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Notes on Quasi-Valuation Rings

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Let R be a commutative ring with a unit. Then R is said to have a few zero-divisors if the total quotient ring of R has only a finite number of maximal ideals. Non zero-divisors in R are said to be regular elements in R, and an ideal A of R is called a regular ideal if A contains a regular element. Let R have a few zero-divisors and A a regular ideal of R. Then it is known that A is generated by regular elements in $R^{(*)}$. A ring R with a few zero-divisors is called a quasi-valuation ring if for any pair (a, b) of regular elements of R we have either $aR \subseteq bR$ or $bR \subseteq aR$. Let R be a quasi-valuation ring and M the ideal of R generated by all the non-unit regular elements in R. Then M is the unique regular maximal ideal of R unless every regular element is a unit in R.

In this paper we shall prove some properties of intersection of a finite number of quasi-valuation rings. Among others we have the following result:

Let $R_1, R_2, \cdots, R_n$ be quasi-valuation rings with the same total quotient ring K. Then $R = \bigcap_{i=1}^{n} R_i$ has K as the total quotient ring.

Let s be a non-unit regular element in R. If there exists a polynomial $f(X) = 1 + \sum_{i=1}^{m-1} h_i X^i + X^m$ with integer coefficients h_i such that $f(s)$ is a unit in R, we call $f(s)$ a unit associated to s.

Lemma 1. *Let R be a semi-local ring with a unit (not necessary Noetherian) and let a be a non-unit regular element in R. Then there exists a unit $f(a)$ associated to a.*

Proof. Let $M_1, M_2, \cdots, M_n$ be all the maximal ideals of R. Since a is not a unit and R has the identity, there is at least one maximal ideal containing a. We denote by A_1 the set of all the maximal ideals of R containing a. If $1+a$ is a unit, then $f(a)=1+a$ is a unit associated to a. If $x_1=1+a$ is not a unit, then the set A_2 of all the maximal ideals of R which contain $1+a$ is not empty and $A_1 \cap A_2 = \phi$, and moreover, $0 \neq ax_1 \in M_i$ for all $M_i \in A_1 \cup A_2$. If $1+ax_1$ is a unit, then $f(a)=1+ax_1$ is a unit associated to a.

If $x_2=1+ax_1$ is not a unit, then we proceed further as above. Since R is a semi-local ring, there exists a positive integer k such that $1+ax_1x_2\cdots x_k$ $(x_i=1+ax_1x_2\cdots x_{i-1})$ is not contained in any one of maximal ideals M_i $(1 \leq i \leq n)$. Hence $f(a)=1+ax_1x_2\cdots x_k$ is a unit associated to a.

(*) cf. E. D. Davis, *Overrings of commutative rings II*, Trans. Amer. Math. Soc. **110** (1964), 196-212.

LEMMA 2. *Let R_1 and R_2 be two non-trivial quasi-valuation rings with the same total quotient ring and let M_1, M_2 be the regular maximal ideals of R_1 and R_2 respectively. Let r be a non-unit regular element in R_1 such that r^{-1} is a non-unit regular element in R_2. Then there exists a unit associated to r in R_1, and moreover for any unit $f(r)$ associated to r in R_1, we have*

(1) ${}^t f(r^{-1})=1+\sum_{i=1}^{N-1} h_{N-i}r^{-i}+r^{-N}$ *is a unit associated to r^{-1} in R_2, where* $f(X)=1+\sum_{i=1}^{N-1} h_i X^i+X^N$.

(2) $r^N/f(r) \in rR_1 \cap R_2$, $r^{-N}/{}^t f(r^{-1}) \in r^{-N}R_2 \cap R_1$.

(3) $(r^N/f(r))(r^{-N}/{}^t f(r^{-1})) \in r^N R_1 \cap r^{-N}R_2 \subseteq M_1 \cap M_2$.

PROOF. Since R_i is a quasi-valuation ring, R_i is a semi-local ring for $i=1, 2$. By Lemma 1 for r there is a unit $f(r)$ associated to r. Let N be the degree of $f(X)$. Since r^N is regular in the total quotient ring K and $f(r)= r^N\,{}^t f(r^{-1})$, ${}^t f(r^{-1})$ is also regular in K and it is contained in R_2. Moreover, ${}^t f(r^{-1})$ is not contained in M_2, and so ${}^t f(r^{-1})$ is a unit associated to r^{-1} in R_2 since R_2 is a quasi-valuation ring. (2) and (3) obvious.

We denote by $S(R)$ the set of all the regular elements in a ring R and $Q(R)$ the total quotient ring of R.

LEMMA 3. *Let R_1 and R_2 be two non-trivial quasi-valuation rings with the same total quotient ring K. Then we have the following three facts:*

(1) $S(R) \subseteq S(R_i)$ $(i=1, 2)$, *where* $R=\bigcap_{i=1}^{2} R_i$.

(2) $Q(R)=K$.

(3) *For any element r in $S(R_1)$ (resp. $S(R_2)$), $rR_1 \cap M_2$ (resp. $rR_2 \cap M_1$) contains a regular element, where M_1 and M_2 are the regular maximal ideals of R_1 and R_2, respectively.*

PROOF. First we shall show that if x is an element in R_1, then there exists a regular element r in K which is contained in R such that $rx \in R$. In fact, assume that $x \notin R_2$. Then there is a regular element a in M_2 such that $ax \in R_2$. If $a \in R_1$, then $a \in R$ and $ax \in R$. If $a \notin R_1$ i.e. $a^{-1} \in M_1$, take a unit $f(a)$ associated to a in R_2.

Let N be the degree of $f(X)$. Then by Lemma 2 $(a^N/f(a))(a^{-N}/{}^t f(a^{-1}))=r$ is a regular element in K and contained in R, and moreover, $rx=(a^N/f(a))\cdot (a^{-N}/{}^t f(a^{-1}))x=(a^{N-1}/f(a))(a^{-N}/{}^t f(a^{-1}))ax \in R$ by Lemma 2.

Let x be an element of R. We denote by $A_1(x)$ the annihilators of x in R_1 and $A(x)$ the annihilators in R. To prove (1) it is sufficient to see that if $\mathrm{A}_1(x) \neq 0$, then $A(x) \neq 0$. In fact, for any y in $A_1(x)$ there is a regular element r in K which is contained in R such that $ry \in R$ by the above remark. Hence $0 \neq ry \in A(x)$.

Since $S(R) \subseteq S(R_1)$ we have $Q(R) \subseteq K$. Hence to show (2) it suffices to show $R_1 \subseteq Q(R)$. But this is immediate from the preceding remark.

To prove (3) it is sufficient to show when r is not a unit in R_1, $rR_1 \cap M_2$ contains a regular element. If $r \in M_2$, then $r \in rR_1 \cap M_2$. If $r \notin R_2$ i.e. $r^{-1} \in M_2$, then by Lemma 2 there is a unit $f(r)$ associated to r in R_1. Let N be the degree of $f(X)$. Then $(r^N/f(r))(r^{-N}/{}^t f(r^{-N})) \in rR_1 \cap M_2$ by Lemma 2. The case where $r \in R_2 - M_2$. Take a regular element b in M_2. Then if $b \in R_1$, $br \in rR_1 \cap M_2$. If $b \notin R_1$ i.e. $b^{-1} \in M_1$, then by Lemma 2 there is a unit $f(b)$ associated to b in R_2 and $b^N/f(b) \in R_1 \cap M_2$, where N is the degree of $f(X)$. Thus $(b^N/f(b))\, r \in rR_1 \cap M_2$.

THEOREM. *Let $R_1, R_2, \cdots, R_n$ be non-trivial quasi-valuation rings with the same total quotient ring K and let $M_1, \cdots, M_n$ be regular maximal ideals of $R_1, \cdots, R_n$. Let us set $R = \bigcap_{i=1}^{n} R_i$. Then the following statements hold:*

(1) $S(R) \subseteq S(R_i)$

(2) $Q(R) = K$

(3) *Let r be an element in $S(R_i)$. Then $rR_i \cap M_i'$ contains a regular element, where $M_i' = \bigcap_{h \neq i} M_h$.*

PROOF. We shall prove the Theorem by induction on n. The case $n=2$ is proved in Lemma 3. We shall set $R_1' = \bigcap_{j \neq i} R_j$ and $M'_{ij} = \bigcap_{\substack{h \neq i, j \\ i \neq j}} M_h$. Then by induction assumotion we have

(1)′ $S(R_i') \subseteq S(R_j)$ for any pair i, j such that $i \neq j$.

(2)′ $Q(R_i') = K$

(3)′ For any regular element r in R_i, $rR_i \cap M'_{ij}$ contains an element which is regular in K.

We shall first prove:

(0) Let x be an element in R_i. Then there is a regular element c in K contained in R such that $cx \in R$.

Without loss of generalities we assume $x \in R_1$. If $x \in R_1'$ we have nothing to prove. If $x \notin R_1'$, take a regular element a in R_1' such that $ax \in R_1'$. By (2)′ a is also regular in K. Hence by (3)′ there is a regular element a_i in $aR_i \cap M'_{1i} \subseteq R_1'$, for $i \geq 2$. Let us set $b = \prod_{j \geq 2} a_j$. We have $b \in M_1'$. Moreover $bx = (\prod_{j \geq 2} a_j)x \in R$ because $a_i x \in R_i$ and a_j and $a_j \in R_i$ for any $i \geq 2$ and $j \geq 2$ such that $j \neq i$. Now assume that $b \in R_1$. Then $b \in R_1 \cap M_1' \subseteq R$, and b answers the question. If $b \notin R_1$ we have $b^{-1} \in M_1$ since b is regular in K. Take a unit $f(b^{-1})$ associated to b^{-1} in R_1. Then by Lemma 2 $c = b^N/{}^t f(b) \in R_1 \cap bR_i$ for any i, where N is the degree of $f(X)$. Hence $c \in R$. Moreover we have $cx = (bx)r_i \in R_i (r_i \in R_i)$ for any $i \geq 2$. Since $cx \in R_1$ we have $cx \in R$. Thus the assertion (0) is completely proved.

On account of the assertion (0) the proof of (1) and (2) is literally the same as the proof of Lemma 3. (1) and (2), and will be omitted. To prove (3) let r be an element in $S(R_1)$. Without loss of generalities we can assume that $r \in M_1$. By (3′) there is a regular element $a \in rR_1 \cap M'_{12}$. If $a \in M_2$, then $a \in rR_1 \cap M'_1$ and the assertion is proved. If $a \notin R_2$, then $a^{-1} \in M_2$. We shall again use Lemma 2, i.e., take a unit $f(a^{-1})$ associated to a^{-1} in R_2. Then we have $c = (a^{-N}/f(a^{-1}))(a^N/{}^t f(a)) \in aR_1 \cap a^{-1}R_2 \subseteq rR_1 \cap M_2$. Moreover $(a^{-N}/f(a^{-1}))\cdot(a^N/{}^t f(a)) = (1/{}^t f(a))(a^N/{}^t f(a)) \in M'_{12}$. Hence $c \in rR_1 \cap M'_1$. Finally assume that a is a unit in R_2. Take an arbitrary regular element b in M_2, and a regular element c in $bR_2 \cap M'_{12}$. If $c \in R_1$, then ca answers the question. If $c \notin R_1$, i.e., $c^{-1} \in M_1$, take a unit $f(c^{-1})$ associated to c^{-1} in R_1. Then we see as before the element $(c^N/{}^t f(c))$ is contained in $cR_2 \cap R_1$ where N is the degree of $f(X)$. Then $(c^N/{}^t f(c))\, a \in aR_1 \cap cR_2 \subseteq rR_1 \cap bR_2 \subseteq rR_1 \cap M_2$. This element is also contained in M'_{12} hence contained in $rR_1 \cap M'_1$. The proof for other indices $i \geq 2$ is the same.

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