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Notes on Hausdorff Dimensions of Cartesian Product Sets

Kaoru HATANO

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§1. In the present paper we shall be concerned with evaluation of upper and lower bounds of fractional dimensions of Cartesian product sets by means of the fractional dimensions of their components. Various results on this problem have been obtained by A. S. Besicovitch, P. A. P. Moran, J. M. Marstrand and M. Ohtsuka ([1], [4], [6]). One of results is the following: for given α, $0\leqq\alpha\leqq n$, and β, $0\leqq\beta\leqq m$, if E_1 is a subset of R^n with dim $(E_1)=\alpha$ and E_2 is a subset of R^m with dim $(E_2)=\beta$, then $\alpha+\beta\leqq\dim(E_1\times E_2)\leqq\min\{n+\beta, m+\alpha\}$.

We use the notation $\Lambda_\alpha(E)$ for the α-dimensional measure of a set E in a Euclidean space. Note that dim $(E)=\alpha$ is equivalent to $\Lambda_{\alpha-\varepsilon}(E)=\infty$ and $\Lambda_{\alpha+\varepsilon}(E)=0$ for any $\varepsilon>0$.

In this note we shall show that for any given α, $0\leqq\alpha<n$, β, $0\leqq\beta<m$, and γ such that $\alpha+\beta<\gamma<\min\{n+\beta, m+\alpha\}$, there exist $E_1\subset R^n$ and $E_2\subset R^m$ which satisfy $0<\Lambda_\alpha(E_1)<\infty$, $0<\Lambda_\beta(E_2)<\infty$ and $0<\Lambda_\gamma(E_1\times E_2)<\infty$.

§2. Let R^n be the n-dimensional Euclidean space and let $h(r)$ be a continuous increasing function of r such that $h(r)>0$ for $r>0$ and $h(\mathrm{o})=0$. The Hausdorff Λ_h-measure of a subset E of R^n is defined as follows. First, for $\rho>0$, we set

$$\Lambda_h^{(\rho)}(E)=\inf\{\sum_{\nu=1}^{\infty}h(d_\nu)\},$$

where the infimum is taken over all coverings of E by at most a countable number of n-dimensional open (or closed) cubes I_ν with the sides $d_\nu\leqq\rho$. Then the Hausdorff measure of E is defined by

$$\Lambda_h(E)=\lim_{\rho\to0}\Lambda_h^{(\rho)}(E).$$

If $h(r)=r^\alpha(\alpha>0)$ (resp. $h(r)=1/\log(1/r)$), then we use the notation Λ_α (resp. Λ_o) instead of Λ_h. It is called the α-dimensional measure (resp. logarithmic measure).

The fractional dimension dim (E) of a subset $E\subset R^n$ is defined by

$$\dim(E)=\inf\{\alpha;\ \Lambda_\alpha(E)=0\}.$$

Let $\mathfrak{A}$ be the family of non empty open subsets in R^n which is determined by the following properties:

(1) any n-dimensional open cube belongs to $\mathfrak{A}$,

(2) if ω_1 and ω_2 belong to $\mathfrak{A}$, then so does $\omega_1 \cup \omega_2$,

(3) if ω is an element of $\mathfrak{A}$, then there exists a finite number of n-dimensional open cubes $I_\nu (\nu=1, 2, \dots, N)$ such that $\omega = \bigcup_{\nu=1}^{N} I_\nu$.

We shall refer the definition of the n-dimensional symmetric generalized Cantor set $E_{(n)}$ constructed by the system $[l, \{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=1}^\infty]$ to our previous paper [2]. In what follows we suppose $l=1$ and leave out l from the system. Also we set $\delta_q = (\lambda_{q-1} - k_q \lambda_q)/(k_q - 1)\ (q \geqq 1)$, where $\lambda_0 = 1$.

§3. LEMMA. (P. A. P. Moran [5]) *Let F be a compact set in R^n and let $\mathfrak{A}$ be the family defined in* §2. *Assume that there exists a non-negative set function Φ on $\mathfrak{A}$ satisfying the following conditions*:

(1) *if* $\omega = \bigcup_{i=1}^{N} \omega_i$, $\omega_i \in \mathfrak{A} (i=1, 2, \dots, N)$, *then* $\Phi(\omega) \leqq \sum_{i=1}^{N} \Phi(\omega_i)$,

(2) *if* $\omega \in \mathfrak{A}$ *contains* F, *then* $\Phi(\omega) \geqq b$, *where b is some positive constant*,

(3) *there exist positive constants a and d_o such that if I is any n-dimensional open cube with the side* $d \leqq d_o$, *then* $\Phi(I) \leqq ah(d)$.

Then $\Lambda_h(F) \geqq b/a$.

Using the Lemma we shall prove the following theorem.

THEOREM 1. *Let $E_{(n)}$ be the n-dimensional symmetric generalized Cantor set constructed by the system* $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=1}^\infty]$. *Then*

(i) $2^{-3n} \varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n \lambda_q^\alpha \leqq \Lambda_\alpha(E_{(n)}) \leqq \varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n \lambda_q^\alpha (0 < \alpha \leqq n)$.

(ii) $2^{-3n} \varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n / (\text{-}\log \lambda_q) \leqq \Lambda_o(E_{(n)}) \leqq \varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n / (\text{-}\log \lambda_q)$.

PROOF. From the definition of the Hausdorff measure the right-hand inequalities are obvious. Hence we shall prove the left-hand inequalities. We shall prove (i). If $\varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n \lambda_q^\alpha = 0$, then $\Lambda_\alpha(E_{(n)}) = \varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n \lambda_q^\alpha = 0$. In this case the inequality (i) is obvious. We set $\varliminf_{q\to\infty} (k_1 k_2 \cdots k_q)^n \lambda_q^\alpha = A > 0$. Let B be an arbitrary number such that $0 < B < A$. Then there exists a positive integer q_0 such that $(k_1 k_2 \cdots k_q)^n \lambda_q^\alpha > B$ for $q \geqq q_0$. We choose sequences $\{\lambda_q'\}_{q=q_0}^\infty$ and $\{\delta_q'\}_{q=q_0+1}^\infty$ such that $(k_1 k_2 \cdots k_q)^n \lambda_q'^\alpha = B$ for $q \geqq q_0$ and $k_q \lambda_q' + (k_q - 1) \delta_q' = \lambda_{q-1}'$ for $q \geqq q_0 + 1$. Clearly $\lambda_q' = B^{1/\alpha} (k_1 k_2 \cdots k_q)^{-n/\alpha} < \lambda_q$ and $\delta_q' = B^{1/\alpha} (k_q^{n/\alpha} - k_q)(k_1 k_2 \cdots k_q)^{-n/\alpha} (k_q - 1)^{-1}$. It is easy to see that the sequence $\{N_q(\omega) \lambda_q'^\alpha\}_{q=q_0}^\infty$ is a decreasing sequence for every $\omega \in \mathfrak{A}$, where $N_q(\omega)$ is the number of n-dimensional closed cubes in the q-th approximation of $E_{(n)}$ which meet ω. Now we define a set function Φ on $\mathfrak{A}$ by $\Phi(\omega) = \lim_{q\to\infty} N_q(\omega) \lambda_q'^\alpha$. We can easily see that Φ satisfies conditions (1) and (2) of the Lemma with $F = E_{(n)}$ and $b = B$.

We set $a=2^{3n}$ and $d_0=\lambda_{q_0+1}$. Let I be any open cube with the side $d\leqq d_0$. Then there exist uniquely determined positive integers $q(\geqq q_0+1)$ and $j(1\leqq j\leqq k_{q+1}-1)$ such that $\lambda_{q+1}<d\leqq\lambda_q$ and $j\lambda_{q+1}+(j-1)\delta_{q+1}<d\leqq(j+1)\lambda_{q+1}+j\delta_{q+1}$. Since $E_{(n)}$ is symmetric, we have $N_{q+1}(I)\leqq(2j)^n$. We observe

$$\Phi(I)\leqq(2j)^n\lambda'^{\alpha}_{q+1}=2^n\lambda'^{\alpha}_{q+1}\leqq 2^n\lambda^{\alpha}_{q+1}\leqq 2^n d^{\alpha}<ad^{\alpha}\qquad \text{if } j=1,$$

and

$$\Phi(I)\leqq(2j)^n\lambda'^{\alpha}_{q+1}\leqq 2^n(j\lambda'_{q+1}+(j-1)\delta'_{q+1})^{\alpha}\qquad \text{if } 2\leqq j\leqq k_{q+1}-1.$$

On the other hand,

$$j\lambda'_{q+1}+(j-1)\delta'_{q+1}\leqq j(\lambda'_{q+1}+\delta'_{q+1})\leqq 2jB^{1/\alpha}(k_1k_2\cdots k_q)^{-n/\alpha}k_{q+1}^{-1}\leqq 2j\lambda_q k_{q+1}^{-1}$$
$$\leqq 2j(\lambda_{q+1}+\delta_{q+1})\leqq 4\{j\lambda_{q+1}+(j-1)\delta_{q+1}\}<4d,\qquad \text{if } 2\leqq j\leqq k_{q+1}-1.$$

Hence $\Phi(I)\leqq 2^n4^{\alpha}d^{\alpha}\leqq 2^{3n}d^{\alpha}=ad^{\alpha}$. Thus Φ satisfies condition (3) of the Lemma. By the Lemma we obtain $\Lambda_{\alpha}(E_{(n)})\geqq 2^{-3n}B$. Since B is an arbitrary number such that $0<B<A$, we have $2^{-3n}\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^n\lambda_q^{\alpha}\leqq\Lambda_{\alpha}(E_{(n)})$. This is the desired inequality. By the same method we obtain the inequality (ii).

PROPOSITION. *Let n and m be positive integers and α and β be positive numbers such that $\alpha<n$ and $\beta<m$. Let E_1 be the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_{2q-1}\}_{q=1}^{\infty}, \{\lambda_{2q-1}\}_{q=1}^{\infty}]$ which satisfies $(k_1k_3\cdots k_{2q-1})^n\lambda_{2q-1}^{\alpha}=1$, $q=1, 2,\cdots$, and let E_2 be the m-dimensional symmetric generalized Cantor set constructed by the system $[\{k_{2q}\}_{q=1}^{\infty}, \{\lambda_{2q}\}_{q=1}^{\infty}]$ which satisfies $(k_2k_4\cdots k_{2q})^m\lambda_{2q}^{\beta}=1$, $q=1, 2,\cdots$. If at least one of the sequences $\{k_{2q-1}\}_{q=1}^{\infty}$, $\{k_{2q}\}_{q=1}^{\infty}$ is bounded, then*

$$\dim(E_1\times E_2)=\alpha+\beta.$$

We can prove this proposition by the method of F. Hausdorff [3], p. 177 and we omit the proof. This proposition shows that the lower bound of the fractional dimension of Cartesian product sets is attained. (cf. Section 1)

THEOREM 2. *Let n and m be positive integers and α, β and γ be positive numbers such that $n\leqq m$, $\alpha<m$ and $\alpha+\beta<\gamma<\min\{n+\beta, m+\alpha\}$. Then there exist subsets $E_1\subset R^n$ and $E_2\subset R^m$ such that $0<\Lambda_{\alpha}(E_1)<\infty$, $0<\Lambda_{\beta}(E_2)<\infty$ and $0<\Lambda_{\gamma}(E_1\times E_2)<\infty$.*

PROOF. We shall take as E_1 the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_{2q-1}\}_{q=1}^{\infty}, \{\lambda_{2q-1}\}_{q=1}^{\infty}]$ which satisfies $(k_1k_3\cdots k_{2q-1})^n\lambda_{2q-1}^{\alpha}=1$, $q=1, 2,\cdots$, and as E_2 the m-dimensional symmetric generalized Cantor set constructed by the system $[\{k_{2q}\}_{q=1}^{\infty}, \{\lambda_{2q}\}_{q=1}^{\infty}]$ which satisfies $(k_2k_4\cdots k_{2q})^m\lambda_{2q}^{\beta}=1$, $q=1, 2,\cdots$. One can easily check that $0<\Lambda_{\alpha}(E_1)$

$<\infty$ and $0<\Lambda_\beta(E_2)<\infty$.

Case 1: $\gamma<n$. We choose two sequences $\{k_{2q-1}\}_{q=1}^\infty$, $\{k_{2q}\}_{q=1}^\infty$ such that $k_{2q+1}=[(k_1k_3\cdots k_{2q-1})^\delta]$[1] and $k_{2q}=[(k_2k_4\cdots k_{2q-2})^{-1}\times(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha)/\alpha m}]$, where $k_0=1$ and $\gamma(\gamma-\alpha-\beta)/\alpha\beta<\delta<n(\gamma-\alpha-\beta)/\alpha\beta$. We can easily check that they satisfy the following conditions:

(1–1) $$(k_2k_4\cdots k_{2q-2})^{m/\beta}k_{2q}\leqq(k_1k_3\cdots k_{2q-1})^{n/\alpha},$$

(1–2) $$(k_1k_3\cdots k_{2q-1})^{n/\alpha}k_{2q+1}\leqq(k_2k_4\cdots k_{2q})^{m/\beta},$$

(1–3) $$(k_2k_4\cdots k_{2q})^{m(\gamma-\beta)/\beta}\leqq M(k_1k_3\cdots k_{2q+1})^n,$$

(1–4) $$(k_2k_4\cdots k_{2q})^m\leqq(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha)/\alpha}\leqq M(k_2k_4\cdots {}_{2q})^m,$$

where M is a positive constant.

Case 2: $n\leqq\gamma<n+\beta-\beta n/m$. Choose $\{k_q\}_{q=1}^\infty$ such that $k_{2q+1}=[2^{-m/\beta}(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha-\beta)/\alpha(n+\beta-\gamma)}]$ and $k_{2q}=[(k_2k_4\cdots k_{2q-2})^{-1}(k_1k_3\cdots k_{2q-1})^{\beta n(n-\alpha)/\alpha m(n+\beta-\gamma)}]$, where $k_0=1$. They satisfy the following conditions:

(2–1) $$(k_2k_4\cdots k_{2q-2})^{m/\beta}k_{2q}\leqq(k_1k_3\cdots k_{2q-1})^{n/\alpha},$$

(2–2) $$(k_1k_3\cdots k_{2q-1})^{n/\alpha}k_{2q+1}\leqq(k_2k_4\cdots k_{2q})^{m/\beta},$$

(2–3) $$(k_1k_3\cdots k_{2q-1})^{n/(\gamma-\alpha)/\alpha}k_{2q+1}^{\gamma-n}\leqq M(k_2k_4\cdots k_{2q})^m,$$

(2–4) $$M^{-1}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha}\leqq(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta}$$
$$\leqq(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha},$$

(2–5) $$(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha-\beta)/\alpha(n+\beta-\gamma)}\leqq Mk_{2q+1},$$

where M is a positive constant

Case 3: $n+\beta-\beta n/m\leqq\gamma<m$. Take the same $\{k_q\}_{q=1}^\infty$ as in case 2. They satisfy the following conditions:

(3–1) $$2(k_2k_4\cdots k_{2q})^{m/\beta}\leqq(k_1k_3\cdots k_{2q-1})^{n/\alpha}k_{2q+1},$$

(3–2) $$2(k_1k_3\cdots k_{2q-1})^{n/\alpha}\leqq(k_2k_4\cdots k_{2q-2})^{m/\beta}k_{2q},$$

(3–3) $$(k_2k_4\cdots k_{2q})^{m(\gamma-\beta)/\beta}\leqq M(k_1k_3\cdots k_{2q-1})^n,$$

(3–4) $$(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta}\leqq(k_1k_3\cdots k_{2q-1})^{(n-\alpha)/\alpha}$$
$$\leqq M(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta},$$

(3–5) $$(k_2k_4\cdots k_{2q2})^{m(m-\beta)/\beta}\leqq M(k_1k_3\cdots k_{2q-1})^{n(m+\alpha-\gamma)/\alpha},$$

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1) The Gaussian notation [x] stands for the greatest integer not exceeding a real number x. In the following we use this notation without explanation.

where M is a positive constant.

Case 4: $m \leqq \gamma < \min\{n+\beta,\ m+\alpha\}$. Choose $\{k_q\}_{q=1}^{\infty}$ such that $k_{2q+1} = [(k_1k_3\cdots k_{2q-1})^{(m+n-\gamma)(\gamma-\alpha-\beta)/(n+\beta-\gamma)(m+\alpha-\gamma)}]$ and $k_{2q} = [k_2k_4\cdots k_{2q-2})^{-1}(k_1k_3\cdots k_{2q-1})^{\beta n(n-\alpha)/\alpha m(n+\beta-\gamma)}]$, where $k_0=1$. They satisfy the following conditions:

(4-1) $$(k_2k_4\cdots k_{2q})^{m/\beta} \leqq M(k_1k_3\cdots k_{2q-1})^{n/\alpha}k_{2q+1},$$

(4-2) $$(k_1k_3\cdots k_{2q+1})^{n/\alpha} \leqq M(k_2k_4\cdots k_{2q})^{m/\beta}k_{2q+2},$$

(4-3) $$(k_2k_4\cdots k_{2q})^{m(m-\beta)/\beta} \leqq M(k_1k_3\cdots k_{2q+1})^{n(m+\alpha-\gamma)/\alpha},$$

(4-4) $$(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta} \leqq (k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha} \leqq M(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta},$$

where M is a positive constant.

First let us prove $\Lambda_\gamma(E_1 \times E_2) < \infty$. In case 1, $E_1 \times E_2$ is covered by $(k_1k_3\cdots k_{2q-1})^n(k_2k_4\cdots k_{2q})^m$ mutually congruent cubes in R^{n+m} with the side λ_{2q-1}. Hence

$$\Lambda_\gamma(E_1 \times E_2) \leqq \underline{\lim}_{q\to\infty}(k_1k_3\cdots k_{2q-1})^n(k_2k_4\cdots k_{2q})^m\lambda_{2q-1}^{\gamma}$$
$$\leqq \underline{\lim}_{q\to\infty}(k_1k_3\cdots k_{2q-1})^{-n(\gamma-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^m \leqq 1 < \infty.$$

In case 2, 3 and 4, $E_1 \times E_2$ is covered by at most $(k_1k_3\cdots k_{2q-1})^n(k_2k_4\cdots k_{2q})^m (2\lambda_{2q-1}/\lambda_{2q})^n$ mutually congruent closed cubes in R^{n+m} with the side λ_{2q}. Hence

$$\Lambda_\gamma(E_1 \times E_2) \leqq \underline{\lim}_{q\to\infty}(k_1k_3\cdots k_{2q-1})^n(k_2k_4\cdots k_{2q})^m(2\lambda_{2q-1}/\lambda_{2q})^n\lambda_{2q}^{\gamma}$$
$$\leqq 2^n\underline{\lim}_{q\to\infty}(k_1k_3\cdots k_{2q-1})^{-n(n-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^{m(n+\beta-\gamma)/\beta} \leqq 2^n < \infty.$$

Next using the Lemma we shall show $\Lambda_\gamma(E_1 \times E_2) > 0$. It is easy to see that $\lim_{q\to\infty} N_q(\omega)\lambda_{2q-1}^{\alpha}\lambda_{2q}^{\beta}$ exists for every $\omega \in \mathfrak{A}$, where $N_q(\omega)$ is the number of product sets of n-dimensional closed cubes in the q-approximation of E_1 and m-dimensional closed cubes in the q-approximation of E_2 which meet ω. Now we define a non negative set function Φ on $\mathfrak{A}$ by $\Phi(\omega) = \lim_{q\to\infty} N_q(\omega)\lambda_{2q-1}^{\alpha}\lambda_{2q}^{\beta}$. We can easily see that Φ satisfies (1) and (2) of the Lemma with $F = E_1 \times E_2$ and $b=1$. We shall show that Φ satisfies (3) of the Lemma in each of the cases. Let I be any open cube with the side $d \leqq \lambda_1$. Then there is a uniquely determined positive integer such that $\lambda_{2q+1} < d \leqq \lambda_{2q-1}$. In case 1, we shall estimate $N_{q+1}(I)$ by means of conditions (1-1) and (1-2), in each of the following four cases.

(i) If $\lambda_{2q+1}<d\leqq\lambda_{2q+2}+\delta_{2q+2}$, then $N_{q+1}(I)\leqq 2^{n+m}$ and

$$\Phi(I)\leqq 2^{n+m}\lambda_{2q+1}^{\alpha}\lambda_{2q+2}^{\beta}\leqq 2^{n+m}M\lambda_{2q+1}^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(ii) If $\lambda_{2q+2}+\delta_{2q+2}<d\leqq\lambda_{2q}$, then

$$N_{q+1}(I)\leqq 2^{n+m}(d/(\lambda_{2q+2}+\delta_{2q+2}))^{m}\leqq 2^{n+m}k_{2q+2}^{m}(k_2k_4\cdots k_{2q})^{m^2/\beta}d^{m-\gamma}d^{\gamma}$$
$$\leqq 2^{n+m}k_{2q+2}^{m}(k_2k_4\cdots k_{2q})^{m\gamma/\beta}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_2k_4\cdots k_{2q})^{m(\gamma-\beta)/\beta}(k_1k_3\cdots k_{2q+1})^{-n}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(iii) If $\lambda_{2q}<d\leqq\lambda_{2q+1}+\delta_{2q+1}$, then $N_{q+1}(I)\leqq 2^{n+m}k_{2q+2}^{m}$ and

$$\Phi(I)\leqq 2^{n+m}\lambda_{2q}^{\gamma}<2^{n+m}Md^{\gamma}.$$

(iv) If $\lambda_{2q+1}+\delta_{2q+1}<d\leqq\lambda_{2q-1}$, then

$$N_{q+1}(I)\leqq 2^{n+m}k_{2q+2}^{m}(d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}d^{n}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n\gamma/\alpha}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha)\alpha}(k_2k_4\cdots k_{2q})^{-m}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}$$

Therefore Φ satisfies condition (3) of the Lemma with $a=2^{n+m}M$ and $d_0=\lambda_1$. By the Lemma, we obtain $\Lambda_{\gamma}(E_1\times E_2)\geqq 2^{-(n+m)}M^{-1}>0$. Thus this case is proved. In case 2, we shall estimate $N_{q+1}(I)$ by means of conditions (2-1) and (2-2), in each of the following four cases.

(i) If $\lambda_{2q+1}<d\leqq\lambda_{2q+2}+\delta_{2q+2}$, then $N_{q+1}(I)\leqq 2^{n+m}$ and

$$\Phi(I)\leqq 2^{n+m}\lambda_{2q+1}^{\alpha}\lambda_{2q+2}^{\beta}\leqq 2^{n+m}M\lambda_{2q+1}^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(ii) If $\lambda_{2q+2}+\delta_{2q+2}<d\leqq\lambda_{2q}$, then

$$N_{q+1}(I)\leqq 2^{n+m}(d/(\lambda_{2q+2}+\delta_{2q+2}))^{m}$$
$$\leqq 2^{n+m}k_{2q+2}^{m}(k_2k_4\cdots k_{2q})^{m^2/\beta}d^{m-\gamma}d^{\gamma}\leqq 2^{n+m}k_{2q+2}^{m}\ (k_2k_4\cdots k_{2q})^{m\gamma/\beta}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_2k_4\cdots k_{2q})^{m(\gamma-\beta)/\beta}(k_1k_3\cdots k_{2q+1})^{-n}d^{\gamma}$$
$$\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)(\gamma-\beta)/\alpha(n+\beta-\gamma)}(k_1k_3\cdots k_{2q+1})^{-n}d^{\gamma}$$
$$\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n^2(\gamma-\alpha-\beta)/\alpha(n+\beta-\gamma)}k_{2q+1}^{-n}d^{\gamma}\leqq 2^{n+m}d^{\gamma}.$$

(iii) If $\lambda_{2q}<d\leqq\lambda_{2q+1}+\delta_{2q+1}$, then $N_{q+1}(I)\leqq 2^{n+m}k_{2q+2}^{m}$ and

$\Phi(I)\leqq 2^{n+m}k_{2q+2}^{m}\lambda_{2q+1}^{\alpha}\lambda_{2q+2}^{\beta}\leqq 2^{n+m}Md^{\gamma}$.

(iv) If $\lambda_{2q+1}+\delta_{2q+1}<d\leqq\lambda_{2q-1}$, then

$$N_{q+1}(I)\leqq 2^{n+m}k_{2q+2}^{m}\ (d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}d^{n-\gamma}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n(\gamma-\alpha)/\alpha}k_{2q+1}^{\gamma-n}(k_2k_4\cdots k_{2q})^{-m}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

Therefore Φ satisfies condition (3) of the Lemma with $a=2^{n+m}M$ and $d_0=\lambda_1$. By the Lemma we obtain $\Lambda_{\gamma}(E_1\times E_2)\geqq 2^{-(n+m)}M^{-1}>0$. Thus this case is proved. In case 3, we shall estimate $N_{q+1}(I)$ by means of conditions (3–1) and (3–2), in each of the following four cases.

(i) If $\lambda_{2q+1}<d\leqq\lambda_{2q+1}+\delta_{2q+1}$, there exists a positive integer $j(1\leqq j\leqq k_{2q+2}-1)$ such that $j(\lambda_{2q+2}+\delta_{2q+2})<d\leqq(j+1)(\lambda_{2q+2}+\delta_{2q+2})$. Then $N_{q+1}(I)\leqq 2^{n+m}j^{m}$ and

$$\Phi(I)\leqq 2^{n+m}j^{m}(k_1k_3\cdots k_{2q+1})^{-n}(k_2k_4\cdots k_{2q+2})^{-m}$$
$$\leqq 2^{n+m}M(jk_{2q+2}^{-1}(k_2k_4\cdots k_{2q})^{-m/\beta})^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(ii) If $\lambda_{2q+1}+\delta_{2q+1}<d\leqq\lambda_{2q}$, then

$$N_{q+1}(I)\leqq(2d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}(2d/(\lambda_{2q+2}+\delta_{2q+2}))^{m}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q})^{m^2/\beta}d^{n+m-\gamma}d^{\gamma}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q})^{-m(n-\gamma)/\beta}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^{-m(n+\beta-\gamma)/\beta}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(iii) If $\lambda_{2q}<d\leqq\lambda_{2q}+\delta_{2q}$, then $N_{q+1}(I)\leqq(2k_{2q+2})^{m}(2d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}$

$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}d^{n-\gamma}d^{\gamma}$$
$$\leqq 2^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q})^{m(\gamma-n)/\beta}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^{-m(n+\beta-\gamma)/\beta}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

(iv) If $\lambda_{2q}+\delta_{2q}<d\leqq\lambda_{2q-1}$, then

$$N_{q+1}(I)\leqq(2d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}(2k_{2q+2}d/(\lambda_{2q}+\delta_{2q}))^{m}$$

$$\leqq 2^{n+m}k_{2q+1}^{n}(k_{2q}k_{2q+2})^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q-2})^{m^2/\beta}d^{n+m-\gamma}d^{\gamma}$$

$$\leqq 2^{n+m}k_{2q+1}^{n}(k_{2q}k_{2q+2})^{m}(k_1k_3\cdots k_{2q-1})^{-n(m-\gamma)/\alpha}(k_2k_4\cdots k_{2q-2})^{m^2/\beta}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}(k_1k_3\cdots k_{2q-1})^{-n(m+\alpha-\gamma)/\alpha}(k_2k_4\cdots k_{2q-2})^{m(m-\beta)/\beta}d^{\gamma}\leqq 2^{n+m}Md^{\gamma}.$$

Therefore Φ satisfies condition (3) of the Lemma with $a=2^{n+m}M$ and $d_0=\lambda_1$. By the Lemma we obtain $\Lambda_\gamma(E_1\times E_2)\geqq 2^{-(n+m)}M^{-1}>0$. Thus this case is proved. In case 4, we shall estimate $N_{q+1}(I)$ by means of conditions (4–1) and (4–2), in each of the following four cases. By the above conditions (4–1) and (4–2), there exists a constant C ($\geqq 1$) such that $\lambda_{2q+1}+\delta_{2q+1}\leqq C\lambda_{2q}$ and $\lambda_{2q}+\delta_{2q}\leqq C\lambda_{2q-1}$.

(i) If $\lambda_{2q+1}<d\leqq\lambda_{2q+1}+\delta_{2q+1}$, then $N_{q+1}(I)\leqq 2^{n}(2Cd/(\lambda_{2q+2}+\delta_{2q+2}))^{m}$

and

$$\Phi(I)\leqq 2^{n+m}C^{m}(k_1k_3\cdots k_{2q+1})^{-n}(k_2k_4\cdots k_{2q})^{m(m-\beta)/\beta}d^{m-\gamma}d^{\gamma}$$

$$\leqq 2^{n+m}C^{m}(k_1k_3\cdots k_{2q+1})^{-n(m+\alpha-\gamma)/\alpha}(k_2k_4\cdots k_{2q})^{m(m-\beta)/\beta}d^{\gamma}\leqq 2^{n+m}C^{m}Md^{\gamma}.$$

(ii) If $C^{-1}(\lambda_{2q+1}+\delta_{2q+1})<d\leqq\lambda_{2q}$, then

$$N_{q+1}(I)\leqq(2Cd/(\lambda_{2q+1}+\delta_{2q+1}))^{n}(2C^2d/(\lambda_{2q+2}+\delta_{2q+2}))^{m}$$

$$\leqq(2C^2)^{n+m}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q})^{m^2/\beta}d^{n+m-\gamma}d^{\gamma}$$

and

$$\Phi(I)\leqq(2C^2)^{n+m}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^{-m(n+\beta-\gamma)/\beta}d^{\gamma}$$

$$\leqq(2C^2)^{n+m}Md^{\gamma}.$$

(iii) If $\lambda_{2q}<d\leqq\lambda_{2q}+\delta_{2q}$, then $N_{q+1}(I)\leqq(2k_{2q+2})^{m}(2Cd/(\lambda_{2q+1}+\delta_{2q+1}))^{n}$

$$\leqq 2^{n+m}C^{n}k_{2q+1}^{n}k_{2q+2}^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}d^{n-\gamma}d^{\gamma}$$

and

$$\Phi(I)\leqq 2^{n+m}C^{n}(k_1k_3\cdots k_{2q-1})^{n(n-\alpha)/\alpha}(k_2k_4\cdots k_{2q})^{-m(n+\beta-\gamma)/\beta}d^{\gamma}$$

$$\leqq 2^{n+m}C^{n}Md^{\gamma}.$$

(iv) If $C^{-1}(\lambda_{2q}+\delta_{2q})<d\leqq\lambda_{2q-1}$, then

$$N_{q+1}(I)\leqq(2C^2d/(\lambda_{2q+1}+\delta_{2q+1}))^{n}(2k_{2q+2}C^2d/(\lambda_{2q}+\delta_{2q}))^{m}$$

$$\leqq(2C^2)^{n+m}k_{2q+1}^{n}(k_{2q}k_{2q+2})^{m}(k_1k_3\cdots k_{2q-1})^{n^2/\alpha}(k_2k_4\cdots k_{2q-2})^{m^2/\beta}d^{n+m-\gamma}d^{\gamma}$$

and

$$\Phi(I) \leqq (2C^2)^{n+m}(k_1k_3\cdots k_{2q-1})^{-n(m+\alpha-\gamma)/\alpha}(k_2k_4\cdots k_{2q-2})^{m(m-\beta)/\beta}d^\gamma$$

$$\leqq (2C^2)^{n+m}Md^\gamma.$$

Therefore Φ satisfies condition (3) of the Lemma with $a=(2C^2)^{n+m}M$ and $d_0=\lambda_1$. By the Lemma we obtain $\Lambda_\gamma(E_1\times E_2)\geqq(2C^2)^{-(n+m)}M^{-1}>0$. Thus the theorem is proved.

§4. We can establish the following theorems.

THEOREM 3. *Let n and m be positive integers and β and γ be positive numbers such that $\beta<\gamma<\min\{n+\beta, m\}$. Then there exist subsets $E_1\subset R^n$ and $E_2\subset R^m$ such that $0<\Lambda_0(E_1)<\infty$, $0<\Lambda_\beta(E_2)<\infty$ and $0<\Lambda_\gamma(E_1\times E_2)<\infty$.*

THEOREM 4. *Let n and m be positive integers. For any given positive number γ smaller than $\min\{n, m\}$, there exist subsets $E_1\subset R^n$ and $E_2\subset R^m$ such that $0<\Lambda_0(E_1)<\infty$, $0<\Lambda_0(E_2)<\infty$ and $0<\Lambda_\gamma(E_1\times E_2)<\infty$.*

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Department of Mathematics,
Faculty of Science,
Hiroshima University