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On Certain Classes of Algebras-II

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The present note owes its origin to some remarks by Professor K. McCrimmon on an earlier paper with the same title [5]. The main object of this note is to show that the restriction on the characteristic of the base field can be dispensed with in Theorem 3.2 (and therefore also in Theorem 3.3) of [5] and that it can be weakened considerably in Proposition 3.4 of [5]. In other words, we prove

A. (i) *An alternative algebra over a field F of arbitrary characteristic is an (A_2')-algebra iff it is a direct sum of a zero ideal, and ideals which are (alternative) division algebras over F.*

(ii) *For an alternative algebra over F, all the properties stated in* [5, Definition 1.1] *are mutually equivalent.*

B. *A Jordan algebra over a field K of characteristic $\neq 2$ is an (A_k)-algebra for $k \geqslant 3$, iff it is either a zero algebra or is the direct sum of its annihilator ideal and the semisimple ideal A^2 such that there exists no nonzero element x in A^2 with $R_x{}^k=0$, for the right multiplication R_x in A.*

Other results proved in this note are in the nature of some further remarks on the classes (A_k) of algebras supplementing those in [5].

The notations of this note are those of [5], and we consider only vector spaces which are finite dimensional over their base fields.

1. The following two lemmas which lead to results A, B are essentially based on an idea suggested by Professor McCrimmon.

Lemma 1.1. *Let A be a power-associative algebra* (see [7, Chapter V] for definition etc.) *over a field F and N be any nilideal of A. Then, for any idempotent $\bar{x}=x+N$ of A/N, there exists an idempotent e in A such that $\bar{e}=\bar{x}$, where $x \rightarrow \bar{x}$ is the canonical homomorphism of A onto A/N.*

Proof. By power-associativity of A/N, since $\bar{x}$ is an idempotent of A/N, $\bar{x}^n=\bar{x}$ for any integer n. Consequently x cannot be nilpotent in A; the associative subalgebra $F[x]$ of A generated by x is nonnil; $F[x]$ contains an idempotent e [7, Proposition 3.3]. We have $e=\sum_{i=1}^{n} a_i x^i$ for a_i in F. $\bar{e}=(\sum a_i)\bar{x}=b\bar{x}$ (say) is an idempotent in A/N where b is a nonzero element of F (since e cannot belong to N). The relations $\bar{e}^2=\bar{e}$, $\bar{x}^2=\bar{x}$ immediately yield $b=1$ and the lemma is proved.

LEMMA 1.2. *Let A be a flexible strictly power-associative algebra over a field of characteristic $\neq 2$ (of arbitrary characteristic in the special case of an alternative algebra), and I be the annihilator ideal of A. If A/I contains an identity 1, then I is a direct summand of A. Further, any ideal B of A with $BA=AB=0$ and with A/B possessing an identity, can only be the annihilator ideal I.*

PROOF. First, suppose A is a flexible strictly power-associative algebra over a field of characteristic $\neq 2$. Then, by Lemma 1.1 there exists an idempotent e of A such that $\bar{e}=e+I$ is the identity element 1 of A/I. Let $A=A_e(2)+A_e(1)+A_e(0)$ be the Peirce decomposition of A relative to e (see [4]). For x in $A_e(0)$, $xe=ex=0$, $\bar{x}\bar{e}=\bar{x}=0$; $x \in I$, the annihilator ideal of A. For y in $A_e(1)$, $ye+ey=y$, $\bar{y}\bar{e}+\bar{e}\bar{y}=\bar{y}$; $2\bar{y}=\bar{y}$, $\bar{y}=0$ and $y \in I$. Thus $A_e(1)+A_e(0) \subseteq I$. For $z \in I$, $ze=ez=0$; $z \in A_e(0)$. Hence we have $I=A_e(0)$; $A_e(1)=0$. $A_e(2)$ and $A_e(0)$ being orthogonal subalgebras of A [4] and $I=A_e(0)$ being the annihilator ideal of A, $A=I \oplus A_e(2) \equiv I \oplus B$ as ideals; I is a direct summand of A.

Secondly, if A is an alternative algebra (with arbitrary characteristic for the base field), the above arguments are easily modified by using the two-sided Peirce decomposition of A relative to e (see [7, (3.14)]).

For the ideal B with $BA=AB=0$, let e' be the idempotent of A (see Lemma 1.1) such that $\tilde{e}'=e'+B$ is the identity of A/B. $B \subseteq I$; if, however, $B \subsetneqq I$, there exists an element x in I such that $x \notin B$. Then $xe'=0$, $\tilde{x}\tilde{e}'=\tilde{x}=0$, i.e. $x \in B$, a contradiction. Thus B can only be equal to I.

PROOFS of A, B. Let A be an alternative (A_2')-algebra over a field of arbitrary characteristic. The radical R of A is then precisely the annihilator ideal I of A (by [5, Lemma 2.12]). Being semisimple, A/I contains an identity element 1 [7, Theorem 3.10], when $A \neq I$. By Lemma 1.2, $A=I \oplus B$, where B is an ideal of A isomorphic to A/I; B is semisimple. By an earlier result [5, Proposition 1.4], I and B are (A_2')-algebras. An appeal to [5, Theorem 3.1] completes the proof of A.

The proof of B is quite similar and is omitted.

2. As observed by Professor McCrimmon, the statement in italics in Remark (i), p. 231 of [5] has now no significance.

The simple direct proof of a part of A(ii) for the special case of semisimple associative algebras given in [5, p. 231, Remark (i)] is applicable even for semisimple alternative algebras; in fact, one can also show using this argument that the property right (A_2) is equivalent to (A_2) for such algebras.

Professor McCrimmon suggests the following simplification in the proof of Proposition 2.8 of [5]. After obtaining $B^{(n)}A=0=AB^{(n)}$, we observe that (i) if $n=1$, $B \subseteq I$, (ii) if $n>1$, we have a contradiction at the stage $B^{(n)}=0$. The last three sentences of the original proof can therefore be deleted.

For a commutative power-associative (A)-algebra over a field of characteristic $\neq 2$, the radical is the annihilator ideal (see [5, p. 230, end of Section 2]). When the algebra is also strictly power-associative, it can be realised as the direct sum of a zero ideal and a semisimple (A)-algebra of the same kind. Thus the determination of (A)-algebras belonging to this class boils down to that of simple (A)-algebras of the same class.

We note that if A is an algebra such that the set $\{R_x, L_x\}_{x \in A}$ does not contain any nonzero nilpotent element, then A is an (A_∞)-algebra and conversely (cf. [8, Proposition 7] and [6, p. 235, Remark]).

3. For a right alternative algebra (in which the identity $R_x{}^2 = R_{x^2}$ holds for all elements; see [1]) and for right properties instead of two-sided properties one can easily prove analogues of some of the results in §§ 1, 2 of [5]. Of these analogues we prove the following, which immediately suggests a direct proof of Lemma 2.12 of [5].

Proposition 3.1. *The radical (maximal nilideal) N of a right alternative right (A_2)-algebra A over a field of characteristic $\neq 2$ is precisely its right annihilator ideal $I_R \equiv \{x \in A \mid Ax = 0\}$.*

Proof. We note first that $R_x{}^k = R_{x^k}$ for any integer k and for all x in A (see [1]). This fact can be used as in the proof of Proposition 1.2 (i) of [5] to deduce an analogue of the same for right properties in A. Thus A is a right (A_k)-algebra for every integer k. Let now $x^n = 0$ for an x in N. Then $R_x{}^n = R_{x^n} = 0$; by right (A_n)-property of A, $R_x = 0$, i.e. $N \subseteq I_R$. On the other hand, $I_R \subseteq N$, the former being a nilideal of A. (That it is an ideal, follows from the identity $z(xy + yx) = (zx)y + (zy)x$ for A [1, p. 319, identity (3)].)

The following result comprising of analogues of Theorems 3.1, 3.3 of [5] is easily proved (either by modifying the proof of [5, Theorem 3.1] or by appealing to the remarks in para 2 of Section 2).

Proposition 3.2. *For a semisimple alternative algebra, the properties right $(A_k)_{k=2,\dots,\infty}$, right (A), right $(B_k)_{k=2,\dots,\infty}$, left $(A_k)_{k=2,\dots,\infty}$, left (A), left $(B_k)_{k=2,\dots,\infty}$ (besides the corresponding two-sided properties) are all mutually equivalent. Any such algebra is a direct sum of ideals which are alternative division algebras over the base field.*

However, we cannot extend Proposition 3.2 further, to get an analogue for right properties of the result A stated in the beginning of the note. In fact, A(i) is not true, as the following example of a right (A)-algebra (which is not even an (A_2')-algebra) shows: The (associative) algebra $\bar{A}$ with basis e, b over a field F, multiplication being defined by $e^2 = e$, $be = b$, $eb = b^2 = 0$. $\bar{A}$ is easily verified to be a right (A)-algebra. Since $L_b^2 = R_b^2 = 0$ and $L_b \neq 0$, $\bar{A}$ is not even an (A_2')-algebra, nor also a left (A_2)-algebra. The radical R of $\bar{A}$ is the ideal $\{b\}$, which is not the annihilator ideal of $\bar{A}$. Evidently, there cannot exist an ideal in $\bar{A}$ complementary to R.

REMARK. An alternative algebra is an (A_2)-algebra iff it is a right (A_2)-algebra and the right annihilator ideal coincides with the annihilator ideal.

4. Finally, we give parallel definitions of properties $(A_k)_{k=2,\dots,\infty}$ and (A) for Lie triple systems and indicate how these properties can be similarly studied.

DEFINITION. Let T be a Lie triple system (see [3] for definition etc.) with the trilinear composition $[x, y, z]$; $z \to [x, y, z] \equiv zD(x,y)$ defines a derivation in T. Then T is said to be an $(A)-((A_k)_{k\geqslant 2}-)$ system if $zD(x, y)^2 = 0 \Rightarrow zD(x, y) = 0$ for x, y, z in A (if $D(x, y)^k = 0 \Rightarrow D(x, y) = 0$).

As for algebras, the following implications are true:

$$(A) \Rightarrow (A_\infty) \Rightarrow \cdots \Rightarrow (A_k) \Rightarrow \cdots \Rightarrow (A_3) \Rightarrow (A_2).$$

This definition is consistent with that for a Lie algebra. For, if A is a Lie algebra with respect to the multiplication $[x, y]$ and T_A is the Lie triple system associated to A, defined with respect to the composition $[x, y, z] = [[x, y], z]$, T_A is an $(A)-((A_k)-)$ system whenever A is an $(A)-((A_k)-)$ algebra. However, a Lie algebra A which is an (A_k)- or (A)-system as a Lie triple system need not be so as an algebra, e.g.: The Lie algebra A with basis a, b, c and multiplication defined by $[a, b] = -[b, a] = c$, the rest of the products being zero is not even an (A_2)-algebra, but T_A being a zero system is trivially an (A)-system. In view of the above observation, the Lie triple system T_A associated to the Lie algebra A considered by Jôichi (see [2, p. 29]) is an (A_2)-system; it is not, however, an (A_3)-system.

We recall that the center C of a Lie triple system T is the ideal $\{x \in T \mid [xTT] = 0\}$. Evidently, $[TCT] = 0 = [TTC]$ and C is a solvable ideal of T.

PROPOSITION 4.1. *The radical (maximal solvable ideal) of an (A_3)-Lie triple system T coincides with its center C. More generally, a solvable ideal B of T is contained in C.*

PROOF. Let $B^{(1)} = B$, $B^{(2)} = [T, B, B], \dots, B^{(k+1)} = [T, B^{(k)}, B^{(k)}], \dots$ be the solvability series of the ideal B. $B^{(k)}$ are ideals of T. Let $B^{(k)} \neq 0$, $B^{(k+1)} = 0$. If $k = 1$, then $[T, B, B] = 0$, so that $[B, T, B] = 0$; since B is an ideal of T, $[B, T, [B, T, T]] = 0$; by (A_3)-property, $[B, T, T] = 0$, $B \subseteq C$. If $k > 1$, we can arrive at a contradiction (as in the proof of [5, Proposition 2.8]; see also Section 2 supra), proving the proposition.

The above proposition is not true for (A_2)-systems. (The example given in [2, p. 29], and cited in the remarks preceding Proposition 4.1 suffices.)

The following result can be proved using the radical splitting theorem for Lie triple systems [3, Theorem 2.21] in the same way as its analogue [6, Theorem 4] for Malcev algebras is proved.

PROPOSITION 4.2. *A Lie triple system T over a field of characteristic zero is an (A_k)-system for $k \geqslant 3$ iff either T is trivial or T is reductive (i.e. T is a direct sum of the center C and the semisimple ideal $[T, T, T]$) and is such that $[T, T, T]$ contains no distinct elements x, y with $D(x, y)^k = 0$.*

We note that if T is a Lie triple system over a field of characteristic zero and $\{D(x, y)\}_{x,y\in T}$ does not contain any nilpotent elements, then T is an (A_∞)-system and it is reductive.

References

[1] A. A. Albert, *On right alternative algebras*, Ann. of Math. **50** (1949), 318-328.

[2] A. Jôichi, *On certain properties of Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I, **31** (1967), 25-33.

[3] W. G. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. **72** (1952), 217-242.

[4] R. H. Oehmke, *On flexible algebras*, Ann. of Math. **68** (1958), 221-230.

[5] T. S. Ravisankar, *On certain classes of algebras*, J. Sci. Hiroshima Univ. Ser. A-I, **32** (1968), 225-232.

[6] T. S. Ravisankar, *On certain classes of Malcev algebras*, ibid., **32** (1968), 233-236.

[7] R. D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York (1967).

[8] M. Sugiura, *On a certain property of Lie algebras*, Sci. Pap. Coll. Gen. Edu. Univ. Tokyo, **5** (1955), 1-12.

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