a continuous function on $\bar{\omega}$ and the map $\overleftarrow{e}'\to l_{\overleftarrow{e}'}(z)\in C(\bar{\omega})$ is also continuous. Now we can write

$$r_{\overleftarrow{e}'}(z)+\sum_{\mu=0}^{k-1}x^{\mu}<r_{\overleftarrow{e}'},\ a_{\mu}(x)>=l_{\overleftarrow{e}'}(z). \tag{1}$$

Taking any distinct points x_j, $j=1, 2, \cdots, k$, in $(-a_1, a_1)$, we consider the system of equations

$$r_{\overleftarrow{e}'}(x_j+iy)+\sum_{\mu=0}^{k-1}x_j^{\mu}<r_{\overleftarrow{e}'},\ a_{\mu}(x)>=l_{\overleftarrow{e}'}(x_j+iy),$$
$$j=1, 2, \cdots, k.$$

Since $\det|x_j^{\mu}|\neq 0$, it follows that each of $<r_{\overleftarrow{e}'},\ a_{\mu}(x)>$, $\mu=1, 2, \cdots, k-1$, is uniquely extended to a continuous function on $[0, b_1]$ and the map $\overleftarrow{e}'\to$ $<r_{\overleftarrow{e}'},\ a_{\mu}(x)>\in C([0, b_1])$ is continuous. From this we see that if we put

$$\tilde{l}_{\overleftarrow{e}'}(z)=l_{\overleftarrow{e}'}(z)-\sum_{\mu=1}^{k-1}x^{\mu}<r_{\overleftarrow{e}'},\ a_{\mu}(x)>, \tag{2}$$

then $\tilde{l}_{\overleftarrow{e}'}(z)$ is uniquely extended to a continuous function on $[-a_1, a_1]\times[0, b_1]$ and that the map $\overleftarrow{e}'\to\tilde{l}_{\overleftarrow{e}'}(z)\in C([-a_1, a_1]\times[0, b_1])$ is continuous. From (1) and (2) we have

$$r_{\overleftarrow{e}'}(z)+<r_{\overleftarrow{e}'},\ a_0(x)>=\tilde{l}_{\overleftarrow{e}'}(z). \tag{3}$$

Let $\Gamma(y)$ be the positively oriented boundary of the rectangle $(-a_1, a_1)\times(y, b_1)$ with $0<y<b_1$. If we put $s_{\overleftarrow{e}'}(y)=<r_{\overleftarrow{e}'},\ a_0(x)>$, then since $r_{\overleftarrow{e}'}(z)$ is holomorphic on Ω_+, we have

$$s_{\vec{e}'}^{+}(y)-s_{\vec{e}'}^{+}(b_1)=\frac{1}{2a_1}\int_{\Gamma(y)}\tilde{l}_{\vec{e}'}^{+}(\zeta)d\zeta.$$

Consequently $s_{\vec{e}'}^{+}(y)-s_{\vec{e}'}^{+}(y')$ tends to 0 uniformly on V° as $y-y'\to 0$. This together with (3) implies that $r_{\vec{e}'}^{+}(x+i\varepsilon)$ converges uniformly for $x\in[-a_1, a_1]$ and $\vec{e}'\in V^\circ$ as $\varepsilon\to+0$. We can write for $z\in J\times(0, y_1)$ with $0<y_1<b_1$ and for any $\vec{e}'\in V^\circ$

$$<\vec{h}(z), \vec{e}'>=\frac{(2k)!}{2\pi i}i^k\int_{\Gamma_0}\frac{r_{\vec{e}'}^{+}(\zeta)}{(\zeta-z)^{2k+1}}d\zeta,$$

where $\Gamma_0=\lim_{y\to+0}\Gamma(y)$. It follows that

$$\sup_{0<y<y_1}\sup_{x\in J}|y^{2k+1}<\vec{h}(z), \vec{e}'>|$$

is bounded. Thus (a) implies (b).

(b)→(c). Let V be any neighbourhood of 0 in E. Consider any interval $J\subset\subset I$ and a positive number $y_0<b$. Put $\Omega_1=J\times(0, y_0)$. Then, by our hypothesis, there exist a positive integer k and a positive constant M such that

$$|y^k<\vec{h}(z), \vec{e}'>|\leqq M \qquad \text{for any} \quad z\in\Omega_1 \quad \text{and} \quad \vec{e}'\in V^\circ.$$

If we consider an E-valued holomorphic function $\vec{g}(z)$ such that $\frac{d^{k+2}}{dz^{k+2}}\vec{g}=\vec{h}(z)$, then a simple calculation will show that there exists a constant M' satisfying

$$|<\vec{g}(z), \vec{e}'>-<\vec{g}(z'), \vec{e}'>|\leqq M'|z-z'|.$$

for $z, z'\in\Omega_1$ and $\vec{e}'\in V^\circ$. Consequently $<\vec{g}(x+i\varepsilon), \vec{e}'>$ converges in $C(J)$ uniformly for $\vec{e}'\in V^\circ$ as $\varepsilon\to+0$. Using the fact that

$$<\vec{h}(x+i\varepsilon), \vec{e}'>=D_x^{k+2}<\vec{g}(x+i\varepsilon), \vec{e}'>,$$

we can conclude that the distributional limit of $\vec{h}(x+i\varepsilon)$ for $\varepsilon\to+0$ exists. Thus (b) implies (c).

(c)→(d). Suppose (c) holds. Let $\mathfrak{B}_1$ be any bounded subset of $\mathcal{D}(I)$ and V any neighbourhood of 0 in E. Given any $\eta>0$, there exists an $\varepsilon_\eta>0$ such that $0<\varepsilon\leqq\varepsilon_\eta$ implies that

$$|\int<\vec{h}(x+i\varepsilon), \vec{e}'>\phi(x)dx-<<\vec{S}, \vec{e}'>, \phi(x)>|<\eta$$

for any $\phi\in\mathfrak{B}_1$ and any $\vec{e}'\in V^\circ$.

Let $\mathfrak{B}_2$ be any bounded subset of $\mathcal{D}(R_+)$ and let $c>1$ be chosen so that every $\psi\in\mathfrak{B}_2$ vanishes identically for $y>c$. We have then with $\varepsilon\leqq\frac{\varepsilon_\eta}{c}$ and $0<y<c$

$$|\int <\vec{h}(x+i\varepsilon y), \vec{e}'>\phi(x)dx - <<\vec{S}, \vec{e}'>, \phi>| < \eta.$$

Consequently, for any $\phi \in \mathfrak{B}_1$ and any $\psi \in \mathfrak{B}_2$ we have

$$|\iint <\vec{h}(x+i\varepsilon y), \vec{e}'>\phi(x)\psi(y)dx\,dy - <<\vec{S}, \vec{e}'>, \phi>\int\psi(y)dy|$$
$$\leqq \eta\int|\psi|\,dy,$$

which means that

$$\lim_{\varepsilon\to+0} <\vec{h}(x+i\varepsilon y), \phi(x)\otimes\psi(y)> = <\vec{S}(x)\otimes Y_y, \phi(x)\otimes\psi(y)>.$$

Thus (c) implies (d).

The implication (d)→(a) is an immediate consequence of the following Proposition 10.

Thus the proof is complete.

Let Ω_- be the domain symmetric to Ω_+ with respect to the real axis and put $\Omega = \Omega_+ \cup I \cup \Omega_-$.

Proposition 10. *If $\vec{T} \in \mathcal{D}'_{\Omega_+}(E)$ has the distributional limit $\lim_{y\to+0}\vec{T}(x, y) = \vec{S}(x)$, then there exists a $\vec{W} \in \mathcal{D}'_{\Omega}(E)$ such that $\vec{W}$ coincides with $\vec{T}$ on Ω_+ and vanishes on Ω_-. Moreover $\vec{W}$ is unique under the condition that $\vec{W} - \vec{S}\otimes Y(y)$ has the section 0 for $y=0$.*

Proof. Let $\rho \in \mathcal{E}(R)$ be equal to 1 on $(2, \infty)$ and 0 on $(-\infty, 1)$. Put $\rho_{(\varepsilon)}(y) = \rho\left(\frac{y}{\varepsilon}\right)$ for $0<\varepsilon\leqq 1$. We consider the distribution $\vec{W}_\varepsilon \in \mathcal{D}'_{\Omega}(E)$ defined by

$$<\vec{W}_\varepsilon, \phi> = <\vec{T}(x, y), \rho_{(\varepsilon)}(y)\phi(x, y)>, \qquad \phi \in \mathcal{D}(\Omega). \tag{4}$$

$\phi \in \mathcal{D}(\Omega)$ has a compact support and so we can find an interval $J \subset\subset I$ and a positive $b_1 < b$ such that $\operatorname{supp}\phi \subset J\times(-b_1, b_1)$. Let V be any neighbourhood of 0 in E. Then there exist by Proposition 7 a positive integer k and a continuous function $F_{\vec{e}'} \in C(R^2)$, $\vec{e}' \in V^\circ$, such that we can write

$$<\vec{T}, \vec{e}'> = <\vec{S}\otimes Y_y, \vec{e}'> + D_x^k D_y^k F_{\vec{e}'}$$

on $\omega = J\times(0, b_1)$, where $F_{\vec{e}'} = o(y^k)$ uniformly on J and V° as $y\to+0$ and $F_{\vec{e}'}=0$ for $y<0$.

If $\operatorname{supp}\phi \subset J\times(-b_1, b_1)$, then using (4) we can write

$$<<\vec{W}_\varepsilon, \vec{e}'>, \phi> = <<\vec{S}\otimes Y_y, \vec{e}'>, \rho_{(\varepsilon)}(y)\phi(x, y)> + <D_x^k D_y^k F_{\vec{e}'}, \rho_{(\varepsilon)}(y)\phi(x, y)>$$
$$= <<\vec{S}(x), \vec{e}'>, \int\rho_{(\varepsilon)}(y)\phi(x, y)dy> + <F_{\vec{e}'}, D_x^k D_y^k\rho_{(\varepsilon)}(y)\phi(x, y)>,$$

which converges uniformly on V° to $<<\vec{S}, \check{e}'>\otimes Y, \phi>+<D_x^k D_y^k F_{\check{e}'}, \phi>$ as $\varepsilon\to+0$. This means, since $\mathcal{D}'_\Omega(E)$ is quasi-complete, that $\lim_{\varepsilon\to+0}\vec{W}_\varepsilon=\vec{W}\in\mathcal{D}'_\Omega(E)$ exists and we have

$$
\begin{aligned}
<\vec{W}, \phi> &= \lim_{\varepsilon\to+0} <\vec{T}, \rho_{(\varepsilon)}(y)\phi(x, y)> \\
&= \lim_{\varepsilon\to+0} <\vec{T}-\vec{S}\otimes Y, \rho_{(\varepsilon)}(y)\phi(x, y)> + <\vec{S}\otimes Y, \phi>.
\end{aligned}
$$

As a result, we see that $\vec{W}$ satisfies the conditions stated in our proposition.

Let $\vec{W}_1$ be another E-valued distribution belonging to $\mathcal{D}'_\Omega(E)$ with all the properties stated in Proposition 10. If we put $\psi(y)=1-\rho(y)-\rho(-y)$, then using Lemma 4 in our previous paper [8, p. 166] we have

$$
\lim_{\varepsilon\to+0} <\vec{W}_1-\vec{S}\otimes Y, \psi_{(\varepsilon)}(y)\phi(x, y)> = 0.
$$

Consequently we can write

$$
\begin{aligned}
<\vec{W}_1, \phi> &= \lim_{\varepsilon\to+0} <\vec{W}_1-\vec{S}\otimes Y, (\rho_{(\varepsilon)}(y)+\check{\rho}_{(\varepsilon)}(y))\phi(x, y)> + <\vec{S}\otimes Y, \phi> \\
&= \lim_{\varepsilon\to+0} <\vec{T}-\vec{S}\otimes Y, \rho_{(\varepsilon)}(y)\phi(x, y)> + <\vec{S}\otimes Y, \phi>, \\
&= <\vec{W}, \phi>,
\end{aligned}
$$

which completes the proof.

The uniquely determined $\vec{W}\in\mathcal{D}'_\Omega(E)$ in the preceding proposition will be called in this paper a canonical extension of $\vec{T}$. We see that $\vec{T}\in\mathcal{D}'_{\Omega_+}(E)$ has the canonical extension if and only if the distributional limit $\lim_{\varepsilon\to+0}\rho_{(\varepsilon)}(y)\vec{T}$ exists in $\mathcal{D}'_\Omega(E)$. The limit will be the canonical extension of $\vec{T}$.

REMARK 1. Let $\vec{h}\in\mathcal{H}_{\Omega_+}(E)$. If $<\vec{h}, \check{e}'>\in\mathcal{H}(\Omega_+)$ is slowly increasing for each $\check{e}'\in E'$, then $\vec{h}$ is slowly increasing in the sense of (b) in Proposition 9. Indeed, this follows from Baire's category theorem.

PROPOSITION 11. *Let $\vec{h}\in\mathcal{D}'_{\Omega_+}(E)$. If the distributional limit $\vec{h}_+=\lim_{\varepsilon\to+0}\vec{h}(x+i\varepsilon)$ exists, then the canonical extension $\vec{g}\in\mathcal{D}'_\Omega(E)$ of $\vec{h}$ satisfies the following equation:*

$$
\frac{\partial\vec{g}}{\partial\bar{z}} = \frac{i}{2}(\vec{h}_+\otimes\delta_y).
$$

PROOF. By Proposition 10 we have $\vec{g}=\lim_{\varepsilon\to+0}\rho_{(\varepsilon)}(y)\vec{h}$. Consequently

$$\frac{\partial \vec{g}}{\partial \bar{z}}=\lim_{\varepsilon\to+0}\left(\frac{\partial \rho_{(\varepsilon)}}{\partial \bar{z}}\vec{h}+\rho_{(\varepsilon)}\frac{\partial \vec{h}}{\partial \bar{z}}\right)$$

$$=\lim_{\varepsilon\to+0}\frac{\partial \rho_{(\varepsilon)}}{\partial \bar{z}}\vec{h}.$$

Now, for any $\phi\in\mathcal{D}(\Omega)$

$$\lim_{\varepsilon\to+0}<\frac{\partial \rho_{(\varepsilon)}}{\partial \bar{z}}\vec{h},\ \phi>=\lim_{\varepsilon\to+0}<\vec{h},\ \frac{\partial \rho_{(\varepsilon)}}{\partial \bar{z}}\phi>$$

$$=\lim_{\varepsilon\to+0}<\vec{h},\ \frac{i}{2}\frac{1}{\varepsilon}\rho'\left(\frac{y}{\varepsilon}\right)\phi>$$

$$=\lim_{\varepsilon\to+0}<\vec{h}(x+i\varepsilon y),\ \frac{i}{2}\rho'(y)\phi(x,\ y)>$$

$$=\frac{i}{2}<\vec{h}_{+},\ \phi(x,\ 0)\int\rho'(y)dy>$$

$$=\frac{i}{2}<\vec{h}_{+}\otimes\delta_{y},\ \phi>,$$

therefore we obtain

$$\frac{\partial \vec{g}}{\partial \bar{z}}=\frac{i}{2}(\vec{h}_{+}\otimes\delta_{y}),$$

which completes the proof.

$\vec{h}_{+}$ in the preceding proposition will also be called the distributional boundary value of $\vec{h}$. For $\vec{T}\in\mathcal{D}'_{\Omega_{-}}(E)$ we define similarly the canonical extension. $\vec{h}\in\mathcal{H}_{\Omega_{-}}(E)$ has the distributional limit (boundary value) $\vec{h}_{-}=\lim_{\varepsilon\to+0}\vec{h}(x-i\varepsilon)$ if and only if the distributional limit $\vec{g}=\lim_{\varepsilon\to+0}\check{\rho}_{(\varepsilon)}(y)\vec{h}$ exists. In this case we can show that

$$\frac{\partial \vec{g}}{\partial \bar{z}}=-\frac{i}{2}(\vec{h}_{-}\otimes\delta_{y}).$$

Using this fact we shall show

THEOREM 1. *Let $\vec{f}\in\mathcal{D}'_{I}(E)$. In order that $\vec{f}$ can be represented in the form $\vec{f}=\vec{f}_1-\vec{f}_2$, where $\vec{f}_1$ and $\vec{f}_2$ are the distributional boundary values of holomorphic functions $\vec{h}_1\in\mathcal{H}_{\Omega_{+}}(E)$ and $\vec{h}_2\in\mathcal{H}_{\Omega_{-}}(E)$ respectively, it is necessary and sufficient that there exists a distribution $\vec{g}\in\mathcal{D}'_{\Omega}(E)$ satisfying the equation*

$$\frac{\partial \vec{g}}{\partial \bar{z}}=\frac{i}{2}(\vec{f}\otimes\delta_{y})$$

on Ω.

PROOF. Suppose that $\vec{f}=\vec{f}_1-\vec{f}_2$ with $\vec{f}_1=\lim_{\varepsilon\to+0}\vec{h}_1(x+i\varepsilon)$ and $\vec{f}_2=\lim_{\varepsilon\to+0}\vec{h}_2(x-i\varepsilon)$.

If we put $\vec{g}=\vec{g}_1+\vec{g}_2$, where $\vec{g}_j$, $j=1, 2$, is the canonical extension of $\vec{h}_j$, then by Proposition 11 we obtain

$$\begin{aligned}\frac{\partial \vec{g}}{\partial \bar{z}}&=\frac{\partial \vec{g}_1}{\partial \bar{z}}+\frac{\partial \vec{g}_2}{\partial \bar{z}}\\&=\frac{i}{2}((\vec{f}_1-\vec{f}_2)\otimes\delta_y))=\frac{i}{2}(\vec{f}\otimes\delta_y).\end{aligned}$$

Conversely, suppose that there exists a distribution $\vec{g}\in\mathcal{D}'_{\Omega}(E)$ such that

$$\frac{\partial \vec{g}}{\partial \bar{z}}=\frac{i}{2}(\vec{f}\otimes\delta_y).$$

As $\dfrac{\partial \vec{g}}{\partial \bar{z}}=0$ on $\Omega_+\cup\Omega_-$, $\vec{g}$ is holomorphic on $\Omega_+\cup\Omega_-$. If we put $\vec{h}_1=\vec{g}|\Omega_+$, $\vec{h}_2=\vec{g}|\Omega_-$, then $\vec{g}\in\mathcal{D}'_{\Omega}(E)$ is the extension of both $\vec{h}_1$ and $\vec{h}_2$. By virtue of Proposition 9 the distributional limits $\vec{f}_1=\lim\limits_{\varepsilon\to+0}\vec{h}_1(x+i\varepsilon)$ and $\vec{f}_2=\lim\limits_{\varepsilon\to+0}\vec{h}_2(x-i\varepsilon)$ exist. Let $\vec{g}_j$, $j=1, 2$, be chosen as before. Then we have

$$\frac{\partial(\vec{g}-(\vec{g}_1+\vec{g}_2))}{\partial \bar{z}}=\frac{i}{2}(\vec{f}-(\vec{f}_1-\vec{f}_2))\otimes\delta_y$$

and $\operatorname{supp}(\vec{g}-(\vec{g}_1+\vec{g}_2))\subset I$. Owing to a lemma of A. Martineau [12, p. 208], we can conclude that $\vec{f}=\vec{f}_1-\vec{f}_2$.

Thus the proof is complete.

4. A version of Carleman's extension principle and its applications to vector-valued holomorphic functions

Let G_1 be a domain in the complex plane $\boldsymbol{C}$. Suppose the boundary ∂G_1 contains an open line segment I such that for any $z_0\in I$ there exists a neighbourhood U of z_0 such that U does not intersect $\partial G_1\backslash I$. Let G_2 be the domain symmetric to G_1 with respect to I, and let us assume that G_1 and G_2 are disjoint. If two holomorphic functions h_1 on G_1 and h_2 on G_2 take on the same boundary values on this line segment from within each domain, then h_1 and h_2 are analytic continuations of each other. The result will be generalized in this section to the vector-valued holomorphic functions.

For the sake of simplicity, we shall take $G_1=\Omega_+=(a, b)\times(0, c)$ and $I=(a, b)$. Put $\Omega=\Omega_+\cup I\cup\Omega_-$.

Let u be a real- or complex-valued harmonic function on Ω_+. First we note that $\lim\limits_{y\to+0}u(x+iy)$ and $\lim\limits_{y\to+0}\dfrac{\partial u(x+iy)}{\partial y}$ exist in the distributional sense if and only if one of these exists in the same sense. This is because we can apply Proposition 8 since $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$. Suppose $\alpha = \lim_{y\to+0} u(x+iy)$, $\beta = \lim_{y\to+0} \frac{\partial u(x+iy)}{\partial y}$ exist. Consider the canonical extension $\tilde{u}$ which was defined by the distributional limit $\lim_{\varepsilon\to+0} \rho_{(\varepsilon)} u$. We know that $\lim_{\varepsilon\to+0} \rho'_{(\varepsilon)} u = \alpha \otimes \delta_y$ and $\lim_{\varepsilon\to+0} \rho'_{(\varepsilon)} \frac{\partial u}{\partial y} = \beta \otimes \delta_y$. Since $\Delta(\rho_{(\varepsilon)} u) = \frac{\partial}{\partial y}(\rho'_{(\varepsilon)} u) + \rho'_{(\varepsilon)} \frac{\partial u}{\partial y}$, it follows by passing to the limit that

$$\Delta \tilde{u} = \alpha \otimes \delta'_y + \beta \otimes \delta_y.$$

Define $u_1(x+iy) = u(x-iy)$ for $y<0$. u_1 is harmonic on Ω_-. Then in a similar way we have

$$\Delta \tilde{u}_1 = -\alpha \otimes \delta'_y + \beta \otimes \delta_y,$$

where $\tilde{u}_1$ is the canonical extension of u_1. Consequently

$$\Delta(\tilde{u} - \tilde{u}_1) = 2\alpha \otimes \delta'_y, \quad \Delta(\tilde{u} + \tilde{u}_1) = 2\beta \otimes \delta_y.$$

Therefore $\tilde{u} - \tilde{u}_1$ (resp. $\tilde{u} + \tilde{u}_1$) is harmonic on Ω if α (resp. β) vanishes on I. This means that the Schwarz reflection principle for harmonic functions remains valid under a weaker assumption than usual. The reasoning shows that the result holds also for the harmonic function $u(x_1, \cdots, x_n, y)$, $n \geqq 1$.

PROPOSITION 12. *Let $h \in \mathcal{H}(\Omega_+)$ and let u, v be the real and the imaginary parts of h.*

(a) *If $\lim_{y\to+0} u(x+iy) = 0$ in the distributional sense, then h can be continued analytically across I into Ω_-.*

(b) *If the distributional limit $\lim_{y\to+0} u(x+iy)$ exists, then the distributional limit $\lim_{\varepsilon\to+0} h(x+i\varepsilon)$ exists.*

(c) *The distributional limit $\lim_{\varepsilon\to+0} u(x+i\varepsilon)$ exists if and only if the distributional limit $\lim_{y\to+0} u(x+iy)$ exists.*

PROOF. (a) In virtue of the Schwarz reflection principle just described, u can be continued to a harmonic function u_1 on Ω. Let v_1 be a conjugate harmonic function of u_1. $h_1 = u_1 + iv_1$ is holomorphic on Ω and we can take $v_1 = v$ on Ω_+, and therefore $h = h_1$ on Ω_+.

(b) Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, we see that $\lim_{y\to+0} \frac{\partial v}{\partial y}(x+iy)$ exists, and so does $\lim_{y\to+0} v(x+iy)$. This implies that $\lim_{y\to+0} h(x+iy)$ exists. Owing to Proposition 9 we can conclude that $\lim_{\varepsilon\to+0} h(x+i\varepsilon)$ exists.

(c) Suppose $\lim_{\varepsilon\to+0} u(x+i\varepsilon)$ exists. We see that $\lim_{y\to+0} u(x+iy)$ exists by the same reasoning as in the proof of the implication (c)$\rightarrow$(d) in Proposition 9. Conversely, suppose $\lim_{y\rightarrow+0} u(x+iy)$ exists, then (b) implies that $\lim_{\varepsilon\rightarrow+0} h(x+i\varepsilon)$ exists and so does $\lim_{\varepsilon\rightarrow+0} u(x+i\varepsilon)$.

Following the proof of Theorem 1 in T. Carleman [3, p. 38] we shall show

PROPOSITION 13 (*Carleman's extension principle*). *Let* $h_1 \in \mathcal{H}(\Omega_+)$ *and* $h_2 \in \mathcal{H}(\Omega_-)$. *If the distributional limit* $\lim_{\varepsilon\rightarrow+0}(h_1(x+i\varepsilon)-h_2(x-i\varepsilon))$ *exists and equals* 0, *then* h_1 *and* h_2 *are analytic continuations of each other.*

PROOF. Consider the function $\Phi(z)=h_1(z)+\overline{h_2(\bar{z})} \in \mathcal{H}(\Omega_+)$. From the assumption it follows that the imaginary part of $\Phi(z)$ converges to 0 in the sense of distribution as $y\rightarrow+0$, and therefore, by Proposition 12, $\Phi(z)$ can be continued analytically to $\psi \in \mathcal{H}(\Omega)$. Similarly the function $H(z)= i(h_1(z)-\overline{h_2(\bar{z})}) \in \mathcal{H}(\Omega_+)$ can be continued analytically to $\omega \in \mathcal{H}(\Omega)$. We can write for $z \in \Omega_+$

$$h_1(z)=\frac{1}{2}(\psi(z)-i\omega(z)) \quad \text{and} \quad \overline{h_2(\bar{z})}=\frac{1}{2}(\psi(z)+i\omega(z)). \tag{1}$$

Combining (1) with the fact that $\psi(z)$ and $\omega(z)$ are real on the real axis, we see that the analytic continuation of h_1 into Ω_- is equal to h_2. Thus the proof is complete.

PROPOSITION 14. *Let* $h_1 \in \mathcal{H}(\Omega_+)$ *and* $h_2 \in \mathcal{H}(\Omega_-)$. *If the distributional limit*

$$\lim_{\varepsilon\rightarrow+0}(h_1(x+i\varepsilon)-h_2(x-i\varepsilon))=f \in \mathcal{D}'(I),$$

exists, then the distributional limits $\lim_{\varepsilon\rightarrow+0} h_1(x+i\varepsilon)=f_1$ *and* $\lim_{\varepsilon\rightarrow+0} h_2(x-i\varepsilon)=f_2$ *exist and* $f=f_1-f_2$.

PROOF. For any interval $J\subset\subset I$ we shall consider a distribution $\alpha f \in \mathcal{D}'(R)$, where α belongs to $\mathcal{D}(I)$ and takes the value 1 on J. There exists a holomorphic function h defined on $\boldsymbol{C}\setminus\operatorname{supp}(\alpha f)$ such that the distributional limits $\lim_{\varepsilon\rightarrow+0} h(x+i\varepsilon)$, $\lim_{\varepsilon\rightarrow+0} h(x-i\varepsilon)$ exist and

$$\lim_{\varepsilon\rightarrow+0} h(x+i\varepsilon)-\lim_{\varepsilon\rightarrow+0} h(x-i\varepsilon)=f,$$

and therefore we have

$$\lim_{\varepsilon\rightarrow+0}((h_1-h)(x+i\varepsilon)-(h_2-h)(x-i\varepsilon))=(1-\alpha)f.$$

Proposition 13 implies that the functions h_1-h and h_2-h have the same analytic continuation h_0 in $\Omega_+\cup J\cup\Omega_-$. Since both h and h_0 have the distributional boundary values on J from within each domain Ω_+ and Ω_-, if follows that h_1

and h_2 have the distributional boundary values f_1 and f_2 respectively and $f_1-f_2=f$ as J can be chosen arbitrarily. Thus the proof is complete.

We are now in a position to generalize these results for the vector-valued holomorphic functions.

THEOREM 2. *Let $\vec{h}_1 \in \mathcal{H}_{\Omega_+}(E)$ and $\vec{h}_2 \in \mathcal{H}_{\Omega_-}(E)$. If for each $\vec{e}' \in E'$ the distributional limit*

$$\lim_{\varepsilon\to+0} <(\vec{h}_1(x+i\varepsilon)-\vec{h}_2(x-i\varepsilon)),\ \vec{e}'> = f_{\vec{e}'} \in \mathcal{D}'(I)$$

exists, then the distributional limits $\lim_{\varepsilon\to+0}\vec{h}_1(x+i\varepsilon)=\vec{f}_1 \in \mathcal{D}'_I(E)$ and $\lim_{\varepsilon\to+0}\vec{h}_2(x-i\varepsilon)=\vec{f}_2 \in \mathcal{D}'_I(E)$ exist and $<\vec{f}_1-\vec{f}_2,\ \vec{e}'>=f_{\vec{e}'}$, and therefore $f_{\vec{e}'}$ can be written in the form $f_{\vec{e}'}=<\vec{f},\ \vec{e}'>$ with $\vec{f} \in \mathcal{D}'_I(E)$.

PROOF. By Proposition 14 the distributional limits $\lim_{\varepsilon\to+0}<\vec{h}(x+i\varepsilon),\vec{e}'>$, $\lim_{\varepsilon\to+0}<\vec{h}(x-i\varepsilon),\ \vec{e}'>$ exist and therefore by Proposition 9 and Remark 1 we can conclude that the distributional limits $\lim_{\varepsilon\to+0}\vec{h}(x+i\varepsilon)=\vec{f}_1 \in \mathcal{D}'_I(E)$, $\lim_{\varepsilon\to+0}\vec{h}(x-i\varepsilon)=\vec{f}_2 \in \mathcal{D}'_I(E)$ exist. We have moreover $<\vec{f}_1-\vec{f}_2,\ \vec{e}'>=f_{\vec{e}'}$ for each $\vec{e}' \in E'$ and hence we can write $f_{\vec{e}'}=<\vec{f},\ \vec{e}'>$ with $\vec{f} \in \mathcal{D}'_I(E)$. Thus the proof is complete.

As an immediate consequence of Theorem 2 we have

COROLLARY. *Let $\vec{h}_1 \in \mathcal{H}_{\Omega_+}(E)$. If for each $\vec{e}' \in E'$ the distributional limit $\lim_{\varepsilon\to+0}<\vec{h}_1(x+i\varepsilon),\ \vec{e}'>=f_{\vec{e}'} \in \mathcal{D}'(I)$ exists, then the distributional limit $\lim_{\varepsilon\to+0}\vec{h}_1(x+i\varepsilon)=\vec{f}_1 \in \mathcal{D}'_I(E)$ exists and $f_{\vec{e}'}=<\vec{f}_1,\ \vec{e}'>$.*

THEOREM 3 (*Carleman's extension principle for vector-valued holomorphic functions*). *Let $\vec{h}_1 \in \mathcal{H}_{\Omega_+}(E)$ and $\vec{h}_2 \in \mathcal{H}_{\Omega_-}(E)$. If for each $\vec{e}' \in E'$ the distributional limit*

$$\lim_{\varepsilon\to+0} <(\vec{h}_1(x+i\varepsilon)-\vec{h}_2(x-i\varepsilon)),\ \vec{e}'> = 0$$

exists, then $\vec{h}_1$ and $\vec{h}_2$ are analytic continuations of each other.

PROOF. Owing to Theorem 2 there exist the distributional limits

$$\vec{f}_1=\lim_{\varepsilon\to+0}\vec{h}_1(x+i\varepsilon),\quad \vec{f}_2=\lim_{\varepsilon\to+0}\vec{h}_2(x-i\varepsilon),$$

and $\vec{f}_1=\vec{f}_2$. It follows then from the proof of the first part of Theorem 1 that $\vec{h}_1$ and $\vec{h}_2$ are analytic continuations of each other.

Let $\boldsymbol{H}(\Omega)$ be the space of harmonic functions on Ω, with the compact convergence topology. We shall denote by $\boldsymbol{H}_\Omega(E)$ the ε-tensor product $\boldsymbol{H}(\Omega)\varepsilon E$, the space of E-valued harmonic functions on Ω.

PROPOSITION 15. *Let* $\vec{u} \in \boldsymbol{H}_{\Omega_+}(E)$. *If for each* $\vec{e}' \in E'$ *the distributional limit* $\lim_{\varepsilon\to+0} < \vec{u}(x+i\varepsilon), \vec{e}'> = 0$ *exists, then* $\vec{u}(z)$ *can be extended to an E-valued harmonic function across I into* Ω_-.

PROOF. Let z_0 be any fixed point of Ω_+ and consider the line integral

$$\vec{v}(z) = \int_{z_0}^{z} \left(-\frac{\partial \vec{u}}{\partial y} dx + \frac{\partial \vec{u}}{\partial x} dy\right),$$

evaluated along any rectifiable curve $L \subset \Omega_+$ joining the point z_0 to the variable point $z \in \Omega_+$. Then $\vec{v}$ is harmonic and satisfies the equations:

$$\frac{\partial \vec{v}}{\partial x} = -\frac{\partial \vec{u}}{\partial y}, \quad \frac{\partial \vec{v}}{\partial y} = \frac{\partial \vec{u}}{\partial x}.$$

If we put $\vec{f}(z) = \vec{u}(z) + i\vec{v}(z)$, then $\vec{f}$ will be holomorphic on Ω_+. Similarly we define $\vec{g}(z) = -\vec{u}(\bar{z}) + i\vec{v}(\bar{z})$, $z \in \Omega_-$, which is holomorphic on Ω_-.

Owing to our hypothesis, we have for each $\vec{e}' \in E'$

$$\lim_{\varepsilon\to+0} <\vec{f}(x+i\varepsilon) - \vec{g}(x-i\varepsilon), \vec{e}'> = 2 \lim_{\varepsilon\to+0} <(\vec{u}(x+i\varepsilon)), \vec{e}'> = 0.$$

In virtue of Theorem 3 there exists an $\vec{h} \in \mathscr{H}_\Omega(E)$ such that $\vec{h}$ is the analytic continuation of both $\vec{f}$ and $\vec{g}$. The E-valued function $\vec{h}(z) - \vec{h}(\bar{z})$ is harmonic on Ω and we have for $z \in \Omega_+$

$$\vec{u}(z) = \frac{1}{2}(\vec{h}(z) - \vec{h}(\bar{z})),$$

which completes the proof.

5. General discussions on representation of vector-valued distributions as the boundary values

The distributional boundary value problem of holomorphic functions has been developed by many authors. Recently A. Martineau [12] has reduced the problem to solving a non-homogeneous Cauchy-Riemann equation. Basing on this line of thought we shall discuss in this section the problem for vector-valued distributions, especially in a Riemann surface. To this end we shall first introduce the notion of the distributional boundary value of a current on a manifold according to the method described in our previous paper [9]. Here a distribution is understood as a current of degree 0 and of even kind.

Let Ω be a non-empty open subset of $R^N = R_x \times R_y^{N-1}$ and assume that $\Omega_0 = \{y \in R^{N-1} : (0, y) \in \Omega\}$ is not empty. We denote by Ω_- the open subset $\{(x, y) \in \Omega : x < 0\}$. Consider a diffeomorphism

$$\chi: \begin{cases} x' = \xi(x, y) \\ y' = \eta(x, y) \end{cases}$$

of Ω onto an open subset $\Omega' \subset R_{x'} \times R_{y'}^{N-1}$, which maps Ω_0 to Ω_0' and Ω_- to Ω'_-. Then the map $\eta_0 = \eta(0, y)$ is a diffeomorphism of Ω_0 onto Ω_0'. The Jacobians of the maps χ and η_0 will be denoted by J_χ and J_{η_0} respectively. For any $T \in \mathcal{D}'(\Omega_-)$ and $S \in \mathcal{D}'(\Omega_0)$ the direct images $T' = \chi T$ and $S' = \eta_0 S$ are defined by the equations:

$$< T'(x', y'), \phi(x', y') > = < T(x, y), |J_\chi| \phi(\xi(x, y), \eta(x, y)) >, \ \phi \in \mathcal{D}(\Omega'_-)$$

and

$$< S'(y'), \psi(y') > = < S(y), |J_{\eta_0}| \psi(\eta_0(y)) >, \qquad \psi \in \mathcal{D}(\Omega_0').$$

Then we can show the following lemma in the same way as in the proof of Lemma 1 in our previous paper [9, p. 175].

Lemma 1. *Let* $T \in \mathcal{D}'(\Omega_-)$.

(a) *If there exists a distribution* $S \in \mathcal{D}'(\Omega_0)$ *such that* $\lim_{\lambda \to +0} T(\lambda x, y) = \check{Y}(x) \otimes S(y)$, *then* $\lim_{\lambda \to +0} T'(\lambda x', y') = \check{Y}(x') \otimes S'(y')$.

(b) *If there exists a distribution* $S \in \mathcal{D}'(\Omega_0)$ *such that* $\lim_{\lambda \to +0} \lambda T(\lambda x, y) = \check{Y}(x) \otimes S(y)$, *then* $S(y) = 0$.

Let $\overset{p}{T}$, $0 \leqq p \leqq N$, be a p-current on Ω_-. If $\lim_{\lambda \to +0} \overset{p}{T}(\lambda x, y) = \check{Y}(x) \otimes \overset{p}{S}(y)$ exists, then $\overset{p}{S}$ will be called the boundary value of $\overset{p}{T}$ on Ω_0 and we write $\overset{p}{S}(y) = \lim_{x \to -0} \overset{p}{T}(x, y)$. By making use of Lemma 1 we can also show that the statement (a) of the same lemma remains valid with distributions replaced by currents.

Let M be a differentiable manifold of dimension N, and let Ω be an open subset with regular boundary. Owing to the preceding discussions, if we follow the same process as done in [9] for the section of currents, we can in an obvious way define the notion of the boundary value on $\partial\Omega$ of currents defined on Ω. The details are omitted.

Let Ω be an open subset in the z-plane intersecting an open interval I with the real axis. Let $\Omega_+ = \{z = x + iy \in \Omega : y > 0\}$ and $\Omega_- = \{z = x + iy \in \Omega : y < 0\}$. Given an $f \in \mathcal{D}'(I)$, to find out $h_1 \in \mathcal{H}(\Omega_+)$ and $h_2 \in \mathcal{H}(\Omega_-)$ such that f is the difference of the boundary values of h_1 and h_2 is tantamount to solving the differential equation:

$$\frac{\partial g}{\partial \bar{z}} = \frac{i}{2}(f \otimes \delta_y). \tag{1}$$

Let $w = \Phi(z)$ be a holomorphic isomorphism of Ω onto an open subset Ω' in the w-plane. Let I', Ω'_+, Ω'_- be the images of I, Ω_+, Ω_- respectively. Ω'_+

and Ω'_- are open subsets with regular boundary I', so that we can speak of the boundary values of holomorphic functions on Ω'_+ or on Ω'_-. Let H be the injection of I into Ω. If we denote by $\overset{0,1}{H}(f)$ the direct image of $\mathcal{D}'(I)$ into $\overset{0,1}{\mathcal{D}'}(\Omega)$ defined by

$$<\overset{0,1}{H}(f), \phi(z)dz> = <f, \phi(x)dx>, \qquad \phi \in \mathcal{D}(\Omega),$$

then, by a simple calculation, we have $\overset{0,1}{H}(f)=\frac{1}{2i}(f\otimes\delta_y)d\bar{z}$, so that we can write (1) in the form

$$\bar{\partial} g = -\overset{0,1}{H}(f), \tag{2}$$

where $\bar{\partial}$ means the coboundary operator: $\bar{\partial} g=\frac{\partial g}{\partial \bar{z}}d\bar{z}$. By the conformal map Φ the equation (1) is transformed into the same form, where H should be understood as the injection I' into Ω'. The solutions of the equation will give rise to holomorphic representations of a given distribution on I' in the sense described above.

Let Ω be any open subset in the z-plane. We consider a real analytic 1-dimensional closed submanifold Γ of Ω which is oriented. Ω is assumed to have the orientation induced by the z-plane. The conformal map allows us to speak of the positive and the negative sides of Γ. As a result, given a distribution $f \in \mathcal{D}'(\Gamma)$, there exists a holomorphic representation of f, that is, f is represented by the difference of the boundary values of a holomorphic function on $\Omega\backslash\Gamma$ from the positive and the negative sides of Γ if and only if the equation of the form (2) admits a solution, where H is the injection $\Gamma\rightarrow\Omega$ and $\overset{0,1}{H}(f)$ means the direct image of f in the sense just described.

The foregoing discussions lead us naturally to the consideration of the problem in a Riemann surface. Let M be a Riemann surface, that is, a connected complex analytic manifold of complex dimension 1. It is orientable and oriented by its complex analytic structure. Let Γ be a real analytic 1-dimensional closed submanifold which is oriented. The positive and the negative sides of Γ will be defined in an obvious fashion. By reducing the discussions to each coordinate neighbourhood of M, we have, as an immediate consequence of the preceding discussions, the following

PROPOSITION 16. *Let M be a Riemann surface, in which a submanifold Γ with the properties just described is given. A distribution f on Γ is holomorphically representable if and only if the equation*

$$\bar{\partial} g = -\overset{0,1}{H}(f) \tag{3}$$

admits a solution $g \in \mathcal{D}'(M)$, where $\overset{0,1}{H}(f) \in \overset{0,1}{\mathcal{D}'}(M)$ denotes the direct image *induced by the injection $\Gamma \to M$.*

As regards the vector-valued distribution $\vec{f} \in \mathcal{D}'_\Gamma(E)$, the map $\overset{0,1}{H}: \mathcal{D}'(\Gamma) \to \overset{0,1}{\mathcal{D}'}(M)$ being continuous, the tensor product $\overset{0,1}{H} \otimes 1: \mathcal{D}'_\Gamma(E) \to \overset{0,1}{\mathcal{D}'_M}(E)$ has a meaning. Taking into account Theorem 1 together with our preceding discussions we shall obviously reach

PROPOSITION 17. *Let M and Γ be as before. An $\vec{f} \in \mathcal{D}'_\Gamma(E)$ is holomorphically representable if and only if the equation*

$$\bar{\partial} \vec{g} = -(\overset{0,1}{H} \otimes 1)(\vec{f}) \tag{4}$$

admits a solution $\vec{g}$ in $\mathcal{D}'_M(E)$.

Let $\mathfrak{H}\mathcal{H}(M)$ stand for the sheaf of the germs of holomorphic functions in M. An analogous notation prefixed with $\mathfrak{H}$ should be understood in a like sense.

Case (i), where M is open. Then the q-th cohomology group $H^q(M, \mathfrak{H}\mathcal{H}(M)) = 0$ for $q \geqq 1$ since M is a Stein manifold. This together with the exact sheaf sequence

$$0 \to \mathfrak{H}\mathcal{H}(M) \xrightarrow{i} \mathfrak{H}\mathcal{D}'(M) \xrightarrow{\bar{\partial}} \mathfrak{H}\overset{0,1}{\mathcal{D}'}(M) \to 0$$

implies that the sequence

$$0 \to \mathcal{H}(M) \xrightarrow{i} \mathcal{D}'(M) \xrightarrow{\bar{\partial}} \overset{0,1}{\mathcal{D}'}(M) \to 0$$

is exact, so that the map $\mathcal{D}'(M) \to \overset{0,1}{\mathcal{D}'}(M)$ is onto. Consequently any $f \in \mathcal{D}'(\Gamma)$ is holomorphically representable. We note that the map $\bar{\partial}: \mathcal{D}'(M) \to \overset{0,1}{\mathcal{D}'}(M)$ is an epimorphism by a theorem of L. Schwartz [20, p. 604].

Case (ii), where M is compact. The image $\bar{\partial}\mathcal{D}'(M)$ is closed in $\overset{0,1}{\mathcal{D}'}(M)$ [19, p. 88], and therefore the polar of the kernel of the map $\bar{\partial}: \overset{1,0}{\mathcal{D}}(M) \to \overset{1,1}{\mathcal{D}}(M)$ with respect to the scalar product between $\overset{0,1}{\mathcal{D}'}(M)$ and $\overset{1,0}{\mathcal{D}}(M)$, while the kernel consists of the holomorphic forms which generate a g-dimensional linear subspace, g being the genus of M. Let $\phi_1, \phi_2, \cdots, \phi_g$ be the linearly independent holomorphic forms. As a result, the equation (3) admits a solution if and only if $\langle f, H^*(\phi_j) \rangle = 0$, $j = 1, 2, \cdots, g$, where $H^*(\phi_j)$ denotes the reciprocal image of ϕ_j. We note that the map $\bar{\partial}: \overset{1,0}{\mathcal{D}}(M) \to \overset{1,1}{\mathcal{D}}(M)$ is a quasi-monomorphism.

THEOREM 4. *Let E be an arbitrary quasi-complete locally convex Hausdorff topological vector space. Then the equation* (4) *admits a solution for any $\vec{f} \in \mathcal{D}'_\Gamma(E)$ if and only if the following conditions are satisfied according to the cases;*

(a) *when M is open, the space of holomorphic functions $\mathcal{H}(M)$ has a topological supplement in the preimage $\bar{\partial}^{-1}(\overset{0,1}{H}(\mathcal{D}'(\Gamma)))$ which is a closed linear subspace of $\mathcal{D}'(M)$,*

(b) *when M is compact, $\vec{f}$ satisfies the relations*

$$<\vec{f}, H^*(\phi_j)> = 0, \qquad j=1, 2, \cdots, g.$$

PROOF. (a) *Necessity.* It is not difficult to verify that $\overset{0,1}{H}$ is a monomorphism so that $\mathcal{L}$, the image $\overset{0,1}{H}(\mathcal{D}'(\Gamma))$, is a closed linear subspace of $\overset{0,1}{\mathcal{D}'}(M)$. Let $\mathcal{K}$ be the pre-image of $\overset{0,1}{H}(\mathcal{D}'(\Gamma))$ by the map $\bar{\partial}$. Then $\mathcal{K}$ is a closed linear subspace of $\mathcal{D}'(M)$ since the map $\bar{\partial}$ is continuous. Let $\bar{\partial}_\kappa$ denote the map $\mathcal{K} \xrightarrow{\bar{\partial}} \mathcal{L}$. If we let $E=\mathcal{L}'$, the strong dual of $\mathcal{L}$, we can choose an $\vec{f}$ so that $-\overset{0,1}{H}(\vec{f})$ may denote the identical map of $\mathcal{L}$. A solution $\vec{g}$ of (4) for such an $\vec{f}$ denotes a continuous linear map $u: \mathcal{L} \to \mathcal{K}$. Then the equation (4) implies that $\bar{\partial}_\kappa \circ u$ is the identical map of $\mathcal{L}$. Consequently $\mathcal{H}(M)$, the kernel of $\bar{\partial}_\kappa$, has a topological supplement in $\mathcal{K}$.

Sufficiency. Suppose $\mathcal{H}(M)$ has a topological supplement $\mathcal{K}_1$ in $\mathcal{K}$. As noted in the proof of case (i), $\bar{\partial}$ is an epimorphism so that $\bar{\partial}_\kappa$ is also an epimorphism and, in turn, the map $\mathcal{K}_1 \xrightarrow{\bar{\partial}} \mathcal{L}$ has the continuous inverse π. We consider $\mathcal{K}_1(E)$ to be a closed linear subspace of $\mathcal{D}'_M(E)$. Consequently if we put, for any $\vec{f} \in \mathcal{D}'_\Gamma(E)$, $\vec{g}=-(\pi\otimes 1)\overset{0,1}{H}(\vec{f})$, then $\vec{g}$ will satisfy the equation (4).

(b). We shall continue to use the same notations. $\mathcal{H}(M)$ is 1-dimensional and therefore has a topological supplement in $\mathcal{K}$. Then in virtue of case (ii) considered above, the same reasoning as in (a) will show that the statement of (b) holds true.

REMARK 2. In case the right-hand side of the equation (4) is taken arbitrarily from $\overset{0,1}{\mathcal{D}'_M}(E)$, we shall have an analogue of Theorem 4. When M is open, however, we can show that the space of holomorphic functions has no topological supplement in $\mathcal{D}'(M)$. In fact, suppose $\mathcal{H}(M)$ has a topological supplement in $\mathcal{D}'(M)$. There exists then a projection $u: \mathcal{D}'(M) \to \mathcal{H}(M)$. u is also considered a $\overset{1,1}{\mathcal{D}}(M)$-valued holomorphic function $\vec{h}$ on M. Let U be a compact neighbourhood of a point of M. $\operatorname{supp}\vec{h}(p)$, $p \in U$, is contained in a fixed compact set $K \subset M$. If $g \in \overset{0,1}{\mathcal{D}'}(M)$ vanishes on a neighbourhood of K, then $u(g)=0$ since $u(g)$ is holomorphic and vanishes on U. Let Ω be a relatively compact open neighbourhood of K. Choose $\alpha \in \mathcal{D}(\Omega)$ such that $\alpha=1$ on a neighbourhood of K. Let us consider a Banach space $C(\bar{\Omega})$ of continuous functions on $\bar{\Omega}$, and define a continuous map $v: C(\bar{\Omega}) \to \mathcal{H}(M)$ by the relation $v(\phi) \doteqdot u(\alpha\phi)$, $\phi \in C(\bar{\Omega})$. If $g \in \mathcal{H}(M)$, then $g \in C(\bar{\Omega})$ and u maps $(1-\alpha)g$ into 0, which implies $v(g)=g$. Thus v is onto, and therefore an epimorphism. Consequently $\mathscr{H}(M)$ is isomorphic with a Banach space. Since $\mathscr{H}(M)$ is nuclear, so $\mathscr{H}(M)$ must be finite-dimensional, which is a contradiction.

PROPOSITION 18. *Let E be an* (**F**)*-space. Suppose M is open. The equation* (4) *admits a solution for any $\vec{f} \in \mathscr{D}'_\Gamma(E)$.*

PROOF. We have only to show the map $\bar{\partial}: \mathscr{D}'_M \hat{\otimes} E \to \overset{0,1}{\mathscr{D}'_M} \hat{\otimes} E$ is onto. The sheaves $\mathfrak{H}(\mathscr{E}_M \hat{\otimes} E)$ and $\mathfrak{H}(\mathscr{D}'_M \hat{\otimes} E)$ are fine, so from the exact sheaf sequences

$$0 \to \mathfrak{H}(\mathscr{H}_M \hat{\otimes} E) \xrightarrow{i} \mathfrak{H}(\mathscr{E}_M \hat{\otimes} E) \xrightarrow{\bar{\partial}} \mathfrak{H}(\overset{0,1}{\mathscr{E}_M} \hat{\otimes} E) \to 0$$

$$0 \to \mathfrak{H}(\mathscr{H}_M \hat{\otimes} E) \xrightarrow{i} \mathfrak{H}(\mathscr{D}'_M \hat{\otimes} E) \xrightarrow{\bar{\partial}} \mathfrak{H}(\overset{0,1}{\mathscr{D}'_M} \hat{\otimes} E) \to 0$$

we have the exact sequences of cohomology groups [6, p. 34]

$$0 \to \mathscr{H}_M \hat{\otimes} E \xrightarrow{i_*} \mathscr{E}_M \hat{\otimes} E \xrightarrow{\bar{\partial}} \overset{0,1}{\mathscr{E}_M} \hat{\otimes} E \xrightarrow{\delta^0_*} H^1(M, \mathfrak{H}(\mathscr{H}_M \hat{\otimes} E)) \to 0$$

$$0 \to \mathscr{H}_M \hat{\otimes} E \xrightarrow{i_*} \mathscr{D}'_M \hat{\otimes} E \xrightarrow{\bar{\partial}_*} \overset{0,1}{\mathscr{D}'_M} \hat{\otimes} E \xrightarrow{\delta^0_*} H^1(M, \mathfrak{H}(\mathscr{H}_M \hat{\otimes} E)) \to 0.$$

Since E is an (**F**)-space, the map $\bar{\partial}_*: \mathscr{E}_M \hat{\otimes} E \to \overset{0,1}{\mathscr{E}_M} \hat{\otimes} E$ is onto [5, p. 38], whence $H^1(M, \mathfrak{H}(\mathscr{H}_M \hat{\otimes} E)) = 0$. This together with the last exact sequence implies that the map $\bar{\partial}_* = \bar{\partial}: \mathscr{D}'_M \hat{\otimes} E \to \mathscr{D}'_M \hat{\otimes} E$ is onto. Thus the proof is complete.

Hitherto we have considered the holomorphic representations for distributions. However, the same reasoning will be applied to the representations for currents of degree 1. In the beginning of this section we have discussed the boundary values of currents on an open subset with regular boundary in a differentiable manifold. From these considerations, given a current $\gamma \in \overset{1}{\mathscr{D}'}(\Gamma)$, the statement that γ is representable by a holomorphic form on $M \setminus \Gamma$ will have a definite meaning. In the following we shall enumerate the analogues of Propositions 16, 17, 18 and Theorem 4 without proofs.

PROPOSITION 16′. *A current $\gamma \in \overset{1}{\mathscr{D}'}(\Gamma)$ is holomorphically representable if and only if the equation*

$$\bar{\partial}\omega = -\overset{1,1}{H}(\gamma) \tag{5}$$

admits a solution $\omega \in \overset{1,0}{\mathscr{D}'}(M)$, where $\overset{1,1}{H}(\gamma) \in \overset{1,1}{\mathscr{D}'}(M)$ denotes the direct image induced by the injection $\Gamma \to M$.

PROPOSITION 17′. *A $\vec{\gamma} \in \overset{1}{\mathscr{D}'_\Gamma}(E)$ is holomorphically representable if and only if the equation*

$$\bar{\partial}\vec{\omega} = -(\overset{1,1}{H} \otimes 1)(\vec{\gamma}) \tag{6}$$

admits a solution $\vec{\omega}$ *in* $\overset{1,0}{\mathcal{D}}{}'_M(E)$.

THEOREM 4′. *Let E be an arbitrary quasi-complete locally convex Hausdorff topological vector space. Then the equation* (6) *admits a solution for any* $\vec{\gamma} \in \overset{1}{\mathcal{D}}{}'_\Gamma(E)$ *if and only if the following conditions are satisfied according to the cases;*

(a) *when M is open, the space of holomorphic forms has a topological supplement in the preimage* $\bar{\partial}^{-1}(\overset{1,1}{H}(\overset{1}{\mathcal{D}}{}'(\Gamma)))$ *which is a closed linear subspace of* $\overset{1,0}{\mathcal{D}}{}'(M)$.

(b) *when M is compact,* $\vec{\gamma}$ *satisfies the relation*

$$\int \vec{\gamma} = 0.$$

PROPOSITION 18′. *Let E be an* (**F**)*-space. Suppose M is open. The equation* (6) *admits a solution for any* $\vec{\gamma} \in \overset{1}{\mathcal{D}}{}'_\Gamma(E)$.

The rest of this section is devoted to the considerations in the z-place $\boldsymbol{C}$, the simplest example of an open Riemann surface.

Given $l \in \mathcal{E}'(\boldsymbol{C})$, we consider the equation: $\bar{\partial} g = l d\bar{z}$ or $\dfrac{\partial g}{\partial \bar{z}} = l$. If we use a fundamental solution $\dfrac{1}{\pi z}$ for $\dfrac{\partial}{\partial \bar{z}}$, we obtain a solution $g = \dfrac{1}{\pi z} * l$, which can be written in the form:

$$g = -\frac{1}{2\pi i z} dz *_1 l d\bar{z},$$

where the symbol $*_1$ means the $*_1$-convolution considered in R. Shiraishi [23, p. 148]. In fact, by the definition of $*_1$ we have

$$\begin{aligned}
\frac{1}{z} dz *_1 l d\bar{z} &= \left(\frac{1}{z} * l\right) *^{-1}(* dz \wedge * d\bar{z}) \\
&= \left(\frac{1}{z} * l\right) *^{-1}(i d\bar{z} \wedge (-i dz)) \\
&= 2i\left(\frac{1}{z} * l\right) *^{-1}(dx \wedge dy) = -2i\left(\frac{1}{z} * l\right).
\end{aligned}$$

We shall first consider the holomorphic representation of a distribution f on a unit circle Γ. Owing to Proposition 16, we are led to solve the equation (3). $\operatorname{supp} \overset{0,1}{H}(f) \subset \Gamma$, so we see that

$$g = \frac{1}{2\pi i z} dz *_1 \overset{0,1}{H}(f) \in \mathcal{D}'(\boldsymbol{C})$$

is a solution. A precise interpretation can be given to this expression in accordance with the idea of F. Norguet [13, p. 15]. Let u be the map: $\boldsymbol{C}\times\Gamma\ni(z,\theta)\to z+e^{i\theta}\in\boldsymbol{C}$. From the definition of $*_1$-convolution, a simple calculation shows that the expression is the direct image of $\frac{1}{2\pi iz}dz\wedge f(\theta)$ by the map u and written $u\left(\frac{1}{2\pi iz}dz\wedge f(\theta)\right)$ in notation. If we decompose u into v and w:

$$\boldsymbol{C}\times\Gamma\xrightarrow{v}\boldsymbol{C}\times\Gamma\xrightarrow{w}\boldsymbol{C},$$

where v maps (z,θ) into $(z+e^{i\theta},\theta)$ and w is a projection, then we can write

$$\frac{1}{2\pi iz}dz*_1\overset{0,1}{H}(f)=w\circ\left(v\left(\frac{1}{2\pi iz}dz\wedge f(\theta)\right)\right),$$

from which, after a calculation, we can conclude the following

PROPOSITION 19. *Let f be any distribution on a unit circle Γ. If we put*

$$g=\frac{1}{2\pi i}\int_\Gamma\frac{f(\theta)}{\zeta-z}d\zeta,\qquad (\textit{Cauchy integral})$$

where the integral means the partial integration in the sense of L. Schwartz [18, p. 130], *then g satisfies the equation*

$$\bar\partial g=-\overset{0,1}{H}(f),$$

and therefore f is holomorphically representable by the holomorphic function g restricted to $\boldsymbol{C}\setminus\Gamma$.

If we consider any vector-valued distribution $\vec{f}$ on a unit circle Γ, we have an analogue of Proposition 19. Theorem 4 implies then that the space of entire functions has a topological supplement in the space $\bar\partial^{-1}(\overset{0,1}{H}(\mathcal{D}'(\Gamma)))$.

Next we consider the case where Γ is the real axis R. Let $\mathcal{D}'_{L^1}$ be the space of summable distributions on R.

DEFINITION 2. *Let k be a positive integer. A distribution $f\in\mathcal{D}'(R)$ is called to be of class k if $f\in(x-i)^k\mathcal{D}'_{L^1}$. We denote by $\mathcal{K}'_k$ the set of distributions of class k.*

$\mathcal{K}'_k$ is assumed to have the image topology by the map $\mathcal{D}'_{L^1}\ni g\to(x-i)^kg\in\mathcal{K}'_k$. The dual space $(\mathcal{K}'_k)'_c$ with the compact convergence topology is $\frac{1}{(x-i)^k}\mathcal{B}$ with the image topology defined by the map $\mathcal{B}_c\ni\phi\to\frac{\phi}{(x-i)^k}\in\frac{1}{(x-i)^k}\mathcal{B}$. We can show that $\mathcal{K}'_k$ is the strong dual of $(\mathcal{K}'_k)'_c$ so that the space $\mathcal{K}'_k$ is essentially the space $\mathcal{O}'_{-k}$ introduced by H. J. Bremermann [2, p. 54].

In the following, for the sake of simplicity, we shall write $\frac{1}{x}$ for $\mathrm{pv}\frac{1}{x}$.

PROPOSITION 20. *Let $f \in \mathcal{D}'(R)$. Then the following conditions are equivalent:*

(1) *f and $\frac{1}{x}$ are composable.*

(2) *$f \in \mathcal{K}_1'$.*

(3) *$f*\phi \in (x-i)L^1$ for any $\phi \in \mathcal{D}$.*

(4) *$f*\phi \in (x-i)L^1$ for any $\phi \in \mathcal{S}$.*

(5) *f and $\frac{1}{x}$ are $\mathcal{S}'$-composable.*

(6) *$f_x \otimes \delta_y$ and $\frac{1}{z}$ are composable.*

(7) *$f_t \cdot \frac{1}{z-t} \in \mathcal{D}'_{x,y,t}$ (the multiplicative product) is partially summable with respect to t.*

PROOF. (1)$\rightarrow$(2). Since f and $\frac{1}{x}$ are composable, we have

$$f\left(\frac{1}{x}*\psi\right) \in \mathcal{D}'_{L^1} \qquad \text{for any} \quad \psi \in \mathcal{D}.$$

Consequently

$$\frac{f}{x-i}\left((x-i)\left(\frac{1}{x}*\psi\right)\right) \in \mathcal{D}'_{L^1}.$$

If we choose $\psi \in \mathcal{D}$ so that $\psi \geqq 0$, but not identically vanishes, then $\frac{1}{x}*\psi \in \mathcal{B}$ and for sufficiently large $|x|$ there exists a positive constant C with $\left|(x-i)\left(\frac{1}{x}*\psi\right)\right| \geqq C$, and hence $f \in \mathcal{K}_1'$.

(2)$\rightarrow$(4)$\rightarrow$(3). f is written in the form $f=(x-i)g$, $g \in \mathcal{D}'_{L^1}$. Then we have for any $\phi \in \mathcal{S}$

$$\begin{aligned} f*\phi &= (xg)*\phi - ig*\phi \\ &= x(g*\phi) - g*x\phi - ig*\phi \\ &= (x-i)(g*\phi) - g*x\phi, \end{aligned}$$

where $x\phi \in \mathcal{S}$ and $g*\phi$, $g*x\phi \in L^1$, and therefore $f*\phi \in (x-i)L^1$. (4)$\rightarrow$(3) is trivial.

(4)$\rightarrow$(5). For any $\psi \in \mathcal{D}$, $(x-i)\left(\frac{1}{x}*\psi\right) \in \mathcal{B}$, therefore for any $\phi \in \mathcal{S}$ we have

$$(f*\phi)\left(\frac{1}{x}*\psi\right)\in L^1,$$

which shows that f and $\frac{1}{x}$ are $\mathcal{S}'$-composable.

(5)→(1) is trivial.

(3)→(2). Let $K=[-1, 1]$. The linear map $\psi\to h=f*\psi$ of $\mathcal{D}_K$ into the Banach space $(x-i)L^1$ with norm $\|h\|=\int\left|\frac{h}{x-i}\right|dx$ is continuous, and hence it is continuous in the topology of $\mathcal{D}_K$ induced by $\mathcal{D}_K^m$ for some positive integer m. We can find $u\in\mathcal{D}_K^m$ and $\xi\in\mathcal{D}_K$ such that

$$\delta=D_x^{m+2}u+\xi.$$

Then

$$f=f*\delta=D^{m+2}(f*u)+f*\xi.$$

Since $f*u, f*\xi\in(x-i)L^1$ and $D^{m+2}(f*u)\in(x-i)\mathcal{D}'_{L^1}$, consequently we see that $f\in\mathcal{K}'_1$.

Therefore we have shown that the conditions (1) through (5) are equivalent.

Before proving the equivalence of conditions (6) and (7), we note that a distribution $g\in\mathcal{D}'(R_x)$ is summable if and only if $g(x)\otimes\delta_y\in\mathcal{D}'_{x,y}$ is summable. In fact, $g(x)\otimes\delta_y\in(\mathcal{D}'_{L^1})_{x,y}$ means that $(g(x)\otimes\delta_y)*(\phi(x)\otimes\psi(y))\in L^1_{x,y}$ for any $\phi\in\mathcal{D}_x$ and $\psi\in\mathcal{D}_y$, that is, $(g*\phi)\psi(y)\in L^1$, which is equivalent to the condition that $g*\phi\in L^1$ for any $\phi\in\mathcal{D}$, and, in turn, to the condition that $g\in\mathcal{D}'_{L^1}$.

(6)⇄(7). The condition (6) means that

$$(f_x\otimes\delta_y)\left(\frac{1}{z}*\phi\right)\in(\mathcal{D}'_{L^1})_{x,y}\qquad\text{for any}\quad\phi\in\mathcal{D}_{x,y},$$

that is,

$$\left(f_x\iint\frac{\phi(t, s)}{x-(t+is)}dtds\right)\otimes\delta_y\in(\mathcal{D}'_{L^1})_{x,y},$$

which is equivalent from the above remark to saying that

$$f_t\iint\frac{\phi(x, y)}{z-t}dxdy\in\mathcal{D}'_{L^1},$$

which means that $f_t\cdot\frac{1}{z-t}\in\mathcal{D}'_{x,y,t}$ is partially summable with respect to t.

(2)⇄(7). For any $\phi\in\mathcal{D}_{x,y}$, if we put $\zeta(x)=\iint\frac{1}{x-(t+is)}dtds$, then $(x-i)\zeta(x)\in\mathcal{B}$, so we see that (2) implies (7). If we take $\phi\geqq 0$, but $\not\equiv 0$, then for sufficiently large $|x|$ there exists a positive constant C with $|(x-i)\zeta(x)| \geqq C$, so we see that (7) implies (2).

Thus the proof is complete.

Now let $f \in \mathcal{K}_1'$. Then we have

$$\overset{0,1}{H}(f)=\frac{1}{2i}(f\otimes\delta_y)d\bar{z}.$$

Proposition 20 implies that $\overset{0,1}{H}(f)$ and $\frac{1}{z}$ are $\mathcal{S}'$-composable and that $\frac{f(t)}{z-t}$ is partially summable with respect to t. With some modifications, we can follow the process given for the preceding case where Γ is a unit circle. And we obtain the following

PROPOSITION 19′. *Let f be a distribution of class* 1. *If we put*

$$g=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt, \qquad (\textit{Cauchy integral})$$

where the integral means the partial integration in the sense of L. Schwartz [18, p. 130], *then g satisfies the equation*

$$\bar{\partial}g=-\overset{0,1}{H}(f),$$

and therefore f is holomorphically representable by the holomorphic function g restricted to $\boldsymbol{C}\backslash R$.

For our later purpose we show

PROPOSITION 21. *Let $f \in \mathcal{K}_1'$. Then the holomorphic function $h(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt$, $z \in \boldsymbol{C}\backslash R$, has the boundary values h_+ and h_-:*

$$h_+=\frac{f}{2}-\frac{1}{2\pi i x}*f=\delta_+*f,$$

$$h_-=-\frac{f}{2}-\frac{1}{2\pi i x}*f=\delta_-*f,$$

where $\delta_+=-\frac{1}{2\pi i}\lim_{\varepsilon\to+0}\frac{1}{x+i\varepsilon}$ and $\delta_-=-\frac{1}{2\pi i}\lim_{\varepsilon\to+0}\frac{1}{x-i\varepsilon}$, the limits being taken in the distributional sense.

PROOF. We define the Fourier transform $\hat{\phi}=\mathcal{F}(\phi)$, $\phi \in \mathcal{S}$, by $\hat{\phi}(\xi)=\int\phi(x)e^{i\xi x}dx$ and therefore the Fourier transform $\hat{u}$, $u \in \mathcal{S}'$, is given by $\langle \hat{u}, \phi\rangle =\langle u, \hat{\phi}\rangle$, $\phi \in \mathcal{S}$. Since f and $\frac{1}{x}$ are $\mathcal{S}'$-composable, it follows from [21,

p. 233] that the multiplicative product of $\mathcal{F}^{-1}(f)$ and $\mathcal{F}^{-1}(\delta_+)=\dfrac{1}{2\pi}Y(\xi)$ exists, Y being the Heaviside function.

Let $\operatorname{Im} z>0$. We can write for any $\varepsilon>0$

$$h_\varepsilon(x)=h(x+i\varepsilon)=f*\frac{1}{2\pi(\varepsilon-ix)},$$

and therefore

$$\mathcal{F}^{-1}(h_\varepsilon)=\mathcal{F}^{-1}(f)\cdot e^{-\varepsilon\xi}Y(\xi)=e^{-\varepsilon\xi}(\mathcal{F}^{-1}(f)Y).$$

Consequently

$$h_\varepsilon(x)=\mathcal{F}(e^{-\varepsilon\xi}(\mathcal{F}^{-1}(f)Y)).$$

Since, for $\varepsilon\to+0$, $e^{-\varepsilon\xi}(\mathcal{F}^{-1}(f)Y)$ converges to $\mathcal{F}^{-1}(f)Y$ in $\mathscr{S}'$, so that h_ε converges to $h_+=\mathcal{F}(\mathcal{F}^{-1}(f)Y)=\delta_+*f$ in $\mathscr{S}'$ as $\varepsilon\to+0$.

Similarly we can show that h_- exists and equals $f*\delta_-$. Thus the proof is complete.

Every $f\in\mathcal{D}'(R)$ is holomorphically representable by holomorphic functions on $\boldsymbol{C}\backslash R$ since there exists a solution $g\in\mathcal{D}'(\boldsymbol{C})$ of the equation $\dfrac{\partial g}{\partial\bar z}=\dfrac{i}{2}(f\otimes\delta_y)$. But this is not true of E-valued distributions if E is taken to be an arbitrary quasi-complete locally convex Hausdorff topological vector space. Otherwise, by Theorem 4 the space $\mathcal{H}(\boldsymbol{C})$ of entire functions would have a topological supplement $\mathcal{W}$ in $\mathcal{K}=\bar\partial^{-1}(\overset{0,1}{H}(\mathcal{D}'(R)))$, a closed linear subspace of $\mathcal{D}'(\boldsymbol{C})$, which consists of the solutions of the equation $\dfrac{\partial g}{\partial\bar z}=\dfrac{i}{2}(f\otimes\delta_y)$ for arbitrary $f\in\mathcal{D}'(R)$. We note that any solution g is holomorphic on $\boldsymbol{C}\backslash R$. The map $\bar\partial\colon\mathcal{K}\to\mathcal{L}=\overset{0,1}{H}(\mathcal{D}'(R))$ being an epimorphism, we see that the map $\mathcal{D}'(R)\ni f\to w\in\mathcal{W}$ with $\dfrac{\partial w}{\partial\bar z}=\dfrac{i}{2}(f\otimes\delta_y)$ is an isomorphism. Let z_0 be a point in $\boldsymbol{C}_+$. The linear form $w\to w(z_0)$ is a continuous one since on $\mathcal{H}(\boldsymbol{C}_+)$ the compact convergence topology coincides with the topology induced by $\mathcal{D}'(\boldsymbol{C}_+)$. Then we can write with a unique $\phi_{z_0}\in\mathcal{D}(R)$

$$w(z_0)=\langle f,\phi_{z_0}\rangle. \tag{7}$$

Let z_0 run through a compact neighbourhood $U\subset\boldsymbol{C}_+$. Then the set $\{\phi_{z_0}\in\mathcal{D}(R)\colon z_0\in U\}$ is compact and therefore contained in some $\mathcal{D}(I)$, I being a finite open interval. If f vanishes on I, but not on R, then by (7) we have $w(z_0)=0$ for every $z_0\in U$, and therefore $w=0$ on $(\boldsymbol{C}\backslash R)\cup I$, where w is holomorphic. Since $f=\lim\limits_{\varepsilon\to+0}(w(x+i\varepsilon)-w(x-i\varepsilon))$ in the distributional sense, we must have $f=0$, a contradiction.

Let us return to an open Riemann surface M. Γ is a real analytic 1-dimensional closed submanifold which need not be connected. When Γ is not compact, the above reasoning will remain valid with some modifications in proving that $\mathcal{H}(M)$ has no topological supplement in $\mathcal{K}$. The map $\mathcal{D}'(R)\to\mathcal{W}$ considered above should be replaced by an isomorphism $\mathcal{D}'(\Gamma)\to\mathcal{W}$, a topological supplement of $\mathcal{H}(M)$ in $\mathcal{K}$. A point $p_0\in M\backslash\Gamma$ is chosen and a linear map $w\to w(p_0)$ will determine a unique form $\phi_{p_0}\in\overset{1}{\mathcal{D}}(\Gamma)$ such that $w(p_0)=<f,\phi_{p_0}>$, and so on. f should be taken so that the corresponding w vanishes identically on $M\backslash\Gamma$. This will be possible since Γ is not compact. On the other hand, however, if Γ is compact, $\mathcal{H}(M)$ has a topological supplement in $\mathcal{K}$. In fact, if we let E be the strong dual of $\mathcal{L}$, then E will be an (**F**)-space and $\mathcal{L}=E'_c$ because of the fact that $\overset{0,1}{H}(\mathcal{D}'(\Gamma))$ is isomorphic with $\mathcal{D}'(\Gamma)$, of which $\overset{1}{\mathcal{D}}(\Gamma)$ is the strong dual. Now it follows from Proposition 18 and the proof of (a) in Theorem 4 that $\mathcal{H}(M)$ has a topological supplement in $\mathcal{K}$. A similar argument is also applicable to $\overset{0,1}{\mathcal{H}}(M)$. Therefore we have the following

THEOREM 5. *In the statements of Theorem* 4 (*resp. Theorem* 4′), *the condition* (a) *is equivalent to the condition*:

(a′) *When M is an open Riemann surface, Γ is compact.*

6. Slowly increasing holomorphic functions on a unit disc

Let $D=\{z:|z|<1\}$ be an open unit disc and let $\Gamma=\partial D$ be its boundary. This section is primarily devoted to the study of the distributional boundary values of elements of $\mathcal{H}(D)$ or $\mathcal{H}_D(E)$, E being a quasi-complete locally convex Hausdorff topological vector space as before.

Let $\vec{u}\in\boldsymbol{H}_D(E)$ be an E-valued harmonic function on D and put $\vec{u}_r(\theta)=\vec{u}(re^{i\theta})$ for $z=re^{i\theta}$, $0\leqq r<1$. $\vec{u}_r$ denotes an element of $\mathcal{D}'_\Gamma(E)$. $\lim_{r\to 1-0}\vec{u}_r$, if it exists in the distributional sense, is called the boundary value of $\vec{u}$. Owing to Proposition 15 together with a conformal map, the existence of the boundary value of $\vec{u}$ is tantamount to the existence of the boundary value of each $<\vec{u}, \vec{e}'>$, $\vec{e}'\in E'$. It follows from a result of L. Simon [24] that if $\vec{f}\in\mathcal{D}'_\Gamma(E)$ is the boundary value of $\vec{u}$, then we can write

$$\vec{u}(z)=\frac{1}{2\pi}\int_\Gamma\frac{(1-r^2)\vec{f}(t)}{1-2r\cos(\theta-t)+r^2}\,dt,\qquad |z|<1 \tag{1}$$

and, conversely, any $\vec{u}$ defined by (1) is harmonic and has the boundary value $\vec{f}$.

Let $\vec{h}\in\mathcal{H}_D(E)$ be an E-valued holomorphic function on D. $\vec{h}$ is also harmonic on D. From the discussions just before, we see that $\vec{h}$ has the boundary value $\vec{f}$ on Γ if and only if $\vec{h}$ is representable by the Poisson integral in the form (1). Clearly $\vec{h}$ is given by the formula

$$\vec{h}(z)=\frac{1}{2\pi i}\int_\Gamma \frac{\vec{f}(\theta)}{\zeta-z}d\zeta. \tag{2}$$

With necessary modifications of the proof of R. Shiraishi [22, p. 103] we have

LEMMA 2. *Let $\vec{f}\in\mathscr{D}'_\Gamma(E)$. If $\vec{f}$ has the value $\vec{f}(0)$ at $z=1$, then $\vec{u}$ defined by* (1) *has the non-tangential value at $z=1$ which is equal to $\vec{f}(0)$.*

Let Γ be a simple analytic arc and $G\subset \boldsymbol{C}$ a domain which is situated on one side of the arc and whose boundary contains an open arc $\Gamma_0\subset\Gamma$.

PROPOSITION 22. *Let $\vec{h}\in\mathscr{H}_G(E)$. Suppose $\vec{h}$ has the boundary value $\vec{f}\in\mathscr{D}'_\Gamma(E)$ on Γ_0. If $\vec{f}$ has the value $\vec{e}_0\in E$ at $z=z_0\in\Gamma_0$, then $\vec{h}$ has the non-tangential value $\vec{e}_0$ at $z=z_0$.*

PROOF. We may assume, if necessary, by means of a conformal map that G is a domain $\subset D$ and that Γ_0 is an open arc of the unit circle Γ and $z_0=1\in\Gamma_0$. We choose an $\alpha\in\mathscr{D}(\Gamma)$ with support $\subset\subset\Gamma_0$ such that $\alpha=1$ on a neighbourhood of $z=1$. Putting $\vec{g}=\alpha\vec{f}$, we can find $\vec{g}_1\in\mathscr{H}_D(E)$ and $\vec{g}_2\in\mathscr{H}_{\bar{D}^c}(E)$ such that $\vec{g}=(\vec{g}_1)_+-(\vec{g}_2)_-$, where $(\vec{g}_1)_+$ and $(\vec{g}_2)_-$ are the boundary values of $\vec{g}_1$ and $\vec{g}_2$ respectively. The functions $\vec{g}_1(z)-\vec{h}(z)$ and $\vec{g}_2(z)$ have the same boundary value on an open arc $\Gamma_1\subset\Gamma_0$ with $1\in\Gamma_1$. Then by Theorem 3 there exists an analytic continuation $\vec{\Phi}(z)$ of $\vec{g}_1(z)-\vec{h}(z)$ into $\vec{g}_2(z)$ across the arc Γ_1. As a result, $(\vec{g}_1)_+$ has the value $\vec{e}_0+\vec{\Phi}(1)$ at $z=1$. It follows from Lemma 2 that $\vec{g}_1(z)$ has the non-tangential value $\vec{e}_0+\vec{\Phi}(1)$ at $z=1$, and therefore $\vec{h}$ has the non-tangential value $\vec{e}_0$ at $z=1$, which completes the proof.

PROPOSITION 23. *Let $\vec{h}\in\mathscr{H}_G(E)$. Suppose $\vec{h}$ has the distributional boundary value $\vec{f}$ on Γ_0. Let $A\subset\Gamma_0$ be a set of positive measure. If $\vec{f}$ has the value* 0 *at every point of A, then $\vec{h}$ vanishes on G.*

PROOF. By virtue of the preceding proposition, $\vec{h}$ has the non-tangential value 0 at every point of A. Owing to Privalov's theorem [15, p. 212], it follows that $\vec{h}$ vanishes on G. The proof is complete.

DEFINITION 3. *$h(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathscr{H}(D)$ is said to be slowly increasing if $\{a_n\}$ is slowly increasing, that is, there exists a positive integer k such that $\{a_n(1+n)^{-k}\}_{n=0,1,2,\dots}$ is bounded.*

Let $\mathfrak{S}_k$ be the set of slowly increasing $h\in\mathscr{H}(D)$ such that $\sup|a|^n(1+n)^{-k}<\infty$. $\mathfrak{S}_k$ is a Banach space with norm: $\|h\|_k=\sup|a|^n(1+n)^{-k}$. $\mathfrak{S}=\bigcup_{k>0}\mathfrak{S}_k$ is defined as the inductive limit of the Banach spaces $\mathfrak{S}_k$. $\mathfrak{S}$ is a nuclear space since $\mathfrak{S}$ is isomorphic to the space of slowly increasing sequences. It is clear that $\mathfrak{S}$ is invariant under differentiation and multiplication.

By means of a conformal map and from Proposition 9, $h \in \mathscr{H}(D)$ has the boundary value on Γ if and only if there exists a positive integer k such that $(1-r)^k h(z)$ is bounded on D. In fact, if $(1-r)^k h(z)$ is bounded, then Cauchy's inequality will show that $\{a_n\}$ is slowly increasing. The converse will follow from the estimate of $h(z)$.

We shall say that $\vec{h} \in \mathscr{H}_D(E)$ is slowly increasing if, given any continuous semi-norm p, there exists a positive integer k such that $(1-r)^k p(\vec{h}(z))$ is bounded on D, or equivalently, if $\{p(\vec{a}_n)\}$ is slowly increasing, where $\vec{a}_n$ denotes the Taylor coefficients of $\vec{h}$. In virtue of Baire's category theorem, $\vec{h}$ is slowly increasing if and only if $\vec{h}$ is scalarly slowly increasing.

As an immediate consequence of these considerations we have

PROPOSITION 24. *$\vec{h} \in \mathscr{H}_D(E)$ has the boundary value on Γ if and only if $\vec{h}$ is slowly increasing.*

It follows from this proposition that $\mathfrak{S}$ has the ε-property. In fact, let $\vec{g} \in \mathscr{D}'_D(E)$ such that $\vec{g}$ is scalarly a slowly increasing holomorphic function, then $\vec{g} \in \mathscr{H}_D(E)$ since $\mathscr{H}(D)$ has the ε-property and therefore $\vec{g} \in \mathfrak{S}(E)$.

REMARK 3. Let $\varepsilon > 0$ be fixed. If $h \in \mathscr{H}(D)$ is slowly increasing if and only if there exists a positive integer k such that

$$(1-r)^k \int_0^{2\pi} |h(re^{i\theta})|^{\varepsilon} d\theta \leqq M, \qquad 0 \leqq r < 1 \tag{3}$$

for some constant M. It is almost trivial that every $h \in \mathfrak{S}$ satisfies the condition (3). Conversely, suppose $h \in \mathscr{H}(D)$ satisfies the condition (2). $|h(z)|^{\varepsilon}$ being subharmonic on D, the inequality

$$|h(re^{i\theta})|^{\varepsilon} \leqq \frac{1}{2\pi} \, |h(\rho e^{it})|^{\varepsilon} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} dt, \qquad \rho = \frac{1}{2}(1+r)$$

yields

$$|h(re^{i\theta})|^{\varepsilon} \leqq \frac{2^{k+1} M}{\pi (1-r)^{k+1}}.$$

Let m be a positive integer $\geqq \dfrac{1}{\varepsilon}(k+1)$. It follows that $(1-r)^m h(z)$ is bounded on D. The condition (3) implies that the Hardy class H_{ε} is contained in $\mathfrak{S}$. In fact, by definition, $h \in H_{\varepsilon}$ if $\lim\limits_{r \to 1-0} \int_0^{2\pi} |h(re^{i\theta})|^{\varepsilon} d\theta < \infty$.

REMARK 4. Let $h \in \mathfrak{S}$. If $h(z)$ has no non-tangential value at any point of Γ, then by Lemma 2 the boundary value of h on Γ has no value at any point of Γ. We can really construct such an h by using N. A. Davydov's theorem: For any sequence $\{a_n\}$ of non-negative numbers satisfying $\lim_{n\to\infty} a_n = \infty$, $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n}=1$, there exists a subsequence $\{a_{n_k}\}$ such that for any sequence $\{\alpha_{n_k}\}$ of real numbers, the function $f(z)=\sum_{k=1}^{\infty} a_{n_k} e^{i\alpha_{n_k}} z^{n_k[a_{n_k}]}$ is holomorphic on the unit disc D and $|f(z)|$ diverges uniformly on a sequence of concentrated rings to infinity [15, p. 119]. As a result, the Nevanlinna class N does not cover $\mathfrak{S}$. In fact, recall that $h \in \mathcal{H}(D)$ is said to belong to the Nevannlinna class N if $\lim_{n\to 1-0}\int_0^{2\pi} \log^+ |h(re^{i\theta})|\, d\theta<\infty$, and that any $h \in N$ has non-tangential value at almost every point on Γ. On the other hand, $e^{\frac{1}{1-z}} \in N$, but $\notin \mathfrak{S}$. We note that N is not invariant under differentiation.

Let us recall the definition of the multiplicative products of distributions defined on a non-empty open subset $\Omega \subset R^N$. Let S, $T \in \mathcal{D}'(\Omega)$. The multiplicative product $S\cdot T$ in the sense of Hirata-Ogata is defined as the distributional limit of $(S*\rho_n)T$ for any δ-sequence $\{\rho_n\}$, if it exists [21, p. 227]. On the other hand, in our previous paper [8, p. 161] we have considered the multiplication invariant under diffeomorphism which covers multiplication in the sense of Hirata-Ogata. The multiplicative product $S\cdot T$ was defined there as the distributional limit of $(S*\rho_n)T$ for any restricted δ-sequence $\{\rho_n\}$, if it exists [21, p. 95]. We note that these multiplications are of local character.

Let us denote by $\mathfrak{S}_+$ the set of the boundary values h_+ of $h \in \mathfrak{S}$. It is clear from the Cauchy integral formula (2) that if we write $h(z)=\sum_{n=0}^{\infty} a_n z^n$, then $h_+=\sum_{n=0}^{\infty} a_n e^{in\theta}$, where the series converges in the distributional sense.

Proposition 25. *For any f_1, $f_2 \in \mathfrak{S}_+$ the multiplicative product $f_1\cdot f_2$ exists. Moreover, f_1, f_2 are the boundary values of h_1, $h_2 \in \mathfrak{S}$ respectively, then $f_1\cdot f_2$ is the boundary value of $h=h_1h_2$.*

Proof. Let

$$f_1=\sum_{\nu=0}^{\infty} a_\nu e^{i\nu\theta}, \quad f_2=\sum_{\nu=0}^{\infty} b_\nu e^{i\nu\theta}. \tag{4}$$

f_1 and f_2 are considered to be periodic distributions on R with period 2π. We shall show that $f_1\cdot f_2$ exists in the sense of Hirata-Ogata. Let ρ_n be any δ-sequence with $\operatorname{supp}\rho_n \subset [-1, 1]$. Since the series (4) converge in the distributional sense, we can write

$$f_1*\rho_n=\sum_{\nu=0}^{\infty} a_\nu e^{i\nu\theta}\int_0^{2\pi} \rho_n(t) e^{-i\nu t}\,dt$$

$$=2\pi\sum_{\nu=0}^{\infty} a_\nu c_{n,\nu} e^{i\nu\theta},$$

and

$$(f_1 * \rho_n) f_2 = \sum_{\nu=0}^{\infty} 2\pi (a_0 c_{n,0} b_\nu + a_1 c_{n,1} b_{\nu-1} + \cdots + a_\nu c_{n,\nu} b_0) e^{i\nu\theta}$$

$$= \sum_{\nu=0}^{\infty} d_{n,\nu} e^{i\nu\theta}, \tag{5}$$

where $c_{n,\nu}$ is the Fourier coefficient of ρ_n as a periodic function with period 2π. $\{c_{n,\nu}\}$ is a rapidly decreasing sequence such that $|c_{n,\nu}| \leqq \frac{1}{2\pi}$ and $\lim_{\nu\to\infty} c_{n,\nu} = \frac{1}{2\pi}$. We shall estimate the coefficients of (4). If $h_1 \in \mathfrak{S}_k$, $h_2 \in \mathfrak{S}_l$ then we have with constants A, B

$$|a_\nu (1+\nu)^{-k}| \leqq A, \quad |b_\nu (1+\nu)^{-l}| \leqq B.$$

Consequently $|d_{n,\nu}| \leqq AB(1+\nu)^{k+l+1}$ and $d_{n,\nu}$ converges to $a_0 b_\nu + a_1 b_{\nu-1} + \cdots + a_\nu b_0$ as $n \to \infty$. It follows that $\lim_{n\to\infty} (f_1 * \rho_n) f_2$ exists in the distributional sense and

$$f_1 \cdot f_2 = \sum_{\nu=0}^{\infty} (a_0 b_\nu + a_1 b_{\nu-1} + \cdots + a_\nu b_0) e^{i\nu\theta}.$$

The last part of the statement is clear. Thus the proof is complete.

THEOREM 6. *Let G and Γ_0 be the same as in Proposition 22. If h_1, $h_2 \in \mathcal{H}(G)$ have the boundary values f_1, f_2 on Γ_0, then the multiplicative product $f_1 \cdot f_2$ exists.*

PROOF. The multiplicative product being of local character, we may assume, if necessary, by means of a conformal map that G is contained in D and Γ_0 is an open arc of the unit circle Γ. We choose an $\alpha \in \mathcal{D}(\Gamma)$ with support $\subset \subset \Gamma_0$ such that $\alpha = 1$ on an open arc $\Gamma_1 \subset \Gamma_0$. Putting $g_1 = \alpha f_1$, $g_2 = \alpha f_2$, we can find $\xi_1(z)$, $\xi_2(z) \in \mathcal{H}(D)$ and $\eta_1(z)$, $\eta_2(z) \in \mathcal{H}(\bar{D}^c)$ such that

$$g_1 = (\xi_1)_+ - (\eta_1)_-, \quad g_2 = (\xi_2)_+ - (\eta_2)_-.$$

There exist analytic continuations $\Phi_1(z)$ and $\Phi_2(z)$ of $\xi_1(z) - h_1(z)$ and $\xi_2(z) - h_2(z)$ respectively across the arc Γ_1 and we have on Γ_1

$$f_1(\theta) = (\xi_1)_+(\theta) - \Phi_1(e^{i\theta}), \quad f_2(\theta) = (\xi_2)_+(\theta) - \Phi_2(e^{i\theta}).$$

It follows from the preceding proposition that the right-hand sides of the equations have the multiplicative product, and therefore $f_1 \cdot f_2$ exists on Γ_1. Since the multiplicative product is of local character and Γ_1 is chosen so as to contain any given point of Γ_0, we can conclude that $f_1 \cdot f_2$ exists on Γ_0. Thus the proof is complete.

REMARK 5. If we consider the set $\mathfrak{M}$ of distributions on Γ_0 consisting of the boundary values of holomorphic functions from the same side of the arc Γ_0, then $\mathfrak{M}$ forms a linear space and, in virtue of Proposition 25, the multiplication defined on $\mathfrak{M}$ is associative.

REMARK 6. Let $\Gamma=\partial D$ be the unit circle. For any $f \in \mathcal{D}'(\Gamma)$, the Cauchy integral (cf. Proposition 19)

$$g=\frac{1}{2\pi i}\int_\Gamma \frac{f(\theta)}{\zeta-z}d\zeta$$

defines holomorphic functions $h_1 \in \mathscr{H}(D)$ and $h_2 \in \mathscr{H}(\bar{D}^c)$ when we restrict g on D and $\bar{D}^c$ respectively. The Fourier expansion of f:

$$f=\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

yields

$$h_1(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad |z|<1$$

$$h_2(z)=\sum_{n=1}^{\infty} a_{-n} z^{-n}, \qquad |z|>1.$$

Therefore

$$(h_1)_+=\sum_{n=0}^{\infty} a_n e^{in\theta},$$

$$(h_2)_-=\sum_{n=1}^{\infty} a_{-n} e^{-in\theta}.$$

We note that $f \in \mathfrak{S}_+$ if and only if $(h_2)_-=0$. Basing on these considerations we can easily verify the following properties:

(i) If $f \in \mathfrak{S}_+$ is real, the f is a constant.

(ii) If $f \in \mathcal{D}'(\Gamma)$ is real, there exists a unique real $g \in \mathcal{D}'(\Gamma)$ within real constants such that $f+ig \in \mathfrak{S}_+$.

(iii) Let $\mathfrak{S}_-$ be the set of the boundary values of holomorphic functions on $\bar{D}^c$. If $g \in \mathfrak{S}_-$ (resp. $\in \mathfrak{S}_+$) has the multiplicative product with every $f \in \mathfrak{S}_+$ (resp. $\in \mathfrak{S}_-$), then $g \in C^\infty(\Gamma)$.

7. Holomorphic functions on a half-plane

Let $\boldsymbol{C}$ be the complex plane with generic point $z=x+iy$ and let $\boldsymbol{C}_+$, $\boldsymbol{C}_-$ be the upper and the lower half planes respectively. This section will be mainly concerned with the $\mathscr{S}'$-boundary values of holomorphic functions on a half plane. The study of the space $\mathscr{K}'_k$, which was introduced in Section 5, will allow us to make some improvements of the results of E. J. Beltrami-M. R. Wohlers [1, p. 77] related to the Hilbert transform pairs.

Let H^+ be the space of $h(z) \in \mathscr{H}(\boldsymbol{C}_+)$ with the property: for any $y_0>0$ there exist a non-negative integer l and a constant $A(y_0)$ such that

$$|h(x+iy)| \leqq A(y_0)|z|^l$$

on the half plane $y \geqq y_0$. We shall denote by H^- the space of $h(z) \in \mathscr{H}(\boldsymbol{C}_-)$ such that $\overline{h(\bar{z})} \in H^+$.

Consider an $h \in H^+$. If we put $h_\varepsilon(x)=h(x+i\varepsilon)$ for $\varepsilon>0$, $h_\varepsilon \in O_M(R) \subset \mathscr{S}'(R)$. If $\lim_{\varepsilon\to+0} h_\varepsilon$ exists in the space $\mathscr{S}'$, then the limit is said to be the $\mathscr{S}'$-boundary value of h. Similarly for $h \in H^-$. Let Ξ be the space of real numbers dual to the space R. Let $\mathscr{D}'_+$ be the space of distributions on Ξ with supports $\subset [0, \infty)$. Similarly we define $\mathscr{D}'_-$. If $g \in \mathscr{D}'_+$ and $e^{-\varepsilon\xi} g$ is a summable distribution for every $\varepsilon>0$, the Fourier-Laplace transform

$$\mathscr{L}[g](z)=\int_{-\infty}^{\infty} e^{iz\xi} g(\xi)d\xi, \qquad \operatorname{Im} z>0$$

is an element of H^+. H^+ is the set of such Fourier-Laplace transforms. It is well known that $\mathscr{L}[g]$ has an $\mathscr{S}'$-boundary value if and only if $g \in \mathscr{S}'(\Xi)$ and then the $\mathscr{S}'$-boundary value is the Fourier transform of g. The Fourier-Laplace transform $\mathscr{L}[g]$ is defined for $g \in \mathscr{D}'_-$ if $e^{\varepsilon\xi} g \in \mathscr{D}'_{L^1}$ for every $\varepsilon>0$ and the same is true of H^-.

From the proof of Proposition 21, the above considerations on Fourier-Laplace transforms yield the following

PROPOSITION 26. *Let $f \in \mathscr{K}'_1$. The Cauchy integral (cf. Proposition 19')*

$$g=\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt$$

defines $h_1 \in H^+$ and $h_2 \in H^-$ when g is restricted on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. h_1 and h_2 have the $\mathscr{S}'$-boundary values

$$(h_1)_+ = \delta_+ * f,$$

$$(h_2)_- = \delta_- * f,$$

and therefore $f=(h_1)_+$ if and only if $\delta_- * f=0$, that is,

$$\operatorname{Im} f = \frac{1}{\pi x} * \operatorname{Re} f,$$

$$\operatorname{Re} f = -\frac{1}{\pi x} * \operatorname{Im} f.$$

REMARK 8. It is easily verified that $\mathscr{K}'_1 \supset \mathscr{D}'_{L^p} \supset L^p$, $1 \leqq p < \infty$.

The Fourier transform $\hat{u}$ of a function $u \in L^p$, $1<p\leqq 2$, is a function $\in L^{p'}$, $p'=\frac{p}{p-1}$, however, if $p>2$, $\hat{u}$ is not a function in general [7, p. 105]. For $f \in \mathcal{K}_1'$, we can show that

$$\int_0^\tau \mathcal{F}^{-1}(f)(\xi)d\xi = \frac{1}{2\pi}\int f(x)\frac{e^{-ix\tau}-1}{-ix}dx, \tag{1}$$

where integrals are taken in the distributional sense and the left-hand side is a continuous function of τ. In fact, first we shall show that the multiplicative product $\mathcal{F}^{-1}(f)\cdot Y(\xi-\tau)$ exists for every τ in the sense of Hirata-Ogata, which will entail that if we consider any g such that $Dg=\mathcal{F}^{-1}(f)$, the multiplicative product $\delta_\tau\cdot g$ exists in the sense of Hirata-Ogata [21, p. 229], so that g will be continuous [21, p. 229]. As $f \in \mathcal{K}_1'$, f and $e^{ix\tau}\delta_+$ are $\mathcal{S}'$-composable, and therefore $\mathcal{F}^{-1}(f)\cdot\mathcal{F}^{-1}(e^{ix\tau}\delta_+)$ exists in the sense of Hirata-Ogata, while we can write $\mathcal{F}^{-1}(e^{ix\tau}\delta_+)=\frac{1}{2\pi}Y(\xi-\tau)$, which was to be proved. By definition, $g(\tau)-g(0)=\int_0^\tau \mathcal{F}^{-1}(f)(\xi)d\xi$, and therefore $\int_0^\tau \mathcal{F}^{-1}(f)(\xi)d\xi$ is a continuous function of τ. For simplicity, let $\tau>0$. Let $\chi_\tau(\xi)=Y(\xi)-Y(\xi-\tau)$ be the characteristic function of the interval $[0,\tau]$. We can write [9, p. 184]

$$\int_0^\tau \mathcal{F}^{-1}(f)(\xi)d\xi = \int_{-\infty}^\infty \chi_\tau(\xi)\mathcal{F}^{-1}(f)(\xi)d\xi.$$

On the other hand, for any δ-sequence $\{\rho_n\}$ we have $\chi_\tau\cdot\mathcal{F}^{-1}(f)=\lim_{n\to\infty}(\chi_\tau*\rho_n)\mathcal{F}^{-1}(f)$ and therefore

$$\begin{aligned}\int_{-\infty}^\infty \chi_\tau(\xi)\mathcal{F}^{-1}(f)(\xi)d\xi &= \lim_{n\to\infty} \langle \mathcal{F}^{-1}(f),\ \chi_\tau*\rho_n\rangle \\ &= \lim_{n\to\infty}\langle f,\ 2\pi\mathcal{F}^{-1}(\chi_\tau)\mathcal{F}^{-1}(\rho_n)\rangle.\end{aligned}$$

Since $\mathcal{F}^{-1}(\chi_\tau)=\frac{1}{2\pi}\frac{e^{-ix\tau}-1}{-ix} \in \mathcal{O}_{-1}$ and $\mathcal{F}^{-1}(\rho_n) \in \mathcal{B}$ converges to $\frac{1}{2\pi}$ in $\mathcal{B}_c$, so $2\pi\mathcal{F}^{-1}(\chi_\tau)\mathcal{F}^{-1}(\rho_n)$ converges to $\mathcal{F}^{-1}(\chi_\tau)$ in $\mathcal{O}_{-1}$ [2, p. 54]. Then from the above equations we obtain (1.) The continuity follows also from the right-hand side of the relation (1), because if $\tau\to\tau_0$, $\frac{e^{-ix\tau}-1}{-ix}$ converges to $\frac{e^{-ix\tau_0}-1}{-ix}$ in $\mathcal{O}_{-1}$ as easily verified.

The following proposition is the analogue of Proposition 20 and will be proved in a similar way

PROPOSITION 27. *Let $f \in \mathcal{D}'(R)$ and k any positive integer. Then the following conditions are equivalent:*

(1) *f and $D^{k-1}\frac{1}{x}$ are composable.*

(2) $f \in \mathcal{K}'_k$.
(3) $f*\phi \in (x-i)^k L^1$ *for any* $\phi \in \mathcal{D}$.
(4) $f*\phi \in (x-i)^k L^1$ *for any* $\phi \in \mathcal{S}$.
(5) f *and* $D^{k-1}\frac{1}{x}$ *are* $\mathcal{S}'$*-composable.*
(6) $f_x \otimes \delta_y$ *and* $\frac{1}{z^k}$ *are composable.*
(7) $f_t \cdot \frac{1}{(z-t)^k} \in \mathcal{D}'_{x,y,t}$ *(the multiplicative product) is partially summable with respect to* t.

PROPOSITION 28. *Let* $f \in \mathcal{K}'_k$, $k \geqq 1$. *The integral*

$$g = \frac{(k-1)!}{2\pi i} \int_\Gamma \frac{f(t)}{(t-z)^k} dt$$

determines $h_1 \in H^+$ *and* $h_2 \in H^-$ *when* g *is restricted on* $\boldsymbol{C}_+$ *and* $\boldsymbol{C}_-$ *respectively. Then the boundary values* $(h_1)_+$ *and* $(h_2)_-$ *exist:*

$$(h_1)_+ = D^{k-1}\delta_+ * f,$$

$$(h_2)_- = D^{k-1}\delta_- * f$$

and $D^{k-1}f = (h_1)_+ - (h_2)_-$. *Therefore* $(h_1)_+ = D^{k-1}f$ *if and only if* $D^{k-1}\delta_- * f = 0$, *that is,*

$$\operatorname{Im} D^{k-1}f = \frac{1}{\pi}\left(D^{k-1}\frac{1}{x} * \operatorname{Re} f\right),$$

$$\operatorname{Re} D^{k-1}f = -\frac{1}{\pi}\left(D^{k-1}\frac{1}{x} * \operatorname{Im} f\right).$$

This is also equivalent to the condition that f *is a boundary value of an* $h \in H^+$.

PROOF. The two tempered distributions are $\mathcal{S}'$-composable, then their Fourier transforms have the multiplicative product in the sense of Hirata-Ogata [21, p. 233]. From this fact together with the formulas $\mathcal{F}^{-1}(D^{k-1}\delta_+) = \frac{1}{2\pi}(i\xi)^{k-1}Y(\xi)$ and $\mathcal{F}^{-1}\left(\frac{(k-1)!}{(x+i\varepsilon)^k}\right) = (-i)^k \xi^{k-1} e^{-\varepsilon\xi} Y(\xi)$ for $\varepsilon > 0$, we have

$$\mathcal{F}^{-1}\left(\frac{(-1)^k(k-1)!}{2\pi i(x+i\varepsilon)^k} * f\right) = e^{-\varepsilon\xi}((i\xi)^{k-1} Y \cdot \mathcal{F}^{-1}(f))$$

$$= e^{-\varepsilon\xi}(\mathcal{F}^{-1}(D^{k-1}\delta_+ * f)).$$

Therefore $h_1 = \mathcal{L}[\mathcal{F}^{-1}(D^{k-1}\delta_+ * f)]$ belongs to H^+ and has the boundary value $D^{k-1}\delta_+ * f$. The same is true of $h_2 = \mathcal{L}[\mathcal{F}^{-1}(D^{k-1}\delta_- * f)]$.

The rest of the statement in our proposition is clear except the last part. f is a boundary value of an $h \in H^+$ if and only if $\mathcal{F}^{-1}(f) \in \mathcal{D}'_+ \cap \mathcal{S}'$. Let

$(h_1)_+ = D^{k-1}f$. Then it follows that $\xi^{k-1}\mathcal{F}^{-1}(f)$ and therefore $\mathcal{F}^{-1}(f) \in \mathcal{D}'_+ \cap \mathcal{S}'$. Conversely, let f be a boundary value of an $h \in H^+$. $\mathcal{F}^{-1}(f) \in \mathcal{D}'_+ \cap \mathcal{S}'$. Since $f \in \mathcal{K}'_k$, $\mathcal{F}^{-1}(f) \cdot \xi^{k-1}(1-Y)$ exists. If we take a δ-sequence $\{\rho_n\}$ such that $\operatorname{supp} \rho_n \subset (0, \infty)$, then $\mathcal{F}^{-1}(f) \cdot \xi^{k-1}(1-Y) = \lim_{n\to\infty}(\mathcal{F}^{-1}(f) * \rho_n)\xi^{k-1}(1-Y) = 0$. Consequently $D^{k-1}\delta_- * f = 0$, which completes the proof.

Proposition 28 is an improvement of a result of E. J. Beltrami-M. R. Wohlers [1, p. 73], where f is taken from $(x-i)^{k-1}\mathcal{D}'_{L^2}$ properly contained in $\mathcal{K}'_k = (x-i)^k \mathcal{D}'_{L^1}$.

Proposition 29. *Let $f \in \mathcal{K}'_k$. If we take $\tilde{f}$ such that $f = x^{k-1}\tilde{f}$, then $\tilde{f} \in \mathcal{K}'_1$. The integral*

$$g = \frac{z^{k-1}}{2\pi i}\int_{-\infty}^{\infty} \frac{\tilde{f}(t)}{t-z}\,dt$$

determines $h_1 \in H^+$ and $h_2 \in H^-$ when g is restricted on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. Then the boundary values $(h_1)_+$ and $(h_2)_-$ exist:

$$(h_1)_+ = x^{k-1}(\delta_+ * \tilde{f}),$$

$$(h_2)_- = x^{k-1}(\delta_- * \tilde{f})$$

and $f = (h_1)_+ - (h_2)_-$. f is a boundary value of an $h \in H^+$ if and only if $x^{k-1}(\delta_- * \tilde{f})$ is a polynomial. And if this is the case, we can choose $\tilde{f}$ so that $x^{k-1}(\delta_- * \tilde{f}) = 0$.

Proof. Evidently $\tilde{f} \in \mathcal{K}'_1$. It follows from Proposition 26 that $(h_1)_+$ and $(h_2)_-$ exist with the formulas described in our proposition. f is a boundary of an $h \in H^+$ if and only if h_2 is a polynomial in z. This will follow from Proposition 30. h_2 is a polynomial if and only if $(h_2)_-$ is a polynomial in x. Let h_2 be a polynomial, then $\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\tilde{f}(t)}{t-iy}\,dt \to 0$ for $y \to -\infty$. Therefore $h_2(z)$ must be of the form

$$h_2(z) = c_1 z^{k-2} + c_2 z^{k-3} + \cdots + c_{k-1}.$$

If we use the formulas

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\delta^{(j)}(t)}{t-z}\,dt = \frac{(-1)^{j+1} j!}{2\pi i} \frac{1}{z^{j+1}}, \qquad \operatorname{Im} z < 0$$

our last statement will easily follow.

We now turn to an extension theorem of Carleman's type for $\mathcal{S}'$-convergence. To do so, we shall first show the following

Lemma 3. *Let $h \in \mathcal{H}(\boldsymbol{C})$ and $I = [-a, a]$ with $a > 0$. If the function*

$$h*'\phi=\int_{-a}^{a}h(x-x'+iy)\phi(x')dx', \qquad \phi\in\mathcal{D}_I$$

is a polynomial in $z=x+iy$, then $h(z)$ is also a polynomial in z.

PROOF. Let $\mathfrak{P}_n$ be the space of polynomials in z of degree $\leq n$. $\mathfrak{P}_n$ is a finite-dimensional Banach space. $\mathcal{D}_I$ is an (**F**)-space and the map: $\mathcal{D}_I\ni\phi\to h*'\phi\in\mathcal{D}'(\boldsymbol{C})$ is continuous. Then, owing to Theorem A of Grothendieck [16, p. 16], the hypothesis $h*'\phi\in\cup\mathfrak{P}_n$ implies $h*'\phi\in$ some $\mathfrak{P}_n$ for every $\phi\in\mathcal{D}_I$. Take a $\rho\in\mathcal{D}_I$ such that $\rho\geqq 0$ and $\int\rho dx=1$. If we put $\rho_\lambda=\frac{1}{\lambda}\rho\left(\frac{x}{\lambda}\right)$ for $0<\lambda<1$, then $h*'\rho_\lambda\in\mathfrak{P}_n$ converges to h in $\mathcal{D}'(\boldsymbol{C})$, which implies that the polynomials $h*'\rho_\lambda\in\mathfrak{P}_n$ converge in $\mathfrak{P}_n$. Therefore $h(z)$ is a polynomial in z, completing the proof.

PROPOSITION 30. *Let $h_1\in H^+$ and $h_2\in H^-$. If $h_1(x+i\varepsilon)-h_2(x-i\varepsilon)$ tends to 0 in $\mathcal{S}'$ as $\varepsilon\to+0$, then h_1 and h_2 are the restrictions of a polynomial in z. Therefore the functions h_1 and h_2 have the same $\mathcal{S}'$-boundary value.*

PROOF. Since $\lim_{\varepsilon\to+0}(h_1(x+i\varepsilon)-h_2(x-i\varepsilon))=0$ in $\mathcal{S}'$, it follows from Proposition 13 that h_1 and h_2 are analytic continuations of each other. Thus there exists $h\in\mathcal{H}(\boldsymbol{C})$ such that h equals h_1 on $y>0$ and h_2 on $y<0$. Let $F(z)=h*'\phi(z)$, $\phi\in\mathcal{D}(R)$. Then $h*'\phi\in\mathcal{H}(\boldsymbol{C})$. Using the fact that $F(x+i\varepsilon)-F(x-i\varepsilon)$ tends to 0 in $\mathcal{O}_M$ as $\varepsilon\to+0$, we can show that

$$|F(z)-F(\bar{z})|\leqq M(1+|z|)^s \tag{2}$$

for some positive M and s, which may depend on ϕ. $F(z)-\overline{F(\bar{z})}$ is an entire function whose real part coincides with that of $F(z)-F(\bar{z})$. This together with (2) implies that $F(z)-\overline{F(\bar{z})}$ is a polynomial in z. Consequently we have with some constants M' and s'

$$\begin{aligned}|F(z)-\overline{F(z)}|&\leqq|F(z)-\overline{F(\bar{z})}|+|\overline{F(\bar{z})}-\overline{F(z)}|\\&\leqq 2M'(1+|z|)^{s'},\end{aligned}$$

which implies that $F(z)$ is a polynomial. Owing to Lemma 3 we see that the function $h(z)\in\mathcal{H}(\boldsymbol{C})$ is a polynomial. Thus the proof is complete.

PROPOSITION 31. *Let $h_1\in H^+$ and $h_2\in H^-$. If there exists an $f\in\mathcal{S}'$ such that*

$$\lim_{\varepsilon\to+0}(h_1(x+i\varepsilon)-h_2(x-i\varepsilon))=f$$

in $\mathcal{S}'$, then h_1 and h_2 have the $\mathcal{S}'$-boundary values f_1, f_2 respectively and $f=f_1-f_2$.

PROOF. If we write $g=\mathcal{F}^{-1}(f)$ in the form $g=g_1-g_2$, $g_1\in\mathcal{D}'_+\cap\mathcal{S}'$,

$g_2 \in \mathcal{D}'_- \cap \mathcal{S}'$, then $\mathcal{L}[g_1]$ and $\mathcal{L}[g_2]$ belong to H^+, H^- respectively and $\mathcal{L}[g_1]$, $\mathcal{L}[g_2]$ have the $\mathcal{S}'$-boundary values and therefore

$$\lim_{\varepsilon\to+0}((h_1-\mathcal{L}[g_1])(x+i\varepsilon)-(h_2-\mathcal{L}[g_2])(x-i\varepsilon))=0.$$

From this together with Proposition 30, it follows that h_1 and h_2 have the $\mathcal{S}'$-boundary values f_1 and f_2 respectively and $f=f_1-f_2$. Thus the proof is complete.

Definition 5. *We shall denote by $H^+(E)$ the space of $\vec{h} \in \mathcal{H}_{C_+}(E)$ such that $<\vec{h}, \vec{e}'> \in H^+$ for every $\vec{e}' \in E'$, and by $H^-(E)$ the space of $\vec{h} \in \mathcal{H}_{C_-}(E)$ such that $\overline{\vec{h}(\bar{z})} \in H^+(E)$.*

Lemma 4. *Let $\vec{h} \in \mathcal{H}_{C_+}(E)$. If $<\vec{h}, \vec{e}'>$ has the $\mathcal{S}'$-boundary value $<\vec{h}, \vec{e}'>_+ = \lim_{\varepsilon\to+0} <\vec{h}(x+i\varepsilon), \vec{e}'>$ for every $\vec{e}' \in E'$, then there exists a unique $\vec{h}_+ \in \mathcal{S}'(E)$ such that $\vec{h}(x+i\varepsilon)$ tends to $\vec{h}_+$ in $\mathcal{S}'(E)$ as $\varepsilon \to +0$.*

Proof. By Corollary to Theorem 2 the map $E'_c \ni \vec{e}' \to <\vec{h}, \vec{e}'>_+ \in \mathcal{D}'_x$ is continuous and $<\vec{h}, \vec{e}'>_+ \in \mathcal{S}'$. Since the space $\mathcal{S}'$ has an ε-property, there exists a unique $\vec{h}_+ \in \mathcal{S}'(E)$ such that $<\vec{h}_+, \vec{e}'> = <\vec{h}, \vec{e}'>_+$. From the relation

$$<\mathcal{F}^{-1}(\vec{h}_+), \vec{e}'> = \mathcal{F}^{-1}(<\vec{h}_+, \vec{e}'>), \qquad \vec{e}' \in E',$$

we see that $\mathcal{F}^{-1}(\vec{h}_+) \in \mathcal{S}'_+(E)$. $\mathcal{F}(e^{-\varepsilon\xi}\mathcal{F}^{-1}(\vec{h}_+)) = \vec{h}(x+i\varepsilon)$, where $e^{-\varepsilon\xi}\mathcal{F}^{-1}(\vec{h}_+)$ tends to $\mathcal{F}^{-1}(\vec{h}_+)$ in $\mathcal{S}'_+(E)$. Therefore $\vec{h}(x+i\varepsilon)$ tends to $\vec{h}_+$ in $\mathcal{S}'(E)$. Thus the proof is complete.

Theorem 7. *Let $\vec{h}_1 \in H^+(E)$ and $\vec{h}_2 \in H^-(E)$, If for every $\vec{e}' \in E'$ there exists a distribution $f_{\vec{e}'} \in \mathcal{S}'$ such that*

$$\lim_{\varepsilon\to+0} <(\vec{h}_1(x+i\varepsilon)-\vec{h}_2(x-i\varepsilon)), \vec{e}'> = f_{\vec{e}'}$$

in $\mathcal{S}'$, then $\vec{h}_1(x+i\varepsilon)$, $\vec{h}_2(x-i\varepsilon)$ have the limits $\vec{f}_1$, $\vec{f}_2$ in $\mathcal{S}'(E)$ respectively as $\varepsilon \to +0$ and $<\vec{f}_1-\vec{f}_2, \vec{e}'>=f_{\vec{e}'}$.

Proof. By virtue of Proposition 31, $\lim_{\varepsilon\to+0} <\vec{h}_1(x+i\varepsilon), \vec{e}'>$ and $\lim_{\varepsilon\to+0} <\vec{h}_2(x-i\varepsilon), \vec{e}'>$ exist in $\mathcal{S}'$ for any $\vec{e}' \in E'$. It follows therefore from Lemma 4 that there exist $\vec{f}_1, \vec{f}_2 \in \mathcal{S}'(E)$ such that

$$\lim_{\varepsilon\to+0} \vec{h}_1(x+i\varepsilon) = \vec{f}_1, \qquad \lim_{\varepsilon\to+0} \vec{h}_2(x-i\varepsilon) = \vec{f}_2$$

in $\mathcal{S}'(E)$. Clearly $<\vec{f}_1-\vec{f}_2, \vec{e}'>=f_{\vec{e}'}$. Thus the proof is complete.

The space $\mathcal{K}'_k(E)$ will be introduced in an obvious way, and we can show the analogues of Proposition 28, but the details will be omitted.

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