

Monotone Limits in Linear Programming Problems

Maretsugu YAMASAKI

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§ 1. Introduction and problem setting

The aim of this paper is to investigate the behavior of values of linear programming problems under some monotone variations of objective functions and constraints.

More precisely, let X and Y be real linear spaces paired under the bilinear functional $((,))_1$, and let Z and W be real linear spaces paired under the bilinear functional $((,))_2$. A (linear) program for these paired spaces is a quintuple (A, P, Q, y_0, z_0) . In this quintuple, A is a linear transformation from X into Z , P is a convex cone in X , Q is a convex cone in Z , y_0 is an element of Y and z_0 is an element of Z . The set S of feasible solutions for the program and the value M of the program are defined by

$$S = \{x \in P; Ax - z_0 \in Q\},$$

and

$$M = \inf \{((x, y_0))_1; x \in S\} \quad \text{if } S \neq \phi,$$

$$M = \infty \quad \text{if } S = \phi,$$

where ϕ denotes the empty set.

Let us denote the weak topology on X by $w(X, Y)$ and the Mackey topology on X by $s(X, Y)$ (cf. [2]). Let R be the set of real numbers and R_0 the set of non-negative real numbers. Let us define P^+ and Q^+ by

$$P^+ = \{y \in Y; ((x, y))_1 \geq 0 \quad \text{for all } x \in P\},$$

$$Q^+ = \{w \in W; ((z, w))_2 \geq 0 \quad \text{for all } z \in Q\}.$$

We say that the program (A, P, Q, y_0, z_0) is *regular* if A is $w(X, Y) - w(Z, W)$ continuous, P is $w(X, Y)$ -closed and Q is $w(Z, W)$ -closed.

We shall investigate some relations between the sequence $\{M_n\}$ of values of programs $(A_n, P_n, Q_n, y_n, z_n)$ and the value M of the program (A, P, Q, y_0, z_0) determined by any one of the following conditions:

$$(I) \quad A_n = A, y_n = y_0, z_n = z_0,$$

$$(I. 1) \quad P_n \subset P_{n+1} \quad \text{and} \quad P = \bigcup_{n=1}^{\infty} P_n,$$

- (I. 2) $Q_n \subset Q_{n+1}$ and $Q = \bigcup_{n=1}^{\infty} Q_n$.
- (II) $P_n = P, Q_n = Q, y_n = y_0$,
 (II. 1) $(A_{n+1} - A_n)(P) \subset Q$ and $\{A_n x\}$ $w(Z, W)$ -converges to Ax for all $x \in X$,
 (II. 2) $z_n - z_{n+1} \in Q$ and $\{z_n\}$ $w(Z, W)$ -converges to z_0 .
- (III) $A_n = A, P_n = P, Q_n = Q, z_n = z_0$,
 $y_n - y_{n+1} \in P^+$ and $\{y_n\}$ $w(Y, X)$ -converges to y_0 .
- (IV) $A_n = A, y_n = y_0, z_n = z_0$,
 (IV. 1) $P_{n+1} \subset P_n$ and $P = \bigcap_{n=1}^{\infty} P_n$,
 (IV. 2) $Q_{n+1} \subset Q_n$ and $Q = \bigcap_{n=1}^{\infty} Q_n$.
- (V) $P_n = P, Q_n = Q, y_n = y_0$,
 (V. 1) $(A_n - A_{n+1})(P) \subset Q$ and $\{A_n x\}$ $w(Z, W)$ -converges to Ax for all $x \in X$,
 (V. 2) $z_{n+1} - z_n \in Q$ and $\{z_n\}$ $w(Z, W)$ -converges to z_0 .
- (VI) $A_n = A, P_n = P, Q_n = Q, z_n = z_0$,
 $y_{n+1} - y_n \in P^+$ and $\{y_n\}$ $w(Y, X)$ -converges to y_0 .

The above problems were partially studied by K. S. Kretschmer [3]. Potential-theoretic forms of the above problems were investigated by M. Ohtsuka [4] and M. Yamasaki [5; 6].

§ 2. Preliminaries

For later use, we shall recall some results in [1], [3] and [6].

Let A be a $w(X, Y) - w(Z, W)$ continuous linear transformation from X into Z in this section. We denote by A^* the adjoint transformation. Namely A^* is a linear transformation from W into Y which is $w(W, Z) - w(Y, X)$ continuous and satisfies $((Ax, w))_2 = ((x, A^*w))_1$ for all $x \in X$ and $w \in W$.

The dual program of the program (A, P, Q, y_0, z_0) is defined as the program $(A^*, Q^+, -P^+, -z_0, y_0)$ for W and Z paired under ${}_2((,))$ and for Y and X paired under ${}_1((,))$. Here the bilinear functionals ${}_2((,))$ and ${}_1((,))$ are defined by ${}_2((w, z)) = ((z, w))_2$ for all $w \in W$ and $z \in Z$ and ${}_1((y, x)) = ((x, y))_1$ for all $y \in Y$ and $x \in X$. The set S^* of feasible solutions for the dual program and the value M^* of the dual program are given by

$$S^* = \{w \in Q^+; y_0 - A^*w \in P^+\},$$

$$\begin{aligned}
 M^* &= \sup \{((z_0, w))_2; w \in S^*\} && \text{if } S^* \neq \phi, \\
 M^* &= -\infty && \text{if } S^* = \phi.
 \end{aligned}$$

We have

PROPOSITION 1.¹⁾ *It is always valid that $M^* \leq M$. If the $s(Z, W)$ -interior Q° of Q is nonempty, if there exists $x \in P$ such that $Ax - z_0 \in Q^\circ$ and if M is finite, then $M = M^* = ((z_0, w))_2$ for some $w \in S^*$.*

We say that a program (A, P, Q, y_0, z_0) is a *finite (linear) program* if X, Y, Z and W are finite dimensional Euclidean spaces, $((,))_1$ and $((,))_2$ are usual inner products and P and Q are polyhedral cones. We have

PROPOSITION 2.²⁾ *Assume that (A, P, Q, y_0, z_0) is a finite program. If $M < \infty$ or $M^* > -\infty$, then $M = M^*$ holds.*

We shall use the following separation theorem.

PROPOSITION 3.³⁾ *Let K and F be convex sets in X and assume that the $s(X, Y)$ -interior K° of K is nonempty and $F \cap K^\circ = \phi$. Then there exists a non-zero $y \in Y$ such that*

$$((u, y))_1 \leq ((x, y))_1$$

for all $u \in F$ and $x \in K$.

§ 3. The case where $M_n \geq M_{n+1}$

Let us define S_n by

$$S_n = \{x \in P_n; A_n x - z_n \in Q_n\}.$$

First we shall prove

THEOREM 1. *Assume that $y_n = y_0$ for all n and that $S_n \subset S_{n+1} \subset S$ and $S \subset \bigcup_{n=1}^\infty S_n$ (the $w(X, Y)$ -closure of $\bigcup_{n=1}^\infty S_n$). Then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.*

PROOF. By our assumption that $S_n \subset S_{n+1} \subset S$ and $y_n = y_0$, we have $M_n \geq M_{n+1} \geq M$ and $\lim_{n \rightarrow \infty} M_n \geq M$. We show that $\lim_{n \rightarrow \infty} M_n \leq M$ in case $M < \infty$.

Let r be any number such that $M < r$. There exists $x \in S$ such that $((x, y_0))_1 < r$. For any $\varepsilon > 0$, there exist n_0 and \bar{x} such that $\bar{x} \in S_{n_0}$ and $((\bar{x}, y_0))_1 < ((x, y_0))_1 + \varepsilon$. Then

$$M_n \leq M_{n_0} \leq ((\bar{x}, y_0))_1 < r + \varepsilon$$

1) See [3] for the case of regular programs and [6] for the present case.
 2) [3], p. 230.
 3) [1], Proposition 1, p. 71.

for all $n \geq n_0$, and hence $\lim_{n \rightarrow \infty} M_n < r + \varepsilon$. Letting first $\varepsilon \rightarrow 0$ and next $r \rightarrow M$, we obtain the desired inequality.

COROLLARY. *If condition (I) is fulfilled, then $\lim_{n \rightarrow \infty} M_n = M$.*

PROOF. By condition (I), we have $S_n \subset S_{n+1}$ and $S = \bigcup_{n=1}^{\infty} S_n$.

Next we assume condition (II). By the relation

$$A_{n+1}x - z_{n+1} = A_nx - z_n + (A_{n+1}x - A_nx) + (z_n - z_{n+1}),$$

we see that $S_n \subset S_{n+1}$ and $M_n \geq M_{n+1}$. If Q is $w(Z, W)$ -closed, then for any $x \in S_n$, $\{A_nx - z_n\}$ $w(Z, W)$ -converges to $Ax - z_0 \in Q$, so that $S_n \subset S$ and $\bigcup_{n=1}^{\infty} S_n \subset S$. It is not always valid that $S \subset \overline{\bigcup_{n=1}^{\infty} S_n}$.

We shall prove

THEOREM 2. *Assume that condition (II) is fulfilled and that Q is $w(Z, W)$ -closed. If $Q^\circ \neq \phi$ and $P \cap A^{-1}(z_0 + Q^\circ) \neq \phi$, then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.*

PROOF. By Theorem 1, it is enough to show that $S \subset \overline{\bigcup_{n=1}^{\infty} S_n}$. Let $\bar{x} \in P \cap A^{-1}(z_0 + Q^\circ)$ and let C be the convex hull of the set $\{A_n\bar{x} - z_n; n = 1, 2, \dots\}$. If $C \cap Q^\circ = \phi$, then there exists $\bar{w} \neq 0$ such that

$$((u, \bar{w}))_2 \leq ((v, \bar{w}))_2$$

for all $u \in C$ and $v \in Q$ by Proposition 3. Since Q is a convex cone, we see easily that $\bar{w} \in Q^+$ and $((u, \bar{w}))_2 \leq 0$ for all $u \in C$. It follows that $((A_n\bar{x} - z_n, \bar{w}))_2 \leq 0$ for all n , and hence $((A\bar{x} - z_0, \bar{w}))_2 \leq 0$. This is a contradiction. Therefore $C \cap Q^\circ \neq \phi$. There exist a finite subset J of natural numbers and positive numbers $\{t_i; i \in J\}$ such that

$$\sum_{i \in J} t_i = 1 \quad \text{and} \quad \sum_{i \in J} t_i(A_i\bar{x} - z_i) \in Q^\circ.$$

Let m be an integer such that $m \geq i$ for all $i \in J$. Then by the relation

$$A_m\bar{x} - z_m = \sum_{i \in J} t_i(A_i\bar{x} - z_i) + \sum_{i \in J} t_i(A_m\bar{x} - A_i\bar{x}) + \sum_{i \in J} t_i(z_i - z_m)$$

we have $A_m\bar{x} - z_m \in Q^\circ$ and $\bar{x} \in \bigcup_{n=1}^{\infty} S_n$. Thus we have shown that $P \cap A^{-1}(z_0 + Q^\circ) \subset \overline{\bigcup_{n=1}^{\infty} S_n}$. Let $x \in S$ and $t \in R, 0 < t < 1$. Then we have

$$x_t = t\bar{x} + (1-t)x \in P,$$

$$Ax_t - z_0 = t(A\bar{x} - z_0) + (1-t)(Ax - z_0) \in Q^\circ,$$

and hence $x_t \in \bigcup_{n=1}^{\infty} S_n$. Letting $t \rightarrow 0$, we conclude that $x \in \overline{\bigcup_{n=1}^{\infty} S_n}$, which completes the proof.

Finally we are concerned with condition (III). We have

THEOREM 3. *Assume that condition (III) is satisfied. Then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.*

PROOF. We have $S_n = S$ and we may assume that $S \neq \emptyset$. Let $M(y) = \inf \{((x, y))_1; x \in S\}$. Then $M_n = M(y_n)$ and $M = M(y_0)$. Since $y_n - y_{n+1} \in P^+$ and P^+ is $w(Y, X)$ -closed, we have $y_n - y_0 \in P^+$ and $M(y_n) \geq M(y_{n+1}) \geq M(y_0)$. Since $M(y)$ is $w(Y, X)$ -upper semicontinuous, we have $M(y_0) \geq \lim_{n \rightarrow \infty} M(y_n)$. Thus we have $\lim_{n \rightarrow \infty} M(y_n) = M(y_0)$.

§ 4. The case where $M_n \leq M_{n+1}$

In this section we always assume that the program $(A_n, P_n, Q_n, y_n, z_n)$ is regular for each n . It follows that the set S_n of feasible solutions is $w(X, Y)$ -closed.

It is easily seen that condition (IV) implies that $S_n \subset S_{n+1} \subset S$ and $S = \bigcap_{n=1}^{\infty} S_n$. If condition (VI) is fulfilled, then $S_n = S$ for all n and $M_n \leq M_{n+1}$. Assume condition (V). Then by the relations

$$A_n x - z_n = (A_{n+1} x - z_{n+1}) + (A_n x - A_{n+1} x) + (z_{n+1} - z_n),$$

$$(A_n - A)(P) \subset Q \quad \text{and} \quad z_0 - z_n \in Q,$$

we have $S \subset S_{n+1} \subset S_n$. Let us assume that $x \in \bigcap_{n=1}^{\infty} S_n$. Then $x \in P$ and $A_n x - z_n \in Q$ for all n . Since Q is $w(Z, W)$ -closed, we have $Ax - z_0 \in Q$ and hence $x \in S$.

Thus we have

THEOREM 4. *Any one of conditions (IV), (V) and (VI) implies that $S \subset S_{n+1} \subset S_n$, $S = \bigcap_{n=1}^{\infty} S_n$ and $M_n \leq M_{n+1} \leq M$.*

We remark that any one of conditions (IV), (V) and (VI) does not necessarily imply that $\lim_{n \rightarrow \infty} M_n = M$. This will be shown by examples in §5 below.

We shall investigate some criteria which assure that $\lim_{n \rightarrow \infty} M_n = M$.

THEOREM 5. *Assume that $y_n = y_0$ for all n , that $S \subset S_{n+1} \subset S_n$ and $S = \bigcap_{n=1}^{\infty} S_n$ and that there is an n_0 such that S_{n_0} is nonempty and $w(X, Y)$ -compact. Then*

it is valid that $\lim_{n \rightarrow \infty} M_n = M$.

PROOF. Since $S \subset S_{n+1} \subset S_n$ and $y_n = y_0$, we have $M_n \leq M_{n+1} \leq M$ and $\lim_{n \rightarrow \infty} M_n \leq M$. It is enough to show that $\lim_{n \rightarrow \infty} M_n \geq M$ in case $\lim_{n \rightarrow \infty} M_n < \infty$. Let r be any number such that $\lim_{n \rightarrow \infty} M_n < r$. For each n , there exists $x_n \in S_n$ such that $((x_n, y_0))_1 < r$. Obviously $x_n \in S_{n_0}$ for all $n \geq n_0$. By our assumption that S_{n_0} is $w(X, Y)$ -compact, there exists a $w(X, Y)$ -convergent subsequence of $\{x_n\}$. Denote it again by $\{x_n\}$ and let x be the limit. Since S_m is $w(X, Y)$ -closed and $x_n \in S_m$ for all $n \geq m$, we have $x \in S_m$. By the arbitrariness of m , we have $x \in S$ and

$$M \leq ((x, y_0))_1 = \lim_{n \rightarrow \infty} ((x_n, y_0))_1 \leq r.$$

By the arbitrariness of r , we obtain the desired inequality.

Next we shall prove

THEOREM 6. Assume that $y_n = y_0 \in (P^+)^\circ$ (the $s(Y, X)$ -interior of P^+) and that any $w(X, Y)$ -bounded set in P is relatively $w(X, Y)$ -compact. If condition (V) is fulfilled, then $\lim_{n \rightarrow \infty} M_n = M$.

PROOF. It is enough to show that $\lim_{n \rightarrow \infty} M_n \geq M$ in case $\lim_{n \rightarrow \infty} M_n < \infty$. Since $y_0 \in P^+$, we have $M_n \geq 0$. There exists $x_n \in S_n$ such that $((x_n, y_0))_1 < M_n + 1/n$. For any $y \in Y$, there is $\varepsilon > 0$ such that $y_0 \pm \varepsilon y \in (P^+)^\circ$. Since $x_n \in P$, we have

$$0 \leq ((x_n, y_0 \pm \varepsilon y))_1 < M_n + 1/n \pm \varepsilon((x_n, y))_1,$$

so that

$$|((x_n, y))_1| < (M_n + 1/n)/\varepsilon \leq (\lim_{n \rightarrow \infty} M_n + 1)/\varepsilon.$$

Namely $\{x_n\}$ is $w(X, Y)$ -bounded. By our assumption, there exists a $w(X, Y)$ -convergent subsequence of $\{x_n\}$. Denote it again by $\{x_n\}$ and let x be the limit. It follows from the same argument as in the proof of Theorem 5 that $x \in S$. Therefore we have

$$M \leq ((x, y_0))_1 = \lim_{n \rightarrow \infty} ((x_n, y_0))_1 \leq \lim_{n \rightarrow \infty} M_n.$$

This completes the proof.

Let us denote

$$S_n^* = \{w \in Q_n^+; y_n - A^*w \in P_n^+\}.$$

If any one of conditions (IV), (V) and (VI) is satisfied, then $S_n^* \subset S_{n+1}^*$ and $M_n^* \leq M_{n+1}^*$. We prepare

LEMMA 1. Assume that the program (A, P, Q, y_0, z_0) is regular and that any one of conditions (V) and (VI) is fulfilled. If $S^* = \overline{\bigcup_{n=1}^{\infty} S_n^*}$ (the $w(W, Z)$ -closure of $\bigcup_{n=1}^{\infty} S_n^*$), then $\lim_{n \rightarrow \infty} M_n^* = M^*$.

PROOF. Since $M_n^* \leq M_{n+1}^* \leq M^*$, it suffices to show that $\lim_{n \rightarrow \infty} M_n^* \geq M^*$ in case $M^* > -\infty$. Let r be any number such that $M^* > r$. There exists $w \in S^*$ such that $((z_0, w))_2 > r$. For any $\varepsilon > 0$, we can find n_0 and \bar{w} such that $\bar{w} \in S_{n_0}^*$ and $((z_{n_0}, \bar{w}))_2 > r - \varepsilon$. It follows for each $n \geq n_0$ that

$$M_n^* \geq M_{n_0}^* \geq ((z_{n_0}, \bar{w}))_2 > r - \varepsilon,$$

and hence $\lim_{n \rightarrow \infty} M_n^* > r - \varepsilon$. By letting first $\varepsilon \rightarrow 0$ and next $r \rightarrow M^*$, we obtain the desired inequality.

LEMMA 2. Assume that $M_n \leq M_{n+1} \leq M$ and $\lim_{n \rightarrow \infty} M_n^* = M$. Then $\lim_{n \rightarrow \infty} M_n = M$.

PROOF. Since $M_n^* \leq M_n$ by Proposition 1, we have

$$M = \lim_{n \rightarrow \infty} M_n^* \leq \lim_{n \rightarrow \infty} M_n \leq M.$$

We shall prove

THEOREM 7. Assume that the program (A, P, Q, y_0, z_0) is regular and that $(P^+)^\circ \neq \phi$ and $Q^+ \cap A^{*-1}(y_0 - (P^+)^\circ) \neq \phi$. If any one of conditions (V) and (VI) is fulfilled, then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.

PROOF. By the dual statement of Theorem 2, we have $S^* = \overline{\bigcup_{n=1}^{\infty} S_n^*}$, so that $\lim_{n \rightarrow \infty} M_n^* = M^*$ by Lemma 1. Since $M = M^*$ by the dual statement of Proposition 1, our assertion follows from Lemma 2.

THEOREM 8. Assume that condition (IV) is fulfilled and that there exists $w \in \bigcup_{n=1}^{\infty} Q_n^+$ such that $y_0 - A^*w \in \bigcup_{n=1}^{\infty} (P_n^+)^\circ$. Then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.

PROOF. We have

$$\lim_{n \rightarrow \infty} M_n^* = m^* = \sup \{((z_0, w))_2; w \in C \text{ and } y_0 - A^*w \in D\},$$

where $C = \bigcup_{n=1}^{\infty} Q_n^+$ and $D = \bigcup_{n=1}^{\infty} P_n^+$ (cf. the corollary of Theorem 1). If there exists $w \in C$ such that $y_0 - A^*w \in D^\circ$ (the $s(Y, X)$ -interior of D), then we have $m^* = M$ by the dual statement of Proposition 1, since $D^+ = P$ and $C^+ = Q$ by condition (IV). By means of Lemma 2, we complete the proof.

Finally we have

THEOREM 9. *Let $(A_n, P_n, Q_n, \gamma_n, z_n)$ be a finite program and assume that $A_n = A$ for all n and condition (V) is fulfilled. Then it is valid that $\lim_{n \rightarrow \infty} M_n = M$.*

PROOF. By Theorem 4, we have $M_n \leq M_{n+1} \leq M$. It is enough to show that $\lim_{n \rightarrow \infty} M_n \geq M$ in case $\lim_{n \rightarrow \infty} M_n < \infty$. It follows from Proposition 2 that $S^* \neq \emptyset$. Let $M^*(z) = \sup\{((z, w))_2; w \in S^*\}$. Then $M^*(z_0) = M$ and $M^*(z_n) = M_n$ by Proposition 2. Since $M^*(z)$ is $w(Z, W)$ -lower semicontinuous, we have

$$\lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} M^*(z_n) \geq M^*(z_0) = M.$$

This completes the proof.

§ 5. Counter examples

We shall show by examples that any one of conditions (II), (IV), (V) and (VI) does not necessarily imply that $\lim_{n \rightarrow \infty} M_n = M$.

For a compact interval F in the real line, let $M(F)$ be the totality of Radon measures of any sign on F , $M^+(F)$ be the subset of $M(F)$ which consists of non-negative Radon measures, $C(F)$ be the totality of finite real-valued continuous functions on F and $C^+(F)$ be the subset of $C(F)$ which consists of non-negative functions. Denote by S_μ the support of $\mu \in M(F)$. We always assume that the real linear spaces $M(F)$ and $C(F)$ are paired under the bilinear functional $((,))$ defined by

$$((\mu, f)) = \int f d\mu \quad \text{for all } \mu \in M(F) \text{ and } f \in C(F).$$

We also assume that two n -dimensional Euclidean spaces R^n are paired under the bilinear functional defined by the ordinary inner product. Denote by R_0^n the positive orthant of R^n .

First we give

EXAMPLE 1. Let $X = Y = Z = W = R^2$, $P = Q = R_0^2$, $\gamma_0 = (1, 0)$ and $z_0 = (1, 0)$. For $x = (x_1, x_2) \in X$, define $A_n x$ and Ax by

$$A_n x = (x_1 + x_2, -x_2/2^n),$$

$$Ax = (x_1 + x_2, 0).$$

Consider the programs $(A_n, P, Q, \gamma_0, z_0)$ and (A, P, Q, γ_0, z_0) . Then $(A_{n+1} - A_n)(P) \subset Q$ and $\{A_n x\}$ converges to Ax . It is easily seen that $M_n = 1 > 0 = M$ for all n .

Kretschmer [3] gave

EXAMPLE 2. Let $F = [0, 1]$, $Z = C(F)$, $W = M(F)$, $X = Y = R$, $P = R_0$, $Q = C^+(F)$, $y_0 = 1$ and $(Ax)(t) = xt$. Define $\{z_n\} \subset Z$ by

$$z_n(t) = \min(t, (1-t)/(n-1)).$$

Then $z_n - z_{n+1} \in Q$ and $\{z_n\}$ $w(Z, W)$ -converges to $z_0 = 0$. Let us consider the programs (A, P, Q, y_0, z_n) and (A, P, Q, y_0, z_0) . It is shown that $M_n = 1 > 0 = M$ for all n .

Examples 1 and 2 show that condition (II) does not always imply that $\lim_{n \rightarrow \infty} M_n = M$.

Next we give

EXAMPLE 3. Let $F = [-1, 1]$, $X = M(F)$, $Y = C(F)$, $Z = W = R^2$, $P = M^+(F)$, $y_0(t) = -\min(t, 0)$, $z_0 = (-1, 0)$ and let

$$A\mu = \left(-\int_{-1}^1 |t| d\mu, -\int_0^1 t^2 d\mu \right)$$

for $\mu \in X$. Define Q_n and Q by

$$Q_n = \{(z_1, z_2); z_1 \leq 0, z_2 \leq 0 \text{ and } z_1/n \leq z_2\},$$

$$Q = \{(z_1, 0); z_1 \leq 0\}.$$

It is clear that $Q_{n+1} \subset Q_n$ and $Q = \bigcap_{n=1}^{\infty} Q_n$. Consider the programs (A, P, Q_n, y_0, z_0) and (A, P, Q, y_0, z_0) . We have $M_n = 0 < 1 = M$ for all n .

EXAMPLE 4. Let $F = [-1, 1]$, $X = M(F)$, $Y = C(F)$, $Z = W = R$, $Q = R_0$, $y_0(t) = -\min(t, 0)$, $z_0 = 1$ and let $A\mu = \int |t| d\mu$. Define P_n and P by

$$P_n = \{\mu \in M^+(F); S\mu \subset [-1, 1/n]\},$$

$$P = \{\mu \in M^+(F); S\mu \subset [-1, 0]\}.$$

It is clear that $P_{n+1} \subset P_n$ and $P = \bigcap_{n=1}^{\infty} P_n$. Consider the programs (A, P_n, Q, y_0, z_0) and (A, P, Q, y_0, z_0) . Then it is valid that $M_n = 0 < 1 = M$ for all n .

Examples 3 and 4 show that condition (IV) does not necessarily imply that $\lim_{n \rightarrow \infty} M_n = M$.

EXAMPLE 5. Let $X = Y = R^2$, $Z = W = R$, $P = R_0^2$, $Q = R_0$, $y_0 = (1, 0)$ and $z_0 = 1$. For $x = (x_1, x_2) \in X$, define $A_n x$ and Ax by

$$A_n x = x_1 + x_2/n \quad \text{and} \quad Ax = x_1.$$

Then $(A_n - A_{n+1})(P) \subset Q$ and $\{A_n x\}$ converges to Ax . Let us consider the

programs (A_n, P, Q, y_0, z_0) and (A, P, Q, y_0, z_0) . It is easily verified that $M_n = 0 < 1 = M$ for all n .

EXAMPLE 6. Let $F = [-1, 1]$, $X = M(F)$, $Y = C(F)$, $Z = W = R^2$, $P = M^+(F)$, $Q = R_0^2$, $y_0(t) = -\min(t, 0)$ and let

$$A\mu = \left(\int_{-1}^1 |t| d\mu, -\int_0^1 t^2 d\mu \right).$$

Define $\{z_n\} \subset Z$ by

$$z_n = (1, -1/n).$$

Then $z_{n+1} - z_n \in Q$ and $\{z_n\}$ converges to $z_0 = (1, 0)$.

Consider the programs (A, P, Q, y_0, z_n) and (A, P, Q, y_0, z_0) . Then we have $M_n = 0 < 1 = M$ for all n .

Examples 5 and 6 show that condition (V) does not always imply that $\lim_{n \rightarrow \infty} M_n = M$.

EXAMPLE 7. Let $F = [0, 1]$, $X = M(F)$, $Y = C(F)$, $Z = W = R$, $P = M^+(F)$, $Q = R_0$, $z_0 = -1$ and $A\mu = -\int t d\mu$. Define $\{y_n\} \subset Y$ by

$$y_n(t) = \max(-t, (t-1)/(n-1)).$$

Then $y_{n+1} - y_n \in P^+ = C^+(F)$ and $\{y_n\}$ $w(Y, X)$ -converges to $y_0 = 0$. Consider the programs (A, P, Q, y_n, z_0) and (A, P, Q, y_0, z_0) . It is shown that $M_n = -1 < 0 = M$ for all n . Namely condition (VI) does not necessarily imply that $\lim_{n \rightarrow \infty} M_n = M$.

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*School of Engineering,
Okayama University*