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Functional Calculus in Locally Convex Algebras

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Introduction

L. Waelbroeck [16] and G.R. Allan [1] have shown that the contour integral technique is available in the case of locally convex algebras. Successively C.R. Ionescu-Tulcea [9] and F-Y. Maeda [11] considered operators in locally convex spaces which possess a functional calculus with functions in certain algebras containing analytic functions.

In the present paper we study the properties of elements in a locally convex algebra having a functional calculus with either analytic or $\mathscr{C}^\infty$-functions.

In §2 we give a perturbation formula generalizing a result contained in [3] (see also [4], II, Th. 1.5). In §3 we study the properties of elements which have a functional calculus by means of spectral distributions ([7]). We show that the regularity problem raised in [6], VI, 5(d) has a negative answer in the locally convex case (§4).

§1. Notations and preliminaries

Throughout, all linear structures are over the complex field Λ; Λ_∞ is the one-point compactification of Λ by ∞; R is the real field and N is the set of all natural numbers.

For any $\sigma\subset\Lambda$, $\sigma\neq\emptyset$, $0\leqq r<\infty$ we put

$$C(\sigma, r)=\{\lambda\in\Lambda;\ \mathrm{dist}\,(\lambda, \sigma)\leqq r\}.$$

If $\sigma=\emptyset$ then we put by definition $C(\emptyset, r)=\emptyset$, $0\leqq r<\infty$.

The closure in Λ (resp. Λ_∞) of a set σ is denoted by $\mathrm{cl}\,\sigma$ (resp. $\mathrm{cl}_\infty\sigma$).

If we put

$$D=\frac{1}{2}\Big(\frac{\partial}{\partial \mathrm{Re}\lambda}+i\frac{\partial}{\partial \mathrm{Im}\lambda}\Big),\qquad \bar{D}=\frac{1}{2}\Big(\frac{\partial}{\partial \mathrm{Re}\lambda}-i\frac{\partial}{\partial \mathrm{Im}\lambda}\Big)$$

then $\mathscr{C}^\infty$ denotes the algebra of all infinitely differentiable complex functions on Λ, endowed with the topology determined by the pseudonorms

$$\varphi\to|\varphi|_{n,K}=2^n\max_{j+k\leqq n}\sup_{\lambda\in K}|D^j\bar{D}^k\varphi(\lambda)|$$

for K compact and $n\in N$.

The letter "A" will denote a sequentially complete locally convex algebra with a unit "e" whose topology is determined by the directed family of pseudonorms, $\{|\cdot|_\alpha\}_{\alpha\in\mathcal{A}}$.

Let $x \in A$. If there is $\lambda \in \Lambda$, $\lambda\neq 0$ such that $\{(\lambda x)^n\}_{n\in N}$ is a bounded set in A then we say that x is a bounded element. The set of all bounded elements in A is denoted by A_0 ([1], Def. 2.1).

The spectrum of x, $\sigma(x)$ is the subset in Λ_∞ defined as follows (see [1], Def. 3.1):

a) $\lambda \in \sigma(x)-\{\infty\} \rightleftarrows \lambda - x$ has no inverse belonging to A_0.

b) $\infty \in \sigma(x) \rightleftarrows x \notin A_0$.

By [1], Prop. 2.6 and Th. 3.6, $\sigma(x)$ is the same as defined by L. Waelbroeck ([16], II, Def. 1.1).

The resolvent set of x, $\rho(x)$ is the complement of $\sigma(x)$ in Λ_∞.

We put also $(\lambda-x)^{-1}=R(\lambda;x)$, $\lambda\in\rho(x)$; $|x|_{sp}=\sup_{\alpha\in\mathcal{A}}\lim_{n\to\infty}|x^n|_\alpha^{\frac{1}{n}}$.

Let now y be another element in A. If L_x (resp. R_x) denotes the left (resp. right) multiplication operator by x in A then we put $[x, y]=L_x-R_y$. Explicitely, this means $[x, y]z=xz-zy$, $z\in A$. $[x, y]$ is a linear operator acting in A. We have:

$$[x, y]^n z=\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}zy^k.$$

If $z=e$ we put

$$(x\backslash y)^{(n)}=[x, y]^n e=\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}y^k.$$

The perturbation radius of x and y, $p(x, y)$ is defined by the equation (see also [3])

$$p(x, y)=\sup_{\alpha\in\mathcal{A}}\max\left(\overline{\lim_{n\to\infty}}|(x\backslash y)^{(n)}|_\alpha^{\frac{1}{n}},\ \overline{\lim_{n\to\infty}}|(y\backslash x)^{(n)}|_\alpha^{\frac{1}{n}}\right).$$

Since $(x\backslash y)^{(n)}=(-1)^n(y\backslash x)^{(n)}=(x-y)^n$ if x commutes with y, we have in this case $p(x, y)=|x-y|_{sp}$.

Without any assumption on commutativity we have

$$p(x, y)=p(y, x),\ p(x, 0)=|x|_{sp},\ p(\lambda x, \lambda y)=|\lambda|p(x, y).$$

§2. Perturbation

In this section we suppose A has continuous product, i.e. for any $\alpha\in\mathcal{A}$

there are $\beta_\alpha \in \mathcal{A}$, $M_\alpha > 0$ such that

$$|xy|_\alpha \leqq M_\alpha |x|_{\beta_\alpha} |y|_{\beta_\alpha}, \qquad x, y \in A.$$

LEMMA 2.1. *For any $x, y, z \in A$ we have*

$$p(x, y) \leqq p(x, z) + p(z, y).$$

PROOF. Using the equality $(x\backslash y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (x\backslash z)^{(n-k)} (z\backslash y)^{(k)}$ (see [6], p. 11) we obtain

$$|(x\backslash y)^{(n)}|_\alpha \leqq M_\alpha \sum_{k=0}^{n} \binom{n}{k} |(x\backslash z)^{(n-k)}|_{\beta_\alpha} |(z\backslash y)^{(k)}|_{\beta_\alpha}.$$

Take $\varepsilon > 0$. By the definition of p we can find $a_\alpha > 0$ such that $|(x\backslash z)^{(n-k)}|_\alpha \leqq a_\alpha (p(x, z)+\varepsilon)^{n-k}$, $|(z\backslash y)^{(k)}|_\alpha \leqq a_\alpha (p(z, y)+\varepsilon)^k$. It follows

$$|(x\backslash y)^{(n)}|_\alpha \leqq M_\alpha a_\alpha^2 (p(x, z)+p(z, y)+2\varepsilon)^n$$

and analogously

$$|(y\backslash x)^{(n)}|_\alpha \leqq M_\alpha a_\alpha^2 (p(x, z)+p(z, y)+2\varepsilon)^n.$$

Hence we infer $p(x, y) \leqq p(x, z)+p(z, y)+2\varepsilon$, for any $\varepsilon > 0$, which finishes the proof.

COROLLARY 2.2. *Let $x \in A_0$, $y \in A$. We have $y \in A_0$ if and only if $p(x, y) < \infty$.*

PROOF. It is easy to see that the relation $y \in A_0$ is equivalent to $|y|_{sp} < \infty$. Thus if $|y|_{sp} < \infty$ we have $p(x, y) \leqq p(x, 0) + p(0, y) = |x|_{sp} + |y|_{sp} < \infty$. Conversely, if $p(x, y) < \infty$ then $|y|_{sp} = p(0, y) \leqq p(0, x) + p(x, y) = |x|_{sp} + p(x, y) < \infty$.

LEMMA 2.3. *Let $x, y \in A$ and $p(x, y) < \infty$. Then we have*

$$\sigma(y) - \{\infty\} \subset C(\sigma(x) - \{\infty\}, p(x, y)).$$

PROOF. If $\sigma(x) - \{\infty\} = \varLambda$ then our inclusion becomes trivial. Let us suppose $\sigma(x) - \{\infty\} \neq \varLambda$ and take $\lambda_0 \in \varLambda - C(\sigma(x) - \{\infty\}, p(x, y))$. We can find $r_2 > r_1 > 0$ such that $r_2 - r_1 > p(x, y)$, $C(\lambda_0, r_2) \cap \sigma(x) = \emptyset$.

Using the equality

$$R^{n+1}(\lambda; x) = \frac{1}{2\pi i} \int_{|\zeta - \lambda_0| = r_2} \frac{R(\zeta; x)}{(\lambda - \zeta)^{n+1}} d\zeta, \; \lambda \in C(\lambda_0, r_1),$$

we get

$$|(y\backslash x)^{(n)} R^{n+1}(\lambda; x)|_\alpha \leqq M'_\alpha \frac{|(y\backslash x)^{(n)}|_{\beta_\alpha}}{(r_2 - r_1)^{n+1}},$$

$$|R^{n+1}(\lambda;\, x)(x\backslash y)^{(n)}|_\alpha \leqq M'_\alpha \frac{|(x\backslash y)^{(n)}|_{\beta_\alpha}}{(r_2-r_1)^{n+1}},$$

where M'_α is some positive constant. It follows that the series $\sum_{n=0}^{\infty}(y\backslash x)^{(n)} R^{n+1}(\lambda;\, x)$, $\sum_{n=0}^{\infty}(-1)^n R^{n+1}(\lambda;\, x)(x\backslash y)^{(n)}$ are uniformly convergent in $C(\lambda_0, r_1)$. By the calculus of [6], I, Th. 2.2 we infer that our series have the same sum $F(\lambda)$ and $F(\lambda)(\lambda-y)=(\lambda-y)F(\lambda)=e$. Hence we get $\lambda_0 \in \rho(y)$ and the desired inclusion results.

COROLLARY 2.4. *Let $x, y \in A$ and $p(x, y) < \infty$. Then we have*

(i) *$\sigma(y)=\{\infty\}$ if and only if $\sigma(x)=\{\infty\}$.*

(ii) *∞ is an isolated point of $\sigma(y)$ if and only if ∞ is an isolated point of $\sigma(x)$.*

(iii) *$p(x, y)=0$ implies $\sigma(y)=\sigma(x)$.*

PROOF. (i) If $\sigma(x)=\{\infty\}$ then $\sigma(y)-\{\infty\}\subset\emptyset$, thus $\sigma(y)=\{\infty\}$. Analogously $\sigma(y)=\{\infty\}\Rightarrow\sigma(x)=\{\infty\}$.

(ii) Suppose ∞ is an isolated point of $\sigma(x)$. Then $\sigma(x)-\{\infty\}$ and $C(\sigma(x)-\{\infty\}, p(x, y))$ are compact sets in Λ. The inclusion $\sigma(y)-\{\infty\}\subset C(\sigma(x)-\{\infty\}, p(x, y))$ shows that also $\sigma(y)-\{\infty\}$ is a compact subset of Λ. By Cor. 2.2 we have $y\notin A_0$. Thus $\infty\in\sigma(y)$ and ∞ is isolated in $\sigma(y)$. Similarly one obtains the converse implication.

(iii) If $p(x, y)=0$ then using the equality $C(\sigma(x)-\{\infty\}, 0)=\sigma(x)-\{\infty\}$ we get $\sigma(y)-\{\infty\}\subset\sigma(x)-\{\infty\}$ and by symmetry $\sigma(x)-\{\infty\}\subset\sigma(y)-\{\infty\}$. Using also Cor. 2.2 we get $\sigma(y)=\sigma(x)$.

THEOREM 2.5. *Let $x, y\in A_0$. Then for any analytic complex function f defined in a neighbourhood of $C(\sigma(x), p(x, y))$ we have*

$$f(y)=\sum_{n=0}^{\infty}\frac{(y\backslash x)^{(n)}}{n!}f^{(n)}(x)=\sum_{n=0}^{\infty}(-1)^n f^{(n)}(x)\frac{(x\backslash y)^{(n)}}{n!}.$$

PROOF. Note that we have $\sigma(y)\subset C(\sigma(x), p(x, y))$ (see Cor. 2.2, Lemma 2.3). Thus we may define $f(y)$. An examination of the proof of Lemma 2.3 shows that we have, for $\lambda\in C(\sigma(x), p(x, y))$

$$R(\lambda;\, y)=\sum_{n=0}^{\infty}(y\backslash x)^{(n)}R^{n+1}(\lambda;\, x)=\sum_{n=0}^{\infty}(-1)^n R^{n+1}(\lambda;\, x)(x\backslash y)^{(n)},$$

the series being uniformly convergent in each compact set.

Multiplying the above equalities by $\frac{1}{2\pi i}f(\lambda)$ and integrating term by term on a suitable contour Γ surrounding the compact set $C(\sigma(x), p(x, y))$ we obtain

$$f(y)=\sum_{n=0}^{\infty}(y\backslash x)^{(n)}\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)R^{n+1}(\lambda;x)d\lambda$$

$$=\sum_{n=0}^{\infty}(-1)^{n}\left\{\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)R^{n+1}(\lambda;x)d\lambda\right\}(x\backslash y)^{(n)}.$$

To finish the proof we have to use the equality

$$\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)R^{n+1}(\lambda;x)d\lambda=\frac{1}{n!}f^{(n)}(x).$$

§ 3. Spectral distributions and generalized scalar elements

In the sequel we suppose that A has separately continuous product. We put $p_n(\lambda)=\lambda^n$, $n\in N\cup\{0\}$.

Definition 3.1. *A spectral distribution* is a multiplicative vector distribution

$$u:\mathcal{C}^\infty\to A$$

such that $u(p_0)=e$.

Definition 3.2. The element $x\in A$ is *a generalized scalar one* if there is a spectral distribution u such that $u(p_1)=x$.

The case of generalized scalar operators in Banach spaces ([7]) shows that to an x it is possible to correspond more than one distributions. Since x can be identified with L_x, if we have $x\in A_0$ then using [11], Prop. 1.3 we infer that any spectral distribution of x gives an extension to $\mathcal{C}^\infty$ of the functional calculus with analytic functions defined in a neighbourhood of $\sigma(x)$.

THEOREM 3.3. *Let u, v be two spectral distributions and let $z\in A$ satisfy $[u(p_1),v(p_1)]z=0$. Then given a vector distribution w, valued in A, there is a map $\alpha\to n_\alpha$ $(:\mathcal{A}\to N\cup\{0\})$ such that*

$$|w(\varphi)([u(\bar{p}_1),v(\bar{p}_1)]^{n_\alpha+k}z)|_\alpha=|([u(\bar{p}_1),v(\bar{p}_1)]^{n_\alpha+k}z)w(\varphi)|_\alpha=0$$

for any $k\in N$, $\varphi\in\mathcal{C}^\infty$.

PROOF. The maps

$$F:(\varphi_1,\varphi_2,\varphi_3)\to w(\varphi_1)u(\varphi_2)zv(\varphi_3)$$

$$G:(\varphi_1,\varphi_2,\varphi_3)\to u(\varphi_1)zv(\varphi_2)w(\varphi_3)$$

defined in $\mathcal{C}^\infty\times\mathcal{C}^\infty\times\mathcal{C}^\infty$ are separately linear and continuous and since $\mathcal{C}^\infty$ is an F-space they are continuous.

Let $\varphi \in \mathcal{C}^{\infty}$. The functions defined by the equations

$$f_{\varphi}(\zeta) = F(\varphi,\ e^{\zeta \bar{p}_1},\ e^{-\zeta \bar{p}_1}),\ g_{\varphi}(\zeta) = G(e^{\zeta \bar{p}_1},\ e^{-\zeta \bar{p}_1},\ \varphi),\ \zeta \in \Lambda$$

are holomorphic. Using the equality $[u(p_1),\ v(p_1)]z = 0$ one obtains

$$f_{\varphi}(\zeta) = F(\varphi,\ e^{-\bar{\zeta} p_1 + \zeta \bar{p}_1},\ e^{-\zeta \bar{p}_1 + \bar{\zeta} p_1}),\ g_{\varphi}(\zeta) = G(e^{-\bar{\zeta} p_1 + \zeta \bar{p}_1},\ e^{-\zeta \bar{p}_1 + \bar{\zeta} p_1},\ \varphi),$$

which, together with the continuity of F and G, implies

$$|f_{\varphi}(\zeta)|_{\alpha} \leqq M_{\alpha} |\varphi|_{m_{\alpha}, K_{\alpha}} (1 + |\zeta|^{2m_{\alpha}}),\ |g_{\varphi}(\zeta)|_{\alpha} \leqq M_{\alpha} |\varphi|_{m_{\alpha}, K_{\alpha}} (1 + |\zeta|^{2m_{\alpha}}),$$

where M_{α}, m_{α},K_{α} are suitably chosen.

It results (by Liouville's theorem) that f_{φ} and g_{φ} are polynomyals of degree at most $2m_{\alpha}$ in the topology of the pseudonorm $|\cdot|_{\alpha}$. To finish the proof we have to use the equalities

$$f_{\varphi}^{(n)}(0) = W(\varphi)([u(\bar{p}_1),\ v(\bar{p}_1)]^n z),$$

$$g_{\varphi}^{(n)}(0) = ([u(\bar{p}_1),\ v(\bar{p}_1)]^n z)\ W(\varphi),\ n \in N \cup \{0\},$$

which can be obtained by induction on n.

COROLLARY 3.4. *Let $x(\in A)$ be a generalized scalar element with the spectral distributions u, v* (i.e. $x = u(p_1) = v(p_1)$). *Then there is a map $\alpha \to n_{\alpha}(: \mathcal{A} \to N \cup \{0\})$ such that for any $\varphi \in \mathcal{C}^{\infty}$ we have*

$$|v(\varphi)(u(\bar{p}_1) \backslash v(\bar{p}_1))^{(n_{\alpha}+k)}|_{\alpha} = |(u(\bar{p}_1) \backslash v(\bar{p}_1))^{(n_{\alpha}+k)} v(\varphi)|_{\alpha} = 0.$$

PROOF. Since we have $[u(p_1),\ v(p_1)]e = 0$, we can apply Th. 3.3 with $z = e$, $w = v$.

THEOREM 3.5. *Let u, v be two spactral distributions and let $z \in A$ satisfy $[u(p_1),\ v(p_1)]z = 0$. Then for any $\varphi \in \mathcal{C}^{\infty}$ we have*

$$[u(\varphi),\ v(\varphi)]z = \sum_{k=1}^{\infty} \frac{[u(\bar{p}_1),\ v(\bar{p}_1)]^k z}{k!} v(D^k \varphi)$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} u(D^k \varphi) \frac{[u(\bar{p}_1),\ v(\bar{p}_1)]^k z}{k!},$$

the series being finite sums depending only on α in the topology of each pseudonorm $|\cdot|_{\alpha}$.

PROOF. We have $L_{u(\bar{p}_1)} = [u(\bar{p}_1),\ v(\bar{p}_1)] + R_{v(\bar{p}_1)}$. Thus for any $n \in N$ one obtains $L_{u(\bar{p}_1)} = \sum_{k=0}^{\infty} \frac{1}{k!} R_{v(D^k \bar{p}_n)} [u(\bar{p}_1),\ v(\bar{p}_1)]^k$. (We have used the properties of the functional calculus with analytic polynomials in the argument λ). Using this equality and our hypothesis, an easy calculus gives us

$$L_{u(q)}z = \sum_{k=0}^{\infty} \frac{[u(\bar{p}_1), v(\bar{p}_1)^k]z}{k!} v(D^k q)$$

for any polynomial q. Since polynomials are dense in $\mathcal{C}^\infty$, using Th. 3.3 we extend the above equality to $\mathcal{C}^\infty$, the series being a finite sum depending only on α in the topology of the pseudonorm $|\cdot|_\alpha$. Thus we have

$$[u(\varphi), v(\varphi)]z = L_{u(\varphi)}z - R_{v(\varphi)}z = \sum_{k=1}^{\infty} \frac{[u(\bar{p}_1), v(\bar{p}_1)]^k z}{k!} v(D^k \varphi).$$

The second equality is obtained analogously starting with the relation $R_{v(\bar{p}_1)} = L_{u(\bar{p}_1)} - [u(\bar{p}_1), -v(\bar{p}_1)]$.

Corollary 3.6. *Let u, v be two spectral distributions and let $z \in A$ satisfy $[u(p_1), v(p_1)]z = 0$. Then there is a map $\alpha \to n_\alpha (: \mathcal{A} \to N \cup \{0\})$ such that for any system $\{\varphi_j\}_1^{n_\alpha + k}$, $k \in N$ we have*

$$\left| \left(\prod_{j=1}^{n_\alpha + k} [u(\varphi_1), v(\varphi_1)] \right) z \right|_\alpha = 0.$$

Proof. Let n_α be given by Th. 3.3 with $w = v$. By Th. 3.5 we have

$$\left(\prod_{j=1}^{n_\alpha + k} [u(\varphi_j), v(\varphi_j)] \right) z$$

$$= \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n_\alpha+k}=1}^{\infty} \frac{[u(\bar{p}_1), v(\bar{p}_1)]^{m_1 + \cdots + m_{n_\alpha + k}}}{m_1! \cdots m_{n_\alpha+k}!} v(D^{m_1}\varphi_1 \cdots D^{m_{n_\alpha+k}} \varphi_{n_\alpha + k}).$$

Since each term in the right hand is 0 in the topology of $|\cdot|_\alpha$ the proof is finished.

Corollary 3.7. *Let $x (\in A)$ be a generalized scalar element with the spectral distributions u, v. Then for any $\varphi \in \mathcal{C}^\infty$ we have $p(u(\varphi), v(\varphi)) = 0$ (see the definition of p in §1).*

Proof. It results by Cor. 3.6.

Corollary 3.8. *Let $x (\in A)$ be a generalized scalar element with the spectral distributions u, v. Then for any $\varphi \in \mathcal{C}^\infty$ we have*

$$u(\varphi) = \sum_{k=0}^{\infty} \frac{(u(\bar{p}_1) \backslash v(\bar{p}_1))^{(k)}}{k!} v(D^k \varphi)$$

$$= \sum_{k=0}^{\infty} (-1)^k v(D^k \varphi) \frac{(v(\bar{p}_1) \backslash u(\bar{p}_1))^{(k)}}{k!},$$

the series being finite sums depending only on α in the topology of each pseudonorm $|\cdot|_\alpha$.

Proof. Our corollary results by Th. 3.5. Indeed we have $u(p_1)=v(p_1)$; thus $[u(p_1), v(p_1)]e=[v(p_1), u(p_1)]e=0$. Consequently

$$[u(\varphi), v(\varphi)]e=\sum_{k=1}^{\infty}\frac{[u(\bar{p}_1), v(\bar{p}_1)]^k e}{k!}v(D^k\varphi),$$

$$[v(\varphi), u(\varphi)]e=\sum_{k=1}^{\infty}(-1)^{k+1}v(D^k\varphi)\frac{[v(\bar{p}_1), u(\bar{p}_1)]^k e}{k!}.$$

Now we have to use the definition of $(x_1\backslash x_2)^{(k)}$, $x_1, x_2\in A$.

Theorem 3.9. *Let u be a spectral distribution. Then for any $\varphi\in\mathcal{C}^\infty$ we have*

(i) *$u(\varphi)$ is a generalized scalar element with the spectral distribution $\psi\to u(\psi o\varphi)$.*

(ii) $\sigma(u(\varphi))=\mathrm{cl}_\infty\varphi$ (supp u).

Proof. (i) It is evident. (ii) Suppose $\sigma(u(\varphi))\not\subset\mathrm{cl}_\infty\varphi$ (supp u). In this case we have $\mathrm{cl}_\infty\varphi$ (supp $u)\neq\Lambda_\infty$ and the function $\zeta\to\varphi_\zeta$ defined by the equation $\varphi_\zeta(\lambda)=(\zeta-\varphi(\lambda))^{-1}$ is holomorphic in $\Lambda-\mathrm{cl}_\infty\varphi$(supp u) if $\lambda\in$ supp u. Using the equalities $u(\varphi_\zeta)(\zeta-u(\varphi))=(\zeta-u(\varphi))u(\varphi_\zeta)=u(p_0)=e$, we obtain $\Lambda_\infty-\mathrm{cl}_\infty\varphi$(supp $u)\subset\rho(u(\varphi))$ which is preposterous. Now if $\sigma(u(\varphi))\neq\mathrm{cl}_\infty\varphi$ (supp u) we can find φ_0 ($\in\mathcal{C}^\infty$) such that φ_0 has compact support, φ(supp φ_0) $\cap\sigma(u(\varphi))=\emptyset$, $u(\varphi_0)\neq0$. Take $\psi(\in\mathcal{C}^\infty)$ with compact support such that $\psi(\lambda)=1$ in a neighbourhood of supp φ_0, φ (supp $\psi)\cap\sigma(u(\varphi))=\emptyset$ and put for $\zeta\notin\varphi$(supp ψ)

$$\psi_\zeta(\lambda)=\begin{cases}\psi(\lambda)(\zeta-\varphi(\lambda))^{-1}, & \lambda\in\text{supp }\psi\\ 0, & \lambda\notin\text{supp }\psi.\end{cases}$$

Using the identity $(\zeta-\varphi)\psi_\zeta\varphi_0=\varphi_0$ we may define the function $f\colon\Lambda\to A$ by the equation

$$f(\zeta)=\begin{cases}R(\zeta;u(\varphi))u(\varphi_0), & \zeta\notin\sigma(u(\varphi))\\ u(\psi_\zeta)u(\varphi_0), & \zeta\notin\varphi(\text{supp }u).\end{cases}$$

Since f is holomorphic and $\lim_{\zeta\to\infty}f(\zeta)=\lim_{\zeta\to\infty}u(\psi_\zeta)u(\varphi_0)=0$, we have $f=0$, which is impossible because $u(\varphi_0)\neq0$. Thus $\sigma(u(\varphi))=\mathrm{cl}_\infty\varphi$ (supp u).

Corollary 3.10. *Let u be a spectral distribution and put $v_\varphi(\psi)=u(\psi o\varphi)$. Then v_φ is a spectral distribution such that supp $v_\varphi=\mathrm{cl}\ \varphi$(supp u).*

PROOF. By Th. 3.9 we have $\mathrm{cl}_\infty\varphi(\mathrm{supp}\ u)=\sigma(u(\varphi))=\sigma(v_\varphi(p_1))=\mathrm{cl}_\infty$ supp v_φ. Since supp $v_\varphi=(\mathrm{cl}_\infty$ supp $v_\varphi)-\{\infty\}$ and cl $\varphi(\mathrm{supp}\ u)=(\mathrm{cl}_\infty\varphi(\mathrm{supp}\ u))-\{\infty\}$, the corollary results.

Let us denote by $\mathcal{E}^\infty(\Lambda^2)$ the algebra of all infinitely differentiable complex functions defined in $\Lambda^2(=\Lambda\times\Lambda=R^4)$ endowed with the topology of the uniform convergence of all derivatives on compact sets.

LEMMA 3.11. *Let u, v be two commuting spectral distributions. Then there is a unique vector distribution*

$$w:\mathcal{E}^\infty(\Lambda^2)\to A$$

such that $w(\varphi\otimes\psi)=u(\varphi)v(\psi)$, φ, $\psi\in\mathcal{E}^\infty$. The vector distribution w is multiplicative and $w(p_0\otimes p_0)=e$.

PROOF. The map $(\varphi,\psi)\to u(\varphi)v(\psi)$ is separately continuous; therefore it is continuous. If we put $w_0(\sum(\varphi_j\otimes\psi_j)=\sum u(\varphi_j)v(\psi_j)$ the map $w_0:\mathcal{E}^\infty\otimes\mathcal{E}^\infty\to A$ is continuous if we endow $\mathcal{E}^\infty\otimes\mathcal{E}^\infty$ with the projective topology. Since we have $\mathcal{E}^\infty\hat{\otimes}\mathcal{E}^\infty=\mathcal{E}^\infty\bar{\otimes}\mathcal{E}^\infty=\mathcal{E}^\infty(\Lambda^2)$ (see [8], II §2, No. 3, Th. 10 and [15], Prop. 28) we can extend w_0 by continuity to a vector distribution w defined in $\mathcal{E}^\infty(\Lambda^2)$.

The uniqueness of w results by the condition $w(\varphi\otimes\psi)=w_0(\varphi\otimes\psi)=u(\varphi)w(\psi)$. Trivially w_0 is multiplicative; therefore w is so and $w(p_0\otimes p_0)=w_0(p_0\otimes p_0)=u(p_0)v(p_0)=e$.

Notation. Let u, v be two commuting spectral distributions. We shall put

$$(u\overset{s}{+}v)(\varphi)=w(\varphi o(p_1\otimes p_0+p_0\otimes p_1)),\qquad \varphi\in\mathcal{E}^\infty$$

$$(u\overset{s}{\times}v)(\varphi)=w(\varphi o(p_1\otimes p_1)),\qquad \varphi\in\mathcal{E}^\infty$$

where w is defined in Lemma 3.11.

If $\mu(\in\Lambda)$ is a given number then we denote by δ_μ the spectral distribution defined by the equation $\delta_\mu(\varphi)=\varphi(\mu)e$ (δ_μ is Dirac's distribution).

LEMMA 3.12. *Let u, v be two commuting spectral distributions. Then $u\overset{s}{+}v$, $u\overset{s}{\times}v$ are spectral distributions such that $(u\overset{s}{+}v)(p_1)=u(p_1)+v(p_1)$, $(u\overset{s}{+}v)(\bar{p}_1)=u(\bar{p}_1)+v(\bar{p}_1)$, $(u\overset{s}{\times}v)(p_1)=u(p_1)v(p_1)$, $(u\overset{s}{\times}v)(\bar{p}_1)=u(\bar{p}_1)v(\bar{p}_1)$.*

PROOF. It results by the definition of $u\overset{s}{+}v$, $u\overset{s}{\times}v$ and by Lemma 3.11.

Notation. Let u be a spectral distribution. We shall put $(\mathrm{Re}\ u)(\varphi)=u(\varphi o\ \mathrm{Re}\ p_1)$, $(\mathrm{Im}\ u)(\varphi)=u(\varphi o\ \mathrm{Im}\ p_1)$. If $0\notin$ supp u then $|u|(\varphi)=u(\varphi o\psi_1)$,

$u_0(\varphi) = u(\varphi o \psi_2)$ where $\psi_1 = |p_1|$, $\psi_2 = \frac{p_1}{|p_1|}$ in a neighbourhood of supp u.

Evidently all the above maps are spectral distributions commuting each other.

Theorem 3.13. *Let u be a spectral distribution. Then we have* $u = \text{Re } u + \delta_i \overset{s}{\times} \text{Im } u$. *If* $0 \notin$ *supp u then the equality* $u = u_0 \overset{s}{\times} |u|$ *also holds.*

Proof. By Lemma 3.12 we have $(\text{Re } u \overset{s}{+} \delta_i \overset{s}{\times} \text{Im } u)(p_1) = u(p_1)$ and $(\text{Re } u \overset{s}{+} \delta_i \overset{s}{\times} \text{Im } u)(\bar{p}_1) = u(\bar{p}_1)$ which suffice for the first equality. The second one results analogously.

Corollary 3.14. *Let $x(\in A)$ be a generalized scalar element. Then there are two commuting scalar elements y, z such that $\sigma(y), \sigma(z) \subset R \cup \{\infty\}$, $x = y + iz$. If $0 \notin \sigma(x)$ then y, z can be chosen in such a way that $\sigma(y) \subset \{\lambda \in \Lambda; |\lambda| = 1\}$, $\sigma(z) \subset \{t \in R; t \geqq 0\} \cup \{\infty\}$, $x = yz$.*

Proof. We have to apply Th. 3.13 and Th. 3.9.

§4. The problem of regularity of spectral distributions

Definition 4.1. The spectral distribution u is called *regular* if it is valued in the bicommutant (i.e. the commutant of the commutant) of $u(p_1)$

Definition 4.2. The generalized scalar element $x(\in A)$ is called a *regular* one if it possesses a regular spectral distribution.

In [6], VI, 5(d) the following problem is raised: let T be a generalized scalar operator in a Banach space; is it a regular one? Concerning this problem we shall exhibit an example of generalized scalar operator in a locally convex space which is not regular (Th. 4.4). A sufficient condition for regularity is given:

Theorem 4.3. *Let $x(\in A)$ be a generalized scalar element such that $\sigma(x) - \{\infty\} \subset R$. Then x is regular.*

Proof. Let u be a spectral distribution of x and put $y = iu\,(\text{Im } p_1)$. Since u is continuous and supp $u \subset R$ (Cor. 3.10), we obtain easily that there is a map $\alpha \to n_\alpha (: \mathcal{A} \to N \cup \{0\})$ such that

$$|y^{n_\alpha + k} u(\varphi)|_\alpha = |u(\text{Im } p_1)^{n_\alpha + k} \varphi|_\alpha = 0,\ k \in N,\ \varphi \in \mathcal{C}^\infty.$$

It follows that the series $v(f) = \sum_{m=0}^{\infty} \frac{y^m}{m!} u(f^{(m)} \circ \text{Re } p_1)$, $f \in \mathcal{C}^\infty(R)$ is a finite sum in the topology of each pseudonorm $|\cdot|_\alpha$.

If f is a polynomial we have $v(f)=f(x)$; thus v is valued in the bicommutant of x. The map v is a multiplicative distribution defined in $\mathcal{O}^\infty(R)$. For any $\varphi \in \mathcal{O}^\infty$ we consider the function $\tilde{\varphi} \in \mathcal{O}^\infty(R)$ defined by the equation $\tilde{\varphi}(t)=\varphi(t)$. If $w(\varphi)=v(\tilde{\varphi})$, $\varphi \in \mathcal{O}^\infty$, then w is a regular spectral distribution of x.

THEOREM 4.4. *Let A be the algebra of all continuous linear operators defined in $\mathcal{O}^\infty$, endowed with the topology of uniform convergence on bounded sets. The operator $T \in A$ defined by the equation $T\psi = p_1\psi$ is a non-regular generalized scalar operator.*

PROOF. The map u defined by the equation $u(\varphi)\psi = \varphi\psi$ is a spectral distribution such that $u(p_1)=T$. Thus T is a generalized scalar operator. Since we have $DT=TD$, $Du(\bar{p}_1)=u(\bar{p}_1)D+e$, u results to be non-regular.

If x possesses a regular spectral distribution v then putting $h=v(\bar{p}_1)p_0$ we obtain $v(\bar{p}_1)\psi = v(\bar{p}_1)u(\psi)p_0 = u(\psi)h = h\psi$, $\psi \in \mathcal{O}^\infty$.

Given n, K, ψ we can find m(see Cor. 3.4) such that $|(u(\bar{p}_1)-v(\bar{p}_1))^m\psi|_{n,k} = |(\bar{p}_1-h)^m\psi|_{n,k}=0$, which implies $\bar{p}_1=h$. It follows that $u=v$, which is impossible because v is regular, and the proof is finished.

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