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On the Odd Order Non-Singular Immersions of Real Projective Spaces

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§1. Introduction

On the problem of finding bounds for dimensions of higher order non-singular immersions of an n-dimensional C^∞ manifold M in Euclidean N-space, Feldman (cf. [7, Theorem 6.2]) has obtained the following general result (cf. also Pohl [10, Theorem 2.4]). Suppose p is a positive integer. Set $C_{n+p,p}-1=\nu(n, p)$.[1)]

THEOREM (1.1) (Feldman) *If either $N\leqq\nu(n, p)-n$ or $N\geqq\nu(n, p)+n$, there is a pth order non-singular immersion of M in Euclidean N-space.*

For $p=1$, (1.1) says that if $N\geqq 2n$, there is an immersion of M in Euclidean $2n$-space, which is the classical Whitney's theorem [15].

Suzuki (cf. [13], [14]) has proved several results on higher order non-singular immersions of projective spaces in Euclidean spaces. The following theorem [13, Theorem (1.2)] is obtained by making use of Stiefel-Whitney classes of higher order tangent bundles of real projective n-space RP^n. Integers $s(n, p)$ and $d(n, p)$ are defined by

$$s(n, p)=\max\left\{i\,|\,0<i\leqq n,\binom{C_{n+p,p}+i-1}{i}\not\equiv 0 \pmod 2\right\}$$

$$d(n, p)=\max\left\{i\,|\,0<i\leqq n,\binom{C_{n+p,p}}{i}\not\equiv 0 \pmod 2\right\}$$

THEOREM (1.2) (Suzuki) *If p is odd, and if $-d(n, p)<k<s(n, p)$, RP^n cannot be immersed in $(\nu(n, p)+k)$-space without affine singularities of order p.*

Theorem (1.2) shows the impossibility of improving Feldman's theorem (1.1) in many cases of real projective spaces (cf. [13, p. 270]).

The purpose of this paper is to establish some necessary conditions for the existence of odd order non-singular immersions of RP^n in Euclidean N-space and to give non-existence theorems of the non-singular immersions by studying homotopical properties of the stunted projective spaces. We obtain

1) $C_{n+p,p}=\binom{n+p}{p}=\dfrac{(n+p)(n+p-1)\cdots(n+1)}{p!}$

the following two results which are partial improvements of Suzuki's theorem (1.2). For integers m and n with $0 \leqq m < n$, let $\varphi(n, m)$ be the numbers of integers s such that $m < s \leqq n$ and $s \equiv 0$, 1, 2 or 4 (mod 8). We write simply $\varphi(n)$ instead of $\varphi(n, 0)$. Define an integer φ by

$$\varphi = \varphi(n, m-1) \qquad \text{if} \quad m \not\equiv 0 \pmod 4,$$

$$\varphi = \varphi(n, m) \qquad \text{if} \quad m \equiv 0 \pmod 4.$$

THEOREM (1.3) *Suppose p is odd. Set $m = s(n, p)$. If*

$$C_{n+p,p} + m \equiv 0 \pmod 8 \quad \textit{and} \quad \not\equiv 0 \pmod{2^{\varphi-1}},$$

then RP^n cannot be immersed in $(\nu(n, p)+m)$-space without affine singularities of order p.

THEOREM (1.4) *Suppose p is odd. Set $m = d(n, p)$. If*

$$C_{n+p,p} - m \equiv 0 \pmod 8 \quad \textit{and} \quad \not\equiv 0 \pmod{2^{\varphi-1}},$$

then RP^n cannot be immersed in $(\nu(n, p)-m)$-space without affine singularities of order p.

After some preparations in §2, we give in §3 some necessary conditions for the existence of odd order non-singular immersions of RP^n in Euclidean N-space. In §4 we establish a sufficient condition (Lemma (4.1)) and a necessary condition (Lemma (4.2)) that two stunted projective spaces RP^n/RP^{m-1} and RP^{n+k}/RP^{m-1+k} are mod 2 S-related. We apply the method of Adem-Gitler [2] to the proof of (4.1), and we make use of the Adams operation [1] for the proof of (4.2). Applying the results obtained in §3 and §4 to the problem of odd order non-singular immersions, we have in §5 some non-existence theorems (Theorems (5.5)-(5.8)). In §6 we notice that James' theorem and Sanderson's theorem on the non-existence of immersions of RP^n in Euclidean space (cf. [9], [2], [11]) are also shown.

§ 2. Preliminaries

Let M be a C^∞ differentiable manifold of dimension n and let $T_p(M)$ be the bundle of pth order tangent vectors on M. Note that $T_1(M)$ is the tangent bundle $T(M)$ of M. The dimension of $T_p(M)$ is

$$C_{n,1} + C_{n+1,2} + \cdots + C_{n+p-1,p} = C_{n+p,p} - 1,$$

which we denote by $\nu(n, p)$. Let R^N be Euclidean N-space and $x_1, \cdots, x_N$ be the coordinates of R^N. Define a bundle homomorphism, called the natural kth order dissection on R^N, D_k: $T_{k+1}(R^N) \longrightarrow T_k(R^N)$ $(k \geqq 1)$ by

$$D_k(X_k + \sum a_{i_1\cdots i_{k+1}}(\partial^{k+1}/\partial x_{i_1}\cdots\partial x_{i_{k+1}})) = X_k,$$

where $X_k \in T_k(R^N)$. Set $D_1 D_2 \cdots D_{p-1} = \nabla_p$. We say that a C^∞ differentiable map $f: M \longrightarrow R^N$ is a *pth order non-singular immersion of M in* R^N if the bundle homomorphism $\nabla_p T_p(f): T_p(M) \longrightarrow T(R^N)$ is injective or surjective on each fiber according as $\nu(n, p) \leqq N$ or $\nu(n, p) \geqq N$ respectively, where $T_p(f): T_p(M) \longrightarrow T_p(R^N)$ is the pth order differential of f. Clearly, the first order non-singular immersion is an immersion or a submersion. The following result is known (cf. [7, Proposition 8.4] or [13, Lemma (2.3)]).

LEMMA (2.1) *Suppose that there is a pth order non-singular immersion of an n-manifold M in Euclidean N-space.*

(1) *If $N \geqq \nu(n, p)$, there exists an $(N - \nu(n, p))$-dimensional vector bundle α over M such that*

$$T_p(M) \oplus \alpha = N,$$

where $\oplus$ denotes the Whitney sum and where N means the N-dimensional trivial bundle over M.

(2) *If $N \leqq \nu(n, p)$, there exists a $(\nu(n, p) - N)$-dimensional vector bundle β over M such that*

$$T_p(M) = \beta \oplus N.$$

Let ξ be (the isomorphism class of) the canonical line bundle over real projective n-space RP^n. The pth order tangent bundle $T_p(RP^n)$ of RP^n is given as follows (cf. [13, p. 274]).

LEMMA (2.2) *In $KO(RP^n)$*

$$T_p(RP^n) = \begin{cases} C_{n+p,p}\,\xi - 1 & \text{if } p \text{ is odd,} \\ C_{n+p,p} - 1 & \text{if } p \text{ is even.} \end{cases}$$

Let $w(\alpha)$ denote the total Stiefel-Whitney class of a vector bundle α.

COROLLARY (2.3) *If p is odd,*

$$w(T_p(RP^n)) = (1 + x)^{C_{n+p,p}}$$

where x is the generator of $H^1(RP^n; Z_2) \cong Z_2$.

§ 3. Necessary conditions for the existence of odd order non-singular immersions of RP^n

Let p be an odd integer > 0. In this section we shall give some necessary conditions for the existence of pth order non-singular immersions of real projective n-space RP^n in Euclidean N-space. Let m and n be integers such that $0<m\leqq n$.

THEOREM (3.1) *If there exists a pth order non-singular immersion of RP^n in $(\nu(n, p)+m)$-space, then the following* (a) *and* (b) *hold.*

(a) *The bundle $(a\cdot 2^{\varphi(n)}-C_{n+p,p})\xi$ has $a\cdot 2^{\varphi(n)}-C_{n+p,p}-m$ independent non-zero sections, where a is a sufficiently large integer.*

(b) *The bundle $(C_{n+p,p}+m)\xi$ has $C_{n+p,p}$ independent non-zero sections.*

THEOREM (3.2) *If there exists a pth order non-singular immersion of RP^n in $(\nu(n, p)-m)$-space, then the following* (c) *and* (d) *hold.*

(c) *The bundle $C_{n+p,p}\xi$ has $C_{n+p,p}-m$ independent non-zero sections.*

(d) *The bundle $(a\cdot 2^{\varphi(n)}-C_{n+p,p}+m)\xi$ has $a\cdot 2^{\varphi(n)}-C_{n+p,p}$ independent non-zero sections, where a is a sufficiently large integer.*

PROOF OF (3.1). (a) If there is a pth order non-singular immersion of RP^n in $(\nu(n, p)+m)$-space, there exists an m-dimensional vector bundle α over RP^n such that

$$T_p(RP^n)\oplus\alpha=\nu(n, p)+m=C_{n+p,p}-1+m$$

by (2.1) (1). Since $T_p(RP^n)=C_{n+p,p}\xi-1$ by (2.2), we have $C_{n+p,p}\xi+\alpha=C_{n+p,p}+m$ in $KO(RP^n)$. $\xi-1$ is a generator of $\widetilde{KO}(RP^n)\cong Z_{2^{\varphi(n)}}$ (cf. [1, Theorem (7.4)]), and so $a\cdot 2^{\varphi(n)}(\xi-1)=0$ for any integer a. Therefore we have

$$a\cdot 2^{\varphi(n)}\xi-C_{n+p,p}\xi-\alpha=a\cdot 2^{\varphi(n)}-C_{n+p,p}-m$$

in $KO(RP^n)$. If we choose a such that $a\cdot 2^{\varphi(n)}-C_{n+p,p}>n$, we obtain

$$\alpha\oplus(a\cdot 2^{\varphi(n)}-C_{n+p,p}-m)=(a\cdot 2^{\varphi(n)}-C_{n+p,p})\xi.$$

(b) Under the assumption, there exists an m-dimensional vector bundle α over RP^n such that

$$C_{n+p,p}\xi\oplus\alpha=C_{n+p,p}+m.$$

Tensoring both sides of this equation with ξ, we have

$$C_{n+p,p}\oplus\alpha\otimes\xi=(C_{n+p,p}+m)\xi$$

since $\xi\otimes\xi=1$. Q.E.D.

PROOF OF (3.2). (c) If there is a pth order non-singular immersion of RP^n in $(\nu(n, p)-m)$-space, there exists an m-dimensional vector bundle β over RP^n such that

$$T_p(RP^n)=\beta\oplus(\nu(n, p)-m)=\beta\oplus(C_{n+p,p}-1-m)$$

by (2.1) (2). Since $T_p(RP^n)=C_{n+p,p}\xi-1$ by (2.2), we have

$$C_{n+p,p}\xi=\beta\oplus(C_{n+p,p}-m).$$

(d) Tensoring both sides of the above equation with ξ, we have

$$C_{n+p,p}=\beta\otimes\xi\oplus(C_{n+p,p}-m)\xi.$$

As $a\cdot 2^{\varphi(n)}(\xi-1)=0$ for any integer a, we obtain

$$(a\cdot 2^{\varphi(n)}-C_{n+p,p}+m)\xi=\beta\otimes\xi\oplus(a\cdot 2^{\varphi(n)}-C_{n+p,p})$$

for a sufficiently large integer a. Q.E.D.

REMARK. The above proofs show that the assumption of Theorem (3.2) may be replaced by the statement: *if $T_p(RP^n)$ has $\nu(n, p)-m$ independent non-zero sections.*

§ 4. Mod 2 S-relations of RP^n/RP^{m-1}

Let S^qX denote the q-fold suspension of a space X, where q is a non-negative integer. It is said that two spaces Y and Z are mod 2 *S-related*, if for some non-negative integers r and t there is a map $S^rY\longrightarrow S^tZ$ which induces isomorphisms of all homology groups with Z_2 coefficients. Let n and m be integers with $0<m\leqq n$. The next lemma is a generalization of Proposition 3.3 of Adem-Gitler [2].

LEMMA (4.1) *Suppose $C_{m+k,m}\not\equiv 0$ (mod 2). If the bundle $(m+k)\xi$ has k independent non-zero sections, then the stunted projective spaces RP^n/RP^{m-1} and RP^{n+k}/RP^{m-1+k} are* mod 2 *S-related.*

PROOF. If the bundle $(m+k)\xi$ has k independent non-zero sections, there is an m-dimensional vector bundle α over RP^n such that $(m+k)\xi=\alpha\oplus k$. For a vector bundle λ over a CW-complex M let M^λ denote the Thom complex of λ. By the theorems of Atiyah [3], we have

$$S^k(RP^n)^\alpha\approx(RP^n)^{\alpha\oplus k}=(RP^n)^{(m+k)\xi}\approx RP^{n+m+k}/RP^{m-1+k},$$

where by $X\approx Y$ we mean that there is a natural homeomorphism of a space X onto a space Y. Let

$$h:\ S^k(RP^n)^\alpha\longrightarrow RP^{n+m+k}/RP^{m-1+k}$$

denote the composite homeomorphism. The total Stiefel-Whitney class $w(\alpha)$ of α is given by

$$w(\alpha)=(1+x)^{m+k}=\sum_{i=0}^{n}C_{m+k,i}x^i$$

where x is the generator of $H^1(RP^n; Z_2) \cong Z_2$. Since $C_{m+k,m} \not\equiv 0 \pmod 2$, $w_m(\alpha) \neq 0$. Therefore the homomorphism

$$\cup w_m(\alpha): H^{q-m}(RP^n; Z_2) \longrightarrow H^q(RP^n; Z_2)$$

which sends an element $y \in H^{q-m}(RP^n; Z_2)$ to an element $y \cup w_m(\alpha) \in H^q(RP^n; Z_2)$ is an isomorphism for each q with $m \leqq q \leqq n$. Thus for the inclusion map $j: RP^n \longrightarrow (RP^n)^\alpha$, defined by the zero-section of α, the induced homomorphism

$$j^*: H^q((RP^n)^\alpha; Z_2) \longrightarrow H^q(RP^n; Z_2)$$

is an isomorphism for any q with $m \leqq q \leqq n$. As $(RP^n)^\alpha$ is $(m-1)$-connected, there is a map f such that the following diagram is homotopy-commutative:

$$\begin{tikzcd} RP^n \arrow[rr, "j"] \arrow[dr, "p"'] & & (RP^n)^\alpha \\ & RP^n/RP^{m-1} \arrow[ur, "f"'] & \end{tikzcd}$$

where p is the projection. Then the induced homomorphism

$$f^*: H^q((RP^n)^\alpha; Z_2) \longrightarrow H^q(RP^n/RP^{m-1}; Z_2)$$

is an isomorphism for each q with $0 \leqq q \leqq n$. Let $S^k f$ denote the k-fold suspension of f. It is easy to see that there exists a map g such that the following diagram is homotopy-commutative:

$$\begin{tikzcd} S^k(RP^n/RP^{m-1}) \arrow[r, "S^k f"] \arrow[d, "g"'] & S^k(RP^n)^\alpha \arrow[d, "h"'] \\ RP^{n+k}/RP^{m-1+k} \arrow[r, "i"] & RP^{n+m+k}/RP^{m-1+k} \end{tikzcd}$$

where i is the inclusion. Then the map g induces isomorphisms of all cohomology groups with Z_2 coefficients, and isomorphisms of all homology groups with Z_2 coefficients (cf. [12, Chapter 5]). Q.E.D.

Let $\varphi(n, m)$ denote the number of integers s such that $m < s \leqq n$ and $s \equiv 0, 1, 2$ or $4 \pmod 8$. We write $\varphi(n)$ instead of $\varphi(n, 0)$. Define an integer φ by

$$\varphi = \varphi(n, m-1) \qquad \text{if} \quad m \not\equiv 0 \pmod 4,$$

$$\varphi = \varphi(n, m) \qquad \text{if} \quad m \equiv 0 \pmod 4.$$

LEMMA (4.2) *Let k be an integer such that $k \equiv 0 \pmod 8$. If the stunted projective spaces RP^n/RP^{m-1} and RP^{n+k}/RP^{m-1+k} are* mod 2 *S-related, then* $k \equiv 0 \pmod{2^{\varphi-1}}$.

PROOF. We may assume $k>0$. First, consider the case $m\not\equiv 0$ (mod 4). Then, according to [1, Theorem 7.4],

$$\widetilde{KO}(RP^n/RP^{m-1})\cong Z_{2^\varphi},\ \varphi=\varphi(n,\ m-1).$$

By the assumption, for some integer $r\geqq 0$ there is a map

$$f\colon S^{k+r}(RP^n/RP^{m-1})\longrightarrow S^r(RP^{n+k}/RP^{m-1+k})$$

which induces isomorphisms of all homology groups with Z_2 coefficients. We may choose r such that $r\equiv 0$ (mod 8). The map f induces isomorphisms of all cohomology groups with Z_2 coefficients (cf. [12, Chapter 5]), and so by the arguments using the Atiyah-Hirzebruch spectral sequence (cf. [4, §2] and [1, §6]) we can see that f induces an isomorphism of the $\widetilde{KO}$-groups. Consider the following diagram:

$$\begin{array}{ccc}
\widetilde{KO}(RP^{n+k}/RP^{m-1+k}) & \xrightarrow{I^{r/8}} & \widetilde{KO}(S^r(RP^{n+k}/RP^{m-1+k}))\\
\downarrow{\scriptstyle \Psi^3} & & \downarrow{\scriptstyle \Psi^3}\\
\widetilde{KO}(RP^{n+k}/RP^{m-1+k}) & \xrightarrow{I^{r/8}} & \widetilde{KO}(S^r(RP^{n+k}/RP^{m-1+k}))
\end{array}$$

where each of the vertical maps Ψ^3 is the Adams operation, and where each of the horizontal maps $I^{r/8}$ is $r/8$ fold composition of the isomorphism I defined by the Bott periodicity [5, Theorem 1]. According to [1, Theorem 7.4], the right-hand map Ψ^3 is the identity. Thus, by [1, Corollary 5.3], we have

$$\Psi^3 I^{r/8}=3^{r/2}I^{r/8}\Psi^3=3^{r/2}I^{r/8}.$$

Therefore the right-hand map Ψ^3 is $3^{r/2}$. Similarly

$$\Psi^3\colon \widetilde{KO}(S^{k+r}(RP^n/RP^{m-1}))\longrightarrow\widetilde{KO}(S^{k+r}(RP^n/RP^{m-1}))$$

is $3^{(k+r)/2}$. Since Ψ^3 is natural for maps [1, Theorem 5.1], we have

$$3^{(k+r)/2}f^*=f^*3^{r/2}=3^{r/2}f^*.$$

Thus $(3^{(k+r)/2}-3^{r/2})(\iota)=0$, where ι is a generator of $\widetilde{KO}(S^{k+r}(RP^n/RP^{m-1}))\cong Z_{2^\varphi}$, $\varphi=\varphi(n,\ m-1)$. Hence $3^{k/2}-1\equiv 0$ (mod $2^{\varphi(n,m-1)}$). Then we have $k\equiv 0$ (mod $2^{\varphi(n,m-1)-1}$). In fact, if $k/2=(2N+1)2^l$, where N is an integer and where l is an integer with $l\leqq\varphi(n,\ m-1)-3$, then $3^{k/2}-1\equiv 2^{l+2}$(mod 2^{l+3}) by [1, Lemma 8.1]. This is impossible.

In case $m\equiv 0$ (mod 4), according to [1, Theorem 7.4]

$$\widetilde{KO}(RP^n/RP^{m-1})\cong Z+\widetilde{KO}(RP^n/RP^m)\cong Z+Z_{2^\varphi},\ \varphi=\varphi(n,\ m).$$

By the assumption, for some integer $r\equiv 0$ (mod 8) there is a map

$$f\colon S^{k+r}(RP^n/RP^{m-1}) \longrightarrow S^r(RP^{n+k}/RP^{m-1+k})$$

which induces isomorphisms of all homology groups with Z_2 coefficients. We may take f for a cellular map. It defines the map

$$f_0\colon S^{k+r}(RP^n/RP^m) \longrightarrow S^r(RP^{n+k}/RP^{m+k})$$

which induces an isomorphism of $\widetilde{KO}$-groups. The rest of the proof is similar to the above case, so we omit the details here. Q.E.D.

§ 5. *p*th order non-singular immersions of RP^n

We set $\nu(n, p) = C_{n+p,p} - 1$. Let p be odd >0 and let m and n be integers such that $0 < m \leqq n$. From (3.1) and (4.1) we have the following two results.

THEOREM (5.1) *Assume* $\binom{C_{n+p,p}+m-1}{m} \not\equiv 0 \pmod 2$. *If there is a pth order non-singular immersion of* RP^n *in* $(\nu(n, p)+m)$*-space, then* RP^n/RP^{m-1} *and* RP^{n+t}/RP^{m-1+t} *are* mod 2 *S-related, where* $t = a\cdot 2^{\varphi(n)} - C_{n+p,p} - m$ (*a is a sufficiently large integer*).

PROOF. According to (3.1) (a), the bundle $(a\cdot 2^{\varphi(n)} - C_{n+p,p})\xi = (m+t)\xi$ has t independent non-zero sections. Since

$$C_{m+t,m} = \binom{a\cdot 2^{\varphi(n)} - C_{n+p,p}}{m} \equiv \binom{-C_{n+p,p}}{m} \equiv \binom{C_{n+p,p}+m-1}{m} \not\equiv 0 \pmod 2,$$

we obtain the desired result by (4.1). Q.E.D.

THEOREM (5.2) *Assume* $\binom{C_{n+p,p}+m}{m} \not\equiv 0 \pmod 2$. *If there is a pth order non-singular immersion of* RP^n *in* $(\nu(n, p)+m)$*-space, then* RP^n/RP^{m-1} *and* RP^{n+s}/RP^{m-1+s} *are* mod 2 *S-related, where* $s = C_{n+p,p}$.

PROOF. According to (3.1) (b), the bundle $(C_{n+p,p}+m)\xi = (m+s)\xi$ has s independent non-zero sections. Since $C_{m+s,m} \not\equiv 0 \pmod 2$, we have the desired result by (4.1). Q.E.D.

From (3.2) and (4.1) we obtain the following two results. The proofs are similar to those of (5.1) and (5.2).

THEOREM (5.3) *Assume* $\binom{C_{n+p,p}}{m} \not\equiv 0 \pmod 2$. *If there is a pth order non-singular immersion of* RP^n *in* $(\nu(n, p)-m)$*-space, then* RP^n/RP^{m-1} *and* RP^{n+r}/RP^{m-1+r} *are* mod 2 *S-related, where* $r = C_{n+p,p} - m$.

THEOREM (5.4) *Assume* $\binom{C_{n+p,p}-1}{m} \not\equiv 0 \pmod 2$. *If there is a pth order* *non-singular immersion of* RP^n *in* $(\nu(n, p)-m)$*-space, then* RP^n/RP^{m-1} *and* RP^{n+v}/RP^{m-1+v} *are* mod 2 *S-related, where* $v=a\cdot 2^{\varphi(n)}-C_{n+p,p}$ *(a is a sufficiently large integer).*

These theorems, combined with Lemma (4.2), yield non-existence theorems of odd order non-singular immersions of RP^n in Euclidean spaces. We have the following four theorems from Theorems (5.1)–(5.4) respectively.

THEOREM (5.5) *Suppose*

(i) $\binom{C_{n+p,p}+m-1}{m} \not\equiv 0 \pmod 2$

(ii) $C_{n+p,p}+m \equiv 0 \pmod 8$ *and* $\not\equiv 0 \pmod{2^{\varphi-1}}$,

then RP^n *cannot be immersed in* $(\nu(n, p)+m)$*-space without affine singularities of order* p.

Theorem (1.3) follows from Theorem (5.5) immediately.

THEOREM (5.6) *Suppose*

(i) $\binom{C_{n+p,p}+m}{m} \not\equiv 0 \pmod 2$

(ii) $C_{n+p,p} \equiv 0 \pmod 8$ *and* $\not\equiv 0 \pmod{2^{\varphi-1}}$,

then RP^n *cannot be immersed in* $(\nu(n, p)+m)$*-space without affine singularities of order* p.

THEOREM (5.7) *Suppose*

(i) $\binom{C_{n+p,p}}{m} \not\equiv 0 \pmod 2$

(ii) $C_{n+p,p}-m \equiv 0 \pmod 8$ *and* $\not\equiv 0 \pmod{2^{\varphi-1}}$,

then RP^n *cannot be immersed in* $(\nu(n, p)-m)$*-space without affine singularities of order* p.

Theorem (1.4) follows from Theorem (5.7) immediately.

THEOREM (5.8) *Suppose*

(i) $\binom{C_{n+p,p}-1}{m} \not\equiv 0 \pmod 2$

(ii) $C_{n+p,p} \equiv 0 \pmod 8$ *and* $\not\equiv 0 \pmod{2^{\varphi-1}}$,

then RP^n *cannot be immersed in* $(\nu(n, p)-m)$*-space without affine singularities of order* p.

§ 6. Remarks

In this section we notice that non-immersion theorems of James and Sanderson (cf. [9], [11]) follow also from (5.2) and (5.5).

It is said that the stunted projective space RP^n/RP^{m-1} is *S-reducible*, if for a sufficiently large integer t, the t-fold suspension of a generator of $H_n(RP^n/RP^{m-1}; Z_2)$ coincides with the image of the fundamental class of $H_{n+t}(S^{n+t}; Z_2)$ by the homomorphism $H_{n+t}(S^{n+t}; Z_2)\longrightarrow H_{n+t}(S^t(RP^n/RP^{m-1}); Z_2)$, which is induced by the natural map $S^{n+t}\longrightarrow S^t(RP^n/RP^{m-1})$. According to [8] and [1], RP^n/RP^{m-1} is S-reducible if and only if $n+1\equiv 0 \pmod{2^{\varphi(n-m)}}$ (cf. [9, (3.1)]).

Set $n+1=(2b+1)2^{c+4d}$, where b, c and d are integers and $0\leqq c\leqq 3$. Define

$$j(n)=2^c+8d.$$

THEOREM (6.1) *Let p be an odd integer >0 and r be an integer >3 such that $2^r>p-1$. If $n=2^r-1$, RP^n cannot be immersed in $(\nu(n, p)+n-j(n))$-space without affine singularities of order p.*

PROOF. Note that

$$C_{n+p,p}=\binom{2^r-1+p}{p}=\binom{2^r+p-1}{p-1}\frac{2^r}{p}$$

Since p is odd and $2^r>p-1$, we have $C_{n+p,p}=N\cdot 2^r$ for some odd integer $N>0$. Set $m=n-j(n)$. Then $0<m<2^r$ as $r>3$, and we get

$$\binom{C_{n+p,p}+m}{m}=\binom{N\cdot 2^r+2^r-1-j(n)}{2^r-1-j(n)}\not\equiv 0 \pmod 2.$$

If there is a pth order non-singular immersion of RP^n in $(\nu(n, p)+m)$-space, RP^n/RP^{m-1} and RP^{n+s}/RP^{m-1+s} are mod 2 S-related by Theorem (5.2), where $s=C_{n+p,p}$. Thus these two stunted projective spaces are both S-reducible or not S-reducible (cf. [9, Lemma (2.1)]). But by the above remark we see that RP^{n+s}/RP^{m-1+s} is S-reducible, while RP^n/RP^{m-1} is not S-reducible. This is a contradiction. Q.E.D.

For $p=1$, Theorem (6.1) says that if $n=2^r-1$, RP^n cannot be immersed in $(2n-q)$-space, where

$$q=2r \qquad \text{if} \quad r\equiv 1, 2 \pmod 4,$$

$$q=2r+1 \qquad \text{if} \quad r\equiv 0 \pmod 4,$$

$$q=2r+2 \qquad \text{if} \quad r\equiv 3 \pmod 4,$$

which is just Theorem (1.1) of [9]. The method of the above proof is due to Adem and Gitler (cf. [2, Theorem 3.4]) who have given a simple proof of James' theorem. Next, we shall give another proof of Theorem (1.1) of [11]. James and Sanderson obtained their results by making use of axial maps.

THEOREM (6.2) (Sanderson) *Let r be an integer >2. RP^n cannot be immersed in $(2^{r+1}-1)$-space, where*

$$n=2^r+r+2 \qquad \text{if} \quad r\not\equiv 1 \pmod 4,$$

$$n=2^r+r+3 \qquad \text{if} \quad r\equiv 1 \pmod 4.$$

PROOF. In Theorem (5.5) we put $p=1$ and $n+m=2^{r+1}-1\,(r>2)$. If $r\not\equiv 1 \pmod 4$, then $m=2^r-r-3>0$, and hence $C_{n+m,m}\not\equiv 0 \pmod 2$. It is easy to see that $\varphi-1=r+2$. Thus we have $n+m+1=2^{r+1}\not\equiv 0 \pmod{2^{\varphi-1}}$, and so we get the desired result by (5.5). In case $r\equiv 1 \pmod 4$, the proof is similar to the above case. Q.E.D.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math., **75** (1962), 603-632.

[2] J. Adem and S. Gitler, *Non-immersion theorems for real projective spaces*, Bol. Soc. Mat. Mexicana, **9** (1964), 37-50.

[3] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc., (3), **11** (1961), 291-310.

[4] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. of Symposia in Pure Mathematics **3**, Differential Geometry, Amer. Math. Soc., 1961, 7-38.

[5] R. Bott, *Quelques remarques sur les théorèmes de periodicité*, Bull. Soc. Math. de France, **87** (1959), 293-310.

[6] E. A. Feldman, *The geometry of immersions*. I, II, Bull. Amer. Math. Soc., I, **69** (1963), 693-698; II, **70** (1964), 600-607.

[7] E. A. Feldman, *The geometry of immersions*. I, II, Trans. Amer. Math. Soc., I, **120** (1965), 185-224; II, **125** (1966), 181-215.

[8] I. M. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc., (3) **9** (1959), 115-140.

[9] I. M. James, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc., **69** (1963), 331-338.

[10] W. F. Pohl, *Differential geometry of higher order*, Topology **1** (1962), 169-211.

[11] B. J. Sanderson, *A non-immersion theorem for real projective space*, Topology **2** (1963), 209-211.

[12] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.

[13] H. Suzuki, *Bounds for dimensions of odd order non-singular immersions of RP^n*, Trans. Amer. Math. Soc., **121** (1966), 269-275.

[14] H. Suzuki, *Characteristic classes of some higher order tangent bundles of complex projective spaces*, J. Math. Soc. Japan, **18** (1966), 386-393.

[15] H. Whitney, *Differentiable manifolds*, Ann. of Math., **37** (1936), 645-680.

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