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A Remark on the Homeomorphism Group of a Manifold

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Let M be a connected separable metric space each point of which has a neighborhood (open set in the metric topology on M) whose closure in M is homeomorphic to $C_n(0; 1)=\{x \in R^n \mid d(x, 0)\leqq 1\}$. Such a space M is simply called an *n-manifold*. Let $G(M)$ denote the group of all homeomorphisms of an n-manifold M, and $G^0(M)$ the subgroup of $G(M)$ generated by all h in $G(M)$ such that, for some *internal* closed n-cell F, $h\mid M-F=$ identity. Let $G^I(M)$ denote the subgroup of $G(M)$ generated by all h in $G(M)$ such that, for some closed n-cell F, $h\mid F=$ identity. Let $P(F)$ be the set of all h in $G(M)$ such that, for some f in $G^0(M)$, $h\mid F=f\mid F$, and $Q(F)$ the set of all h in $G(M)$ such that, for some f in $G^0(M)$, $h(F)=f(F)$ and $f^{-1}h\mid B$ is in $G^0(B)$, where F is an *internal* closed n-cell in M and $B=$ Bndy F. Let $T(F)$ be the set of all cells in M tame with respect to F.

It has been proved in [1] that, for an n-manifold $M(n\leqq 3)$, there is an internal closed n-cell F_0 in M such that, for any h in $G(M)$, there is an f in $G^0(M)$ such that $f(F_0)=h(F_0)$ (setwise), and then F_0 is called a *pivot* cell. In connection with such a pivot cell the following theorems have been obtained in [1].

Theorem 11 (Fisher). *Let M be a manifold,* dim $M\leqq 3$, *and let F_0 be a pivot cell in M. For every F in $T(F_0)$, $P(F)$ is a normal subgroup of $G(M)$. For each F in $T(F_0)$, $P(F)=P(F_0)$.*

Theorem 12 (Fisher). *Let M be a manifold,* dim $M\leqq 3$, *and let F_0 be a pivot cell in M. For any F in $T(F_0)$, $P(F)=G^I(M)$. Hence an h in $G(M)$ is in $G^I(M)$, so that $h=h_1\cdots h_k$, h_i the identity inside some closed n-cell F_i in M ($n=$* dim M), *if and only if for any n-cell F in M tame with respect to F_0, there is a deformation f of M such that $f(x)=h(x)$ for every x in F.*

Although the existence of the pivot cell in M is unknown for dim $M\geqq 4$, in this note we shall show that, for any internal closed n-cell F in an n-manifold $M(1\leqq n<\infty)$, the above theorems can be generalized as the following theorem and its corollary.

Theorem. *Let M be an n-manifold, and let F_0 and F be any internal closed n-cells in M. Then $P(F_0)=P(F)$ and $P(F)$ is a normal subgroup of $G(M)$.*

Proof. According to the theorem 1 in [1], there is an f in $G^0(M)$ such that $f(F_0)\subset F$, so $P(F)\subset P(f(F_0))$. For h in $P(f(F_0))$, there is a g in $G^0(M)$ such that $h|f(F_0)=g|f(F_0)$, and then $hf|F_0=gf|F_0$ and $gf\in G^0(M)$, so that $hf\in P(F_0)$. By the technique in the proof of the theorem 11 in [1], which can be followed in the case of an n-manifold, we can prove that $P(F_0)$ is a subgroup of $G(M)$. Evidently f is an element of $P(F_0)$, so that h is an element of $P(F_0)$. This shows $P(f(F_0))\subset P(F_0)$, and then $P(F)\subset P(F_0)$. It follows from the arbitrariness of F_0 and F that $P(F_0)\subset P(F)$, and thus $P(F_0)=P(F)$.

Since $P(F_0)=P(f(F_0))$ for f in $G(M)$, for h in $P(F_0)$ there is a g in $G^0(M)$ such that $h|f(F_0)=g|f(F_0)$, and then $f^{-1}hf|F_0=f^{-1}gf|F_0$. $G^0(M)$ is a normal subgroup of $G(M)$ ([1], p. 198), so that $f^{-1}gf\in G^0(M)$. Therefore $f^{-1}hf\in P(F_0)$. q. e. d.

COROLLARY. *Let M be an n-manifold, and let F be any internal closed n-cell in M. Then $P(F)=G^I(M)$. And an h in $G(M)$ is in $G^I(M)$ if and only if, for each internal closed n-cell F, there is an f in $G^0(M)$ such that $h|F=f|F$.*

PROOF. By using the theorem, the method of proof of the theorem 12 in [1] may be followed for an n-manifold.

Using the method of the proof of the theorem 13 in [1], it holds that, for any internal closed n-cell F in an n-manifold M, $P(F)=Q(F)$.

Reference

[1] G. M. Fisher, *On the grcup of all hcmecmorphisms of a manifcld*, Trans. Amer. Math. Soc., **97** (1960), 193-212.

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