HIROSHIMA MATH. J.
1 (1971), 349-372

On One-step Methods Utilizing the Second Derivative

Hisayoshi SHINTANI

(Received September 20, 1971)

1. Introduction

Given a differential equation

(1.1) $$y' = f(x, y)$$

and the initial condition $y(x_0) = y_0$, where the function

(1.2) $$g(x, y) = f_x(x, y) + f(x, y) f_y(x, y)$$

is assumed to be sufficiently smooth. Let

(1.3) $$x_i = x_0 + ih, \ y_i = y(x_i) \quad (i = 1, 2, \dots),$$

where h is a small increment in x and $y(x)$ is the solution to the given initial value problem. We are concerned with the case where the approximate values z_i of y_i $(i = 1, 2, \dots)$ are computed by means of the one-step methods, and put

(1.4) $$T(x_0, y_0; h) = z_1 - y_1.$$

The one-step method of order p with μ stages for approximating y_1 can be expressed as follows:

(1.5) $$z_1 = y_0 + h \sum_{i=1}^{\mu} q_i t_i,$$

where

(1.6) $$T(x_0, y_0; h) = O(h^{p+1}),$$

(1.7) $$t_i = f(x_0 + a_i h, \ y_0 + h \sum_{j=1}^{\mu} b_{ij} t_j),$$

(1.8) $$\sum_{j=1}^{\mu} b_{ij} = a_i \quad (i = 1, 2, \dots, \mu).$$

The method is called *explicit* when $b_{ij} = 0$ for $j \geqq i$. It is well known [2][1] that

1) Numbers in square brackets refer to the references listed at the end of this paper.

the explicit one-step methods of order p ($p=1, 2, 3, 4$) exist only for $\mu \geqq p$, that the methods of order p ($p=5, 6$) exist only for $\mu \geqq p+1$, and that the method of order 7 exists only for $\mu \geqq 9$.

In this paper, we are concerned with the one-step methods that utilize not only $f(x, y)$ but also $g(x, y)$ [22]. We consider first the explicit one-step methods of the type

$$z_1 = y_0 + hk_0 + h^2 \sum_{i=1}^{r} p_i l_i, \tag{1.9}$$

where

$$k_0 = f(x_0, y_0), \tag{1.10}$$

$$l_i = g(x_0 + a_i h,\ y_0 + a_i h k_0 + h^2 \sum_{j=1}^{i-1} b_{ij} l_j) \quad (i=1, 2, \cdots, r). \tag{1.11}$$

Next, considering

$$u_1 = z_1 - y_0 - hk_0 \tag{1.12}$$

as an unknown, we are concerned with the implicit one-step methods of the type A

$$z_1 = y_0 + hk_0 + h^2 \sum_{i=1}^{r} p_i l_i \tag{1.13}$$

and those of the type B

$$z_1 = y_0 + hk_0 + p_0 h(k_1 - k_0) + h^2 \sum_{i=1}^{r} p_i l_i, \tag{1.14}$$

where

$$l_i = g(x_0 + a_i h,\ y_0 + a_i h k_0 + h^2 \sum_{j=1}^{i-1} b_{ij} l_j + c_i u_1) \quad (i=1, 2, \cdots, r), \tag{1.15}$$

$$k_1 = f(x_1,\ y_0 + hk_0 + u_1). \tag{1.16}$$

The methods of the type B are considered because k_1 is used as k_0 in the next step of integration.

The object of this paper is to show that the explicit methods of order $r+2$ exist for $r=1, 2, 3, 4, 5$, that the implicit methods of the type A of order $r+3$ exist for $r=2, 3, 4$ and that those of the type B of order $r+3$ exist for $r=1, 2, 3, 4$. To reduce the number of evaluations of the functions in the implicit methods, the auxiliary formulas for approximating u_1 are given both for the single-step process and for the two-step process. Finally numerical examples are presented.

Computations have been carried out by the use of TOSBAC 3400 at Department of Mathematics, Faculty of Science, Hiroshima University. The author wishes to acknowledge the assistance of Mr. Kenji Tomoeda who has written the program.

2. Preliminaries

2.1 Notations

Let D be a differential operator defined by

$$D=\frac{\partial}{\partial x}+k_0\frac{\partial}{\partial y} \tag{2.1}$$

and put

$$D^i g(x_0, y_0)=Z_i, \quad D^i g_y(x_0, y_0)=Y_i, \quad D^i g_{yy}(x_0, y_0)=X_i, \tag{2.2}$$
$$D^i g_{yyy}(x_0, y_0)=W_i \quad (i=0, 1, 2,\dots).$$

Then $g(x_0+ah, y_0+ahk_0+V)$ and $y_0^{(i)}=y^{(i)}(x_0)$ $(i=1, 2,\dots)$ can be expanded as follows:

$$g(x_0+ah, y_0+ahk_0+V)=\sum_{j\geqq 0}\frac{1}{j!}a^jh^jZ_j+V\sum_{j\geqq 0}\frac{1}{j!}a^jh^jY_j \tag{2.3}$$
$$+V^2\sum_{j\geqq 0}\frac{1}{j!2}a^jh^jX_j+V^3\sum_{j\geqq 0}\frac{1}{j!6}a^jh^jW_j+\cdots,$$

$$y_0^{(1)}=k_0,\ y_0^{(2)}=Z_0,\ y_0^{(3)}=Z_1,\ y_0^{(4)}=Z_2+Z_0Y_0, \tag{2.4}$$

$$y_0^{(5)}=Z_3+3Z_0Y_1+Z_1Y_0, \tag{2.5}$$

$$y_0^{(6)}=Z_4+6Z_0Y_2+4Z_1Y_1+Z_2Y_0+Z_0Y_0^2+3Z_0^2X_0, \tag{2.6}$$

$$y_0^{(7)}=Z_5+10Z_0Y_3+10Z_1Y_2+5Z_2Y_1+Z_3Y_0+8Z_0Y_0Y_1 \tag{2.7}$$
$$+Z_1Y_0^2+10Z_0Z_1X_0+15Z_0^2X_1,$$

$$y_0^{(8)}=Z_6+15Z_0Y_4+20Z_1Y_3+15Z_2Y_2+6Z_3Y_1+Z_4Y_0+21Z_0Y_0Y_2 \tag{2.8}$$
$$+10Z_1Y_0Y_1+18Z_0Y_1^2+Z_2Y_0^2+Z_0Y_0^3+18Z_0^2Y_0X_0+15Z_0Z_2X_0$$
$$+60Z_0Z_1X_1+10Z_1^2X_0+45Z_0^2X_2+15Z_0^3W_0.$$

Since the formula (1.14) includes the formulas (1.9) and (1.13) as special cases, we are concerned with it throughout this section. Put for simplicity

(2.9) $$d_{ij}=i(i+1)\sum_{k=1}^{j-1} a_k^{i-1}b_{jk}+c_j \quad (i, j=1, 2,\cdots, r),$$

(2.10) $$e_{ij}=(i+2)(i+3)\sum_{k=1}^{j-1} a_k^{i-1}d_{1k}b_{jk}+c_j,$$

(2.11) $$l_{ij}=(i+3)(i+4)\sum_{k=1}^{j-1} a_k^{i-1}d_{2k}b_{jk}+c_j,$$

(2.12) $$m_{ij}=(i+4)(i+5)\sum_{k=1}^{j-1} a_k^{i-1}d_{3k}b_{jk}+c_j,$$

(2.13) $$q_{ij}=(i+4)(i+5)\sum_{k=1}^{j-1} a_k^{i-1}d_{1k}^2b_{jk}+c_j.$$

(2.14) $$r_{ij}=(i+4)(i+5)\sum_{k=1}^{j-1} a_k^{i-1}e_{1k}b_{jk}+c_j.$$

Then, by substituting the expression

$$u_1=\frac{1}{2!}h^2y_0^{(2)}+\frac{1}{3!}h^3y_0^{(3)}+\cdots$$

into l_i $(i=1, 2,\cdots, r)$ and k_1, z_1 in (1.14) can be expanded as follows:

(2.15) $$\begin{aligned} z_1&=y_0+hk_0+p_0h(k_1-k_0)+h^2\sum_{i=1}^{r} p_il_i \\ &=y_0+hk_0+h^2A_0Z_0+h^3A_1Z_1+\frac{1}{2!}h^4(A_2Z_2+A_3Z_0Y_0) \\ &\quad+\frac{1}{3!}h^5(A_4Z_3+3A_5Z_0Y_1+A_6Z_1Y_0)+\frac{1}{4!}h^6(B_1Z_4+6B_2Z_0Y_2 \\ &\quad+4B_3Z_1Y_1+B_4Z_2Y_0+B_5Z_0Y_0^2+3B_6Z_0^2X_0)+\frac{1}{5!}h^7(C_1Z_5 \\ &\quad+10C_2Z_0Y_3+10C_3Z_1Y_2+5C_4Z_2Y_1+C_5Z_3Y_0+8C_6Z_0Y_0Y_1 \\ &\quad+C_7Z_1Y_0^2+10C_8Z_0Z_1X_0+15C_9Z_0^2X_1)+\frac{1}{6!}h^8(D_1Z_6 \\ &\quad+15D_2Z_0Y_4+20D_3Z_1Y_3+15D_4Z_2Y_2+6D_5Z_3Y_1+D_6Z_4Y_0 \\ &\quad+21D_7Z_0Y_0Y_2+10D_8Z_1Y_0Y_1+18D_9Z_0Y_1^2+D_{10}Z_2Y_0^2 \\ &\quad+D_{11}Z_0Y_0^3+18D_{12}Z_0^2Y_0X_0+15D_{13}Z_0Z_2X_0+60D_{14}Z_0Z_1X_1 \\ &\quad+10D_{15}Z_1^2X_0+45D_{16}Z_0^2X_2+15D_{17}Z_0^3W_0)+\cdots, \end{aligned}$$

where

(2.16) $$A_0=p_0+\sum_{i=1}^{r}p_i,\ A_1=\frac{1}{2}p_0+\sum_{i=1}^{r}a_ip_i,$$

(2.17) $$A_2=\frac{1}{3}p_0+\sum_{i=1}^{r}a_i^2p_i,\ A_3=A_2+\Sigma(d_{1i}-a_i^2)p_i,$$

(2.18) $$A_4=\frac{1}{4}p_0+\Sigma a_i^3p_i,\ A_5=A_4+\Sigma a_i(d_{1i}-a_i^2)p_i,$$

$$A_6=A_4+\Sigma(d_{2i}-a_i^3)p_i,$$

(2.19) $$B_1=\frac{1}{5}p_0+\Sigma a_i^4p_i,\ B_2=B_1+\Sigma a_i^2(d_{1i}-a_i^2)p_i,$$

$$B_3=B_1+\Sigma a_i(d_{2i}-a_i^3)p_i,\quad B_4=B_1+\Sigma(d_{3i}-a_i^4)p_i,$$

$$B_5=B_1+\Sigma(e_{1i}-a_i^4)p_i,\ B_6=B_1+\Sigma(d_{1i}^2-a_i^4)p_i,$$

(2.20) $$C_1=\frac{1}{6}p_0+\Sigma a_i^5p_i,\ C_2=C_1+\Sigma a_i^3(d_{1i}-a_i^2)p_i,$$

$$C_3=C_1+\Sigma a_i^2(d_{2i}-a_i^3)p_i,\ C_4=C_1+\Sigma a_i(d_{3i}-a_i^4)p_i,$$

$$C_5=C_1+\Sigma(d_{4i}-a_i^5)p_i,\ C_7=C_1+\Sigma(l_{1i}-a_i^5)p_i,$$

$$C_6=C_1+\frac{3}{8}\Sigma(e_{2i}-a_i^5)p_i+\frac{5}{8}\Sigma a_i(e_{1i}-a_i^4)p_i,$$

$$C_8=C_1+\Sigma(d_{1i}d_{2i}-a_i^5)p_i,\ C_9=C_1+\Sigma a_i(d_{1i}^2-a_i^4)p_i,$$

(2.21) $$D_1=\frac{1}{7}p_0+\Sigma a_i^6p_i,\ D_2=D_1+\Sigma a_i^4(d_{1i}-a_i^2)p_i,$$

$$D_3=D_1+\Sigma a_i^3(d_{2i}-a_i^3)p_i,\ D_4=D_1+\Sigma a_i^2(d_{3i}-a_i^4)p_i,$$

$$D_5=D_1+\Sigma a_i(d_{4i}-a_i^5)p_i,\ D_6=D_1+\Sigma(d_{5i}-a_i^6)p_i,$$

$$D_7=D_1+\frac{2}{7}\Sigma(e_{3i}-a_i^6)p_i+\frac{5}{7}\Sigma a_i^2(e_{1i}-a_i^4)p_i,$$

$$D_8=D_1+\frac{2}{5}\Sigma(l_{2i}-a_i^6)p_i+\frac{3}{5}\Sigma a_i(l_{1i}-a_i^5)p_i,$$

$$D_9=D_1+\Sigma a_i(e_{2i}-a_i^5)p_i,\ D_{10}=D_1+\Sigma(m_{1i}-a_i^6)p_i,$$

$$D_{11}=D_1+\Sigma(r_{1i}-a_i^6)p_i,\ D_{13}=D_1+\Sigma(d_{1i}d_{3i}-a_i^6)p_i,$$

$$D_{12}=D_1+\frac{1}{6}\Sigma(q_{1i}-a_i^6)p_i+\frac{5}{6}\Sigma(d_{1i}e_{1i}-a_i^6)p_i,$$

$$D_{14}=D_1+\Sigma a_i(d_{1i}d_{2i}-a_i^5)p_i,\ D_{15}=D_1+\Sigma(d_{2i}^2-a_i^6)p_i,$$

$$D_{16}=D_1+\Sigma a_i^2(d_{1i}^2-a_i^4)p_i,\ D_{17}=D_1+\Sigma(d_{1i}^3-a_i^6)p_i.$$

2.2 Special cases

If we impose the condition that

$$d_{1j}=a_j^2 \quad (j=1, 2,\cdots, r), \tag{2.22}$$

then it follows that

$$e_{ij}=d_{i+2,j},\quad q_{ij}=d_{i+4,j},\quad r_{ij}=m_{ij}, \tag{2.23}$$

$$c_1=a_1^2,\ \ 2\sum_{k=1}^{j-1} b_{jk}+c_j=a_j^2 \quad (j=2, 3,\cdots, r), \tag{2.24}$$

$$A_3=A_2,\ A_5=A_4,\ B_2=B_6=B_1,\ B_5=B_4,\ C_2=C_9=C_1, \tag{2.25}$$

$$C_6=\frac{1}{8}(5C_4+3C_5),\ C_8=C_3,\ D_2=D_{16}=D_{17}=D_1,$$

$$D_7=\frac{1}{7}(5D_4+2D_6),\ D_9=D_5,\ D_{11}=D_{10},\ D_{12}=\frac{1}{6}(5D_4+D_6),$$

$$D_{13}=D_4,\ D_{14}=D_3.$$

In addition to the condition (2.22), if we impose further the condition that

$$d_{2j}=a_j^3 \quad (j=1, 2,\cdots, r), \tag{2.26}$$

then it follows that

$$l_{ij}=d_{i+3,j}, \tag{2.27}$$

$$a_1(a_1-1)=0,\ \ 2\sum_{k=1}^{j-1}(3a_k-1)b_{jk}=a_j^2(a_j-1) \quad (j=2, 3,\cdots, r), \tag{2.28}$$

$$A_6=A_4,\ B_3=B_1,\ C_3=C_8=C_1,\ C_7=C_5, \tag{2.29}$$

$$D_3=D_{14}=D_{15}=D_1,\ D_8=\frac{1}{5}(3D_5+2D_6).$$

Hence, for the choice $a_1=0$, it must be valid that

$$c_1=0,\ b_{21}=\frac{1}{2}a_2^2(1-a_2),\ c_2=a_2^3; \tag{2.30}$$

for the choice $a_1=1$, it must hold that

(2.31) $$c_1=1,\ b_{21}=\frac{1}{4}a_2^2(a_2-1),\ c_2=\frac{1}{2}a_2^2(3-a_2).$$

Besides the conditions (2.22) and (2.26), if we impose the condition that

(2.32) $$d_{3j}=a_j^4 \quad (j=1, 2, 3,\cdots, r),$$

then it follows that

(2.33) $$m_{ij}=r_{ij}=d_{i+4,j},$$

(2.34) $$a_2(a_2-1)\,[a_2(3a_1-1)-3a_1(2a_1-1)]=0,$$

(2.35) $$6\sum_{k=1}^{j-1} a_k(2a_k-1)b_{jk}=a_j^3(a_j-1) \quad (j=2, 3\cdots, r),$$

(2.36) $$B_4=B_5=B_1,\ C_4=C_1,\ D_4=D_{13}=D_1,\ D_{11}=D_{10}=D_6.$$

Hence, for the choice $a_1=0$, it must be valid that

(2.37) $$c_1=0,\ a_2=c_2=1,\ b_{21}=0;$$

the choice $a_1=1$ leads to the condition $a_2=\frac{3}{2}$, so that this case will be excluded usually because $a_2>1$.

2.3 Explicit formulas

In the case of explicit formulas, since

(2.38) $$p_0=0,\ c_i=0 \quad (i=1, 2,\cdots, r),$$

it follows that

(2.39) $$d_{i1}=0,\ e_{i1}=e_{i2}=0,\ l_{i1}=l_{i2}=0,\ m_{i1}=m_{i2}=0,$$
$$q_{i1}=q_{i2}=0,\ r_{i1}=r_{i2}=r_{i3}=0.$$

If we impose the condition (2.22), then it follows that

(2.40) $$a_1=0, \quad b_{21}=\frac{1}{2}a_2^2,$$

(2.41) $$d_{k2}=0\ (k=2, 3,\cdots, r),\ l_{i3}=m_{i3}=0 \quad (i=1, 2,\cdots, r),$$

and the conditions (2.23), (2.24) and (2.25) are valid

In addition to the condition (2.22), if we impose the condition that

(2.42) $$p_2=0,\ d_{2j}=a_j^3 \quad (j=3, 4,\cdots, r),$$

it follows that

(2.43) $$A_6=A_4,\ B_3=B_1,\ C_3=C_1,\ D_3=D_{15}=D_1,$$

$$(2.44)\qquad a_2^2\left[C_5-\frac{5}{3}(a_3+a_4)B_4+\frac{10}{3}a_3a_4A_4\right]+(a_4-a_2)(a_3-a_2)(C_7-C_5)$$

$$=20a_2^2\sum_{j=6}^{r}p_j\sum_{k=5}^{j-1}a_k(a_k-a_3)(a_k-a_4)b_{jk}.$$

3. Explicit formulas

3.1 Formula E-3

The formula of order 3 for $r=1$ is determined uniquely as follows:

$$(3.1)\qquad a_1=\frac{1}{3},\quad p_1=\frac{1}{2},$$

$$(3.2)\qquad T(x_0,\ y_0;\ h)=\frac{1}{4!}h^4\left(-\frac{1}{3}Z_2-Z_0Y_0\right)+\boldsymbol{O}(h^5).$$

3.2 Formula E-4

To obtain the formula of order 4 for $r=2$, we require that

$$T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^5),\quad A_4=\frac{1}{20}.$$

Then it follows that

$$a_1,\ a_2=\frac{4\pm\sqrt{6}}{10}.$$

In particular, we have the formulas as follows:

$$(3.3)\qquad a_1=\frac{4-\sqrt{6}}{10},\quad a_2=\frac{4+\sqrt{6}}{10},\quad b_{21}=\frac{9+\sqrt{6}}{50},$$

$$p_1=\frac{9+\sqrt{6}}{36},\quad p_2=\frac{9-\sqrt{6}}{36},$$

$$(3.4)\qquad T(x_0,\ y_0,\ h)=\frac{1}{5!}h^5\frac{(\sqrt{6}-2)}{2}(Z_0Y_1-Z_1Y_0)+\boldsymbol{O}(h^6).$$

3.3 Formula E-5

Under the condition (2.22), to obtain the formula of order 5 for $r=3$, we require that

$$T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^6),\quad B_1=\frac{1}{30}.$$

Then it follows that

$$a_2,\ a_3=\frac{5\pm\sqrt{5}}{10}.$$

In particular, we have the following formulas:

$$(3.5)\qquad a_1=0,\ a_2=\frac{5-\sqrt{5}}{10},\ b_{21}=\frac{3-\sqrt{5}}{20},$$

$$a_3=\frac{5+\sqrt{5}}{10},\ b_{31}=0,\ b_{32}=\frac{3+\sqrt{5}}{20},$$

$$p_1=\frac{1}{12},\ p_2=\frac{5+\sqrt{5}}{24},\ p_3=\frac{5-\sqrt{5}}{24},$$

$$(3.6)\qquad T(x_0,\ y_0;\ h)=-\frac{1}{6!}h^6\frac{12}{5}(3\sqrt{5}-5)(Z_2Y_0-2Z_1Y_1+Z_0Y_0^2)+\boldsymbol{O}(h^7).$$

3.4 Formula E-6

Under the condition (2.22), to have the formula of order 6 for $r=4$, we require that

$$T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^7),\ C_1=\frac{1}{42},\ D_1=\frac{1}{56}.$$

Then it follows that

$$a_2,\ a_3,\ a_4=\frac{1}{2},\ \frac{7\pm\sqrt{21}}{14}.$$

In particular, we have the formulas as follows:

$$(3.7)\qquad a_1=0,\ a_2=\frac{7-\sqrt{21}}{14},\ b_{21}=\frac{5-\sqrt{21}}{28},$$

$$a_3=\frac{1}{2},\ b_{31}=\frac{3-\sqrt{21}}{192},\ b_{32}=\frac{21+\sqrt{21}}{192},$$

$$a_4=\frac{7+\sqrt{21}}{14},\ b_{41}=\frac{21+5\sqrt{21}}{294},\ b_{42}=\frac{\sqrt{21}-3}{84},\ b_{43}=\frac{21+\sqrt{21}}{147},$$

$$p_1=\frac{1}{20},\ p_2=\frac{7(7+\sqrt{21})}{360},\ p_3=\frac{8}{45},\ p_4=\frac{7(7-\sqrt{21})}{360},$$

$$(3.8)\qquad T(x_0,\ y_0;\ h)=\frac{1}{7!}h^7\left[\frac{(\sqrt{21}-5)}{12}(3Z_1Y_2-6Z_2Y_1+Z_3Y_0+3Z_0Z_1X_0\right.$$

$$-3Z_0Y_0Y_1)+\frac{1}{24}(95-21\sqrt{21})Z_1Y_0^2\Big]+\boldsymbol{O}(h^8).$$

3.5 Formula E-7

Under the conditions (2.22) and (2.42), to obtain the formula of order 7 for $r=5$, we require that $T(x_0, y_0; h)=\boldsymbol{O}(h^8)$. Then, by (2.44), it follows that

$$a_3,\ a_4=\frac{3\pm\sqrt{2}}{7},\quad a_5=1.$$

In particular, we have the formulas as follows:

$$(3.9)\quad a_1=0,\ a_2=\frac{1}{2},\ b_{21}=\frac{1}{8},$$

$$a_3=\frac{3-\sqrt{2}}{7},\ b_{31}=\frac{141-68\sqrt{2}}{2058},\ b_{32}=\frac{45-29\sqrt{2}}{1029},$$

$$a_4=\frac{3+\sqrt{2}}{7},\ b_{41}=\frac{255+50\sqrt{2}}{14406},\ b_{42}=\frac{195-103\sqrt{2}}{7203},\ b_{43}=\frac{162+173\sqrt{2}}{2401},$$

$$a_5=1,\ b_{51}=\frac{\sqrt{2}-1}{2},\ b_{52}=\frac{3\sqrt{2}-5}{3},\ b_{53}=\frac{5-3\sqrt{2}}{6},\ b_{54}=\frac{11-6\sqrt{2}}{6},$$

$$p_1=\frac{1}{15},\ p_2=0,\ p_3=\frac{51+10\sqrt{2}}{240},\ p_4=\frac{51-10\sqrt{2}}{240},\ p_5=\frac{1}{120},$$

$$(3.10)\quad T(x_0, y_0; h)=\frac{1}{8!}h^8\Big[\frac{1}{105}(Z_6+15Z_0Y_4+20Z_1Y_3+60Z_0Z_1X_1$$

$$+10Z_1^2X_0+45Z_0^2X_2+15Z_0^3W_0)-\frac{6}{49}(60-43\sqrt{2})(Z_2Y_2+Z_0Z_2X_0)$$

$$+\frac{2}{21}(11-6\sqrt{2})(Z_3Y_1+3Z_0Y_1^2)+\frac{1}{21}Z_4Y_0$$

$$+\frac{1}{49}(258\sqrt{2}-346)Z_0Y_0Y_2+\frac{1}{21}(83-48\sqrt{2})Z_1Y_0Y_1+$$

$$+\frac{1}{7}(2\sqrt{2}-1)(Z_2Y_0^2+Z_0Y_0^3)+\frac{1}{49}(258\sqrt{2}-353)Z_0^2Y_0X_0\Big]+$$

$$+\boldsymbol{O}(h^9).$$

4. Implicit formulas of the type A

4.1 Formula IA-3

The formula of order 3 for $r=1$ is determined as follows:

$$a_1=\frac{1}{3},\quad c_1=\frac{1}{6},\quad p_1=\frac{1}{2}, \tag{4.1}$$

$$T(x_0,\ y_0;\ h)=-\frac{1}{4!}h^4\frac{1}{3}Z_2+\boldsymbol{O}(h^5). \tag{4.2}$$

4.2 Formula IA-4

Under the condition $c_1=0$, to obtain the formula of order 4 for $r=2$, we require that $T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^5)$. Then it follows that

$$a_1,\ a_2=\frac{4\pm\sqrt{6}}{10}.$$

In particular, we have the formulas as follows:

$$a_1=\frac{4-\sqrt{6}}{10},\quad c_1=0,\quad a_2=\frac{4+\sqrt{6}}{10},\quad b_{21}=\frac{36+29\sqrt{6}}{625}, \tag{4.3}$$

$$c_2=\frac{153-33\sqrt{6}}{625},\quad p_1=\frac{9+\sqrt{6}}{36},\quad p_2=\frac{9-\sqrt{6}}{36},$$

$$T(x_0,\ y_0;\ h)=\frac{1}{5!}h^5\frac{(\sqrt{6}-2)}{2}Z_0Y_1+\boldsymbol{O}(h^6). \tag{4.4}$$

4.3 Formula IA-5

To obtain the formula of order 5 for $r=2$,we require that $T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^6)$. Then it follows that

$$a_1,\ a_2=\frac{4\pm\sqrt{6}}{10}.$$

In particular, we have the following formulas:

$$a_1=\frac{4-\sqrt{6}}{10},\quad c_1=\frac{11-4\sqrt{6}}{50},\quad a_2=\frac{4+\sqrt{6}}{10},\quad b_{21}=\frac{36+29\sqrt{6}}{625}, \tag{4.5}$$

$$c_2=\frac{131-16\sqrt{6}}{1250},\quad p_1=\frac{9+\sqrt{6}}{36},\quad p_2=\frac{9-\sqrt{6}}{36},$$

(4.6) $$T(x_0, y_0; h)=-\frac{1}{6!}h^6\Big[\frac{1}{20}(Z_4+6Z_0Y_2+2Z_2Y_0+2Z_0Y_0^2+3Z_0^2X_0)$$
$$+\frac{2}{5}(\sqrt{6}-1)Z_1Y_1\Big]+O(h^7).$$

4.4 Formula IA-6

Under the conditions (2.22) and (2.26), to obtain the formula of order 7 for $r=3$, we require that $T(x_0, y_0; h)=O(h^7)$. Then it follows that

$$a_2,\ a_3=\frac{5\pm\sqrt{5}}{10}.$$

In particular, we have the formulas as follows:

(4.7) $$a_1=0,\ c_1=0,\ a_2=\frac{5-\sqrt{5}}{10},\ b_{21}=\frac{5-\sqrt{5}}{100},\ c_2=\frac{5-2\sqrt{5}}{25},$$
$$a_3=\frac{5+\sqrt{5}}{10},\ b_{31}=\frac{5+3\sqrt{5}}{300},\ b_{32}=\frac{5+3\sqrt{5}}{60},\ c_3=\frac{5-\sqrt{5}}{50},$$
$$p_1=\frac{1}{12},\ p_2=\frac{5+\sqrt{5}}{24},\ p_3=\frac{5-\sqrt{5}}{24},$$

(4.8) $$T(x_0, y_0; h)=-\frac{1}{7!}h^7\Big[\frac{1}{50}(Z_5+10Z_0Y_3+10Z_1Y_2+10Z_0Z_1X_0$$
$$+15Z_0^2X_1)+\frac{1}{15}(Z_3Y_0+Z_1Y_0^2)+\frac{1}{20}(7\sqrt{5}-5)Z_2Y_1+$$
$$+\frac{1}{20}(7\sqrt{5}-1)Z_0Y_0Y_1\Big]+O(h^8).$$

4.5 Formula IA-7

Under the conditions (2.22) and (2.26), to have the formula of order 7 for $r=4$, we require that

$$T(x_0, y_0; h)=O(h^8),\quad D_1=\frac{1}{56}.$$

Then it follows that

$$a_2,\ a_3,\ a_4=\frac{1}{2},\ \frac{7\pm\sqrt{21}}{14}.$$

In particular, we obtain the following formulas:

(4.9) $$a_1=0,\ c_1=0,\ a_2=\frac{7-\sqrt{21}}{14},\ b_{21}=\frac{7-\sqrt{21}}{196},\ c_2=\frac{14-3\sqrt{21}}{49},$$

$$a_3=\frac{1}{2},\ b_{31}=\frac{1}{96},\ b_{32}=\frac{7+3\sqrt{21}}{192},\ c_3=\frac{5-\sqrt{21}}{32},$$

$$a_4=\frac{7+\sqrt{21}}{14},\ b_{41}=\frac{133+37\sqrt{21}}{4116},\ b_{42}=\frac{5+\sqrt{21}}{84},$$

$$b_{43}=\frac{42+22\sqrt{21}}{1029},\ c_4=\frac{63-9\sqrt{21}}{686},$$

$$p_1=\frac{1}{20},\ p_2=\frac{7(7+\sqrt{21})}{360},\ p_3=\frac{8}{45},\ p_4=\frac{7(7-\sqrt{21})}{360},$$

(4.10) $$T(x_0,\ y_0;\ h)=\frac{1}{8!}h^8\Big[\frac{1}{14}(7-\sqrt{21})(Z_2Y_2+Z_0Y_0Y_2+Z_0^2Y_0X_0$$

$$+Z_0Z_2X_0)-\frac{2}{315}(63-10\sqrt{21})(Z_3Y_1+Z_1Y_0Y_1+3Z_0Y_1^2)$$

$$+(\sqrt{21}-2)(Z_2Y_0^2+Z_0Y_0^3)\Big]+O(h^9).$$

5. Implicit formulas of the type B

5.1 Formula IB-3

Under the conditions (2.22) and $a_1=0$, we have the following formula of order 3 for $r=1$:

(5.1) $$a_1=c_1=0,\ p_0=\frac{1}{3},\ p_1=\frac{1}{6},$$

(5.2) $$T(x_0,\ y_0;\ h)=\frac{1}{4!}h^4\frac{1}{3}(Z_2+Z_0Y_0)+O(h^5).$$

5.2 Formulas of order 4

5.2.1 Formula IB-4-1

Under the condition (2.22), to obtain the formula of order 4 for $r=1$, we require that $T(x_0,\ y_0;\ h)=O(h^5)$. Then it follows that $a_1=\frac{3\pm\sqrt{3}}{6}$. To make the coefficients in the principal error term as small as possible, we choose $a_1=\frac{3-\sqrt{3}}{6}$. Then it follows that

(5.3) $a_1=\frac{3-\sqrt{3}}{6},\quad c_1=\frac{2-\sqrt{3}}{6},\quad p_0=\frac{3-\sqrt{3}}{6},\quad p_1=\frac{\sqrt{3}}{6},$

(5.4) $T(x_0,\ y_0;\ h)=\frac{1}{5!}h^5\left[\frac{1}{9}Z_3+\frac{1}{3}Z_0Y_1-\frac{1}{18}(3-\sqrt{3})Z_1Y_0\right]+\boldsymbol{O}(h^6).$

5.2.2 Formula IB-4-2

Under the conditions (2.22), (2.26) and (2.32), we have the following formula of order 4 for $r=2$:

(5.5)
$$a_1=c_1=0,\ a_2=c_2=1,\ b_{21}=0,$$
$$p_0=\frac{1}{2},\quad p_1=\frac{1}{12},\quad p_2=-\frac{1}{12},$$

(5.6) $T(x_0,\ y_0;\ h)=-\frac{1}{5!}h^5\frac{1}{6}(Z_3+3Z_0Y_1+Z_1Y_0)+\boldsymbol{O}(h^6).$

5.3 Formulas of order 5

5.3.1 Formula IB-5-1

Under the condition (2.22), to obtain the formula of order 5 for $r=2$, we require that

$$T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^6),\ B_1=\frac{1}{30}.$$

Then it follows that

$$a_1,\ a_2=\frac{5\pm\sqrt{15}}{10}.$$

In particular, we have the formulas as follows:

(5.7) $a_1=\frac{5-\sqrt{15}}{10},\ c_1=\frac{4-\sqrt{15}}{10},\ a_2=\frac{5+\sqrt{15}}{10},\ b_{21}=\frac{9+\sqrt{15}}{220},\ c_2=\frac{7+2\sqrt{15}}{22},$

$$p_0=\frac{1}{2},\ p_1=\frac{\sqrt{15}}{36},\ p_2=-\frac{\sqrt{15}}{36},$$

(5.8) $T(x_0,\ y_0;\ h)=-\frac{1}{6!}h^6\left[\frac{1}{10}(5-\sqrt{15})\ Z_1Y_1+\frac{1}{110}(5+3\sqrt{15})(Z_2Y_0\right.$

$$\left.+Z_0Y_0^2)\right]+\boldsymbol{O}(h^7).$$

5.3.2 Formula IB-5-2

Under the conditions (2.22), (2.26) and $a_1=0$, we require that $T(x_0,\ y_0;\ h)$

$=\boldsymbol{O}(h^6)$. Then it follows that $a_2=\dfrac{6\pm\sqrt{6}}{10}$. To make the coefficients in the principal error term as small as possible, we take $a_2=\dfrac{6-\sqrt{6}}{10}$. Then we have the formulas as follows:

(5.9) $$a_1=c_1=0,\ a_2=\frac{6-\sqrt{6}}{10},\ b_{21}=\frac{48-3\sqrt{6}}{1000},\ c_2=\frac{162-57\sqrt{6}}{500},$$

$$p_0=\frac{4-\sqrt{6}}{10},\ p_1=\frac{6+\sqrt{6}}{90},\ p_2=\frac{3+8\sqrt{6}}{90},$$

(5.10) $$T(x_0,\ y_0;\ h)=\frac{1}{6!}h^6\frac{1}{20}[Z_4+6Z_0Y_2+4Z_1Y_1+3Z_0^2X_0$$

$$+(3\sqrt{6}-2)(Z_2Y_0+Z_0Y_0^2)]+\boldsymbol{O}(h^7).$$

5.4 Formula IB-6

Under the conditions (2.22), (2.26) and (2.32), to have the formula of order 6 for $r=3$, we require that $T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^7)$. Then it follows that $a_3=\dfrac{5\pm\sqrt{5}}{10}$. In particular, we obtain the following formulas:

(5.11) $$a_1=c_1=0,\ a_2=c_2=1,\ b_{21}=0,$$

$$a_3=\frac{5-\sqrt{5}}{10},\ b_{31}=\frac{9-\sqrt{5}}{300},\ b_{32}=\frac{\sqrt{5}-3}{300},\ c_3=\frac{13-5\sqrt{5}}{50},$$

$$p_0=\frac{5-\sqrt{5}}{10},\ p_1=\frac{5+\sqrt{5}}{120},\ p_2=\frac{\sqrt{5}-5}{120},\ p_3=\frac{\sqrt{5}}{12},$$

(5.12) $$T(x_0,\ y_0;\ h)=-\frac{1}{7!}h^7\Big[\frac{1}{50}(Z_5+10Z_0Y_3+10Z_1Y_2+5Z_2Y_1$$

$$+10Z_0Z_1X_0+15Z_0^2X_0)+\frac{1}{150}(21\sqrt{5}-25)(Z_3Y_0+Z_1Y_0^2)$$

$$+\frac{1}{50}(21\sqrt{5}-20)Z_0Y_0Y_1\Big]+\boldsymbol{O}(h^8).$$

5.5 Formula IB-7

Under the conditions (2.22), (2.26) and (2.32), to have the formula of order 7 for $r=4$, we require that

$$T(x_0,\ y_0;\ h)=\boldsymbol{O}(h^8),\quad D_1=\frac{1}{56}.$$

Then it follows that

$$a_3,\ a_4=\frac{7\pm\sqrt{21}}{14}.$$

In particular, we have the formulas as follows:

$$\text{(5.13)}\qquad a_1=c_1=0,\ a_2=c_2=1,\ b_{21}=0,$$

$$a_3=\frac{7-\sqrt{21}}{14},\ b_{31}=\frac{11-\sqrt{21}}{588},\ b_{32}=\frac{\sqrt{21}-5}{588},\quad c_3=\frac{33-7\sqrt{21}}{98},$$

$$a_4=\frac{7+\sqrt{21}}{14},\ b_{41}=\frac{86-9\sqrt{21}}{4998},\ b_{42}=\frac{13\sqrt{21}-145}{9996},\ b_{43}=\frac{75+5\sqrt{21}}{1428},$$

$$c_4=\frac{411+109\sqrt{21}}{1666},$$

$$p_0=\frac{1}{2},\quad p_1=\frac{1}{40},\ p_2=-\frac{1}{40},\quad p_3=\frac{7\sqrt{21}}{360},\ p_4=-\frac{7\sqrt{21}}{360},$$

$$\text{(5.14)}\qquad T(x_0,\ y_0;\ h)=\frac{1}{8!}h^8\Big[\frac{2}{105}(28-5\sqrt{21})(Z_3Y_1+3Z_0Y_1^2)$$

$$+\frac{1}{357}(7+5\sqrt{21})\ (Z_4Y_0+6Z_0Y_0Y_2+Z_2Y_0^2+Z_0Y_0^3+3Z_0^2Y_0X_0)$$

$$+\frac{2}{255}(78-5\sqrt{21})\ Z_1Y_0Y_1\Big]+O(h^9).$$

6. Numerical examples

The initial value problem

$$\text{(6.1)}\qquad y'=y,\ y(0)=1$$

is solved with step-size $h=0.25$ by the explicit methods, the implicit methods of the type A and those of the type B. The errors in the numerical solutions are listed in the tables 1, 2 and 3 respectively.

7. Auxiliary formulas

In the implicit methods, usually u_1 is solved by the iteration method from the equation

$$\text{(7.1)}\qquad u_1=h^2\sum_{i=1}^{r}p_il_i,$$

or

$$u_1 = p_0 h(k_1 - k_0) + h^2 \sum_{i=1}^{r} p_i l_i. \tag{7.2}$$

To start the iteration, it is necessary to give the initial approximations to u_1 for computing $l_i\,(i=1, 2, \cdots, r)$ and k_1. Hence we construct the auxiliary formulas of the form

$$w_1^{(s)} = h^2 \sum_{i=1}^{s} q_{si} l_i \quad (0 \leqq s \leqq r) \tag{7.3}$$

for approximating u_1.

Table 1. Explicit methods

formula / x	E-3	E-4	E-5	E-6	E-7
0.25	−1.71 E-04	−2.18 E-06	−7.04 E-08	−7.42 E-10	1.66 E-10
0.50	−4.40 E-04	−5.60 E-06	−1.81 E-07	−1.94 E-09	2.76 E-10
0.75	−8.47 E-04	−1.08 E-05	−3.48 E-07	−3.78 E-09	4.66 E-10
1.00	−1.45 E-03	−1.85 E-05	−5.96 E-07	−6.49 E-09	7.57 E-10
1.25	−2.33 E-03	−2.96 E-05	−9.57 E-07	−1.04 E-08	1.25 E-09
1.50	−3.59 E-03	−4.57 E-05	−1.47 E-06	−1.61 E-08	1.86 E-09
1.75	−5.37 E-03	−6.84 E-05	−2.21 E-06	−2.41 E-08	2.79 E-09
2.00	−7.88 E-03	−1.00 E-04	−3.24 E-06	−3.54 E-08	4.02 E-09
2.25	−1.14 E-02	−1.45 E-04	−4.68 E-06	−5.13 E-08	5.59 E-09
2.50	−1.62 E-02	−2.07 E-04	−6.68 E-06	−7.31 E-08	8.15 E-09
2.75	−2.29 E-02	−2.92 E-04	−9.44 E-06	−1.03 E-07	1.15 E-08
3.00	−3.21 E-02	−4.09 E-04	−1.32 E-05	−1.45 E-07	1.61 E-08
3.25	−4.47 E-02	−5.69 E-04	−1.84 E-05	−2.01 E-07	2.24 E-08
3.50	−6.18 E-02	−7.87 E-04	−2.54 E-05	−2.79 E-07	3.07 E-08
3.75	−8.50 E-02	−1.08 E-03	−3.50 E-05	−3.84 E-07	4.19 E-08
4.00	−1.16 E-01	−1.48 E-03	−4.79 E-05	−5.26 E-07	5.73 E-08

Table 2. Implicit methods of the type A

formula x	IA-3	IA-4	IA-5	IA-6	IA-7
0. 25	6. 00 E -06	−1. 27 E -07	−2. 71 E -08	−5. 68 E -10	5. 82 E -11
0. 50	1. 54 E -05	−3. 26 E -07	−6. 97 E -08	−1. 50 E -09	1. 16 E -10
0. 75	2. 97 E -05	−6. 28 E -07	−1. 34 E -07	−2. 97 E -09	1. 46 E -10
1. 00	5. 08 E -05	−1. 07 E -06	−2. 30 E -07	−5. 09 E -09	2. 04 E -10
1. 25	8. 15 E -05	−1. 72 E -06	−3. 69 E -07	−8. 15 E -09	3. 49 E -10
1. 50	1. 26 E -04	−2. 66 E -06	−5. 69 E -07	−1. 26 E -08	4. 66 E -10
1. 75	1. 88 E -04	−3. 98 E -06	−8. 53 E -07	−1. 89 E -08	6. 40 E -10
2. 00	2. 76 E -04	−5. 84 E -06	−1. 25 E -06	−2. 79 E -08	8. 15 E -10
2. 25	3. 99 E -04	−8. 44 E -06	−1. 81 E -06	−4. 04 E -08	1. 05 E -09
2. 50	5. 69 E -04	−1. 20 E -05	−2. 58 E -06	−5. 74 E -08	1. 75 E -09
2. 75	8. 04 E -04	−1. 70 E -05	−3. 64 E -06	−8. 11 E -08	2. 44 E -09
3. 00	1. 13 E -03	−2. 38 E -05	−5. 10 E -06	−1. 14 E -07	3. 26 E -09
3. 25	1. 56 E -03	−3. 31 E -05	−7. 10 E -06	−1. 59 E -07	4. 42 E -09
3. 50	2. 17 E -03	−4. 58 E -05	−9. 81 E -06	−2. 20 E -07	6. 05 E -09
3. 75	2. 98 E -03	−6. 30 E -05	−1. 35 E -05	−3. 04 E -07	7. 92 E -09
4. 00	4. 08 E -03	−8. 63 E -05	−1. 85 E -05	−4. 15 E -07	1. 07 E -08

Table 3. Implicit methods of the type B

formula x	IB-3	IB-4-1	IB-4-2	IB-5-1	IB-5-2	IB-6	IB-7
0. 25	6. 55 E -05	2. 95 E -06	−1. 75 E -06	−6. 61 E -08	1. 02 E -07	−2. 10 E -09	7. 28 E -11
0. 50	1. 68 E -04	7. 58 E -06	−4. 49 E -06	−1. 70 E -07	2. 61 E -07	−5. 40 E -09	1. 46 E -10
0. 75	3. 24 E -04	1. 46 E -05	−8. 65 E -06	−3. 27 E -07	5. 03 E -07	−1. 05 E -08	2. 04 E -10
1. 00	5. 55 E -04	2. 50 E -05	−1. 48 E -05	−5. 60 E -07	8. 62 E -07	−1. 80 E -08	3. 20 E -10
1. 25	8. 90 E -04	4. 01 E -05	−2. 38 E -05	−8. 99 E -07	1. 38 E -06	−2. 88 E -08	5. 53 E -10
1. 50	1. 37 E -03	6. 18 E -05	−3. 66 E -05	−1. 39 E -06	2. 13 E -06	−4. 44 E -08	7. 57 E -10
1. 75	2. 05 E -03	9. 26 E -05	−5. 48 E -05	−2. 08 E -06	3. 19 E -06	−6. 66 E -C8	1. 11 E -09
2. 00	3. 02 E -03	1. 36 E -04	−8. 05 E -05	−3. 05 E -06	4. 69 E -06	−9. 78 E -08	1. 51 E -09
2. 25	4. 36 E -03	1. 96 E -04	−1. 16 E -04	−4. 40 E -06	6. 77 E -06	−1. 41 E -07	1. 98 E -09
2. 50	6. 22 E -03	2. 80 E -04	−1. 66 E -04	−6. 28 E -06	9. 66 E -06	−2. 02 E -07	3. 14 E -09
2. 75	8. 78 E -03	3. 96 E -04	−2. 34 E -04	−8. 87 E -06	1. 36 E -05	−2. 85 E -07	4. 42 E -09
3. 00	1. 23 E -02	5. 54 E -04	−3. 28 E -04	−1. 24 E -05	1. 91 E -05	−3. 99 E -07	6. 05 E -09
3. 25	1. 71 E -02	7. 71 E -04	−4. 56 E -04	−1. 73 E -05	2. 66 E -05	−5. 55 E -07	8. 38 E -09
3. 50	2. 37 E -02	1. 07 E -03	−6. 31 E -04	−2. 39 E -05	3. 67 E -05	−7. 68 E -07	1. 12 E -08
3. 75	3. 25 E -02	1. 47 E -03	−8. 68 E -04	−3. 29 E -05	5. 06 E -05	−1. 06 E -06	1. 49 E -08
4. 00	4. 46 E -02	2. 01 E -03	−1. 19 E -03	−4. 50 E -05	6. 92 E -05	−1. 45 E -06	2. 10 E -08

If the same step-size h is used again in the second step, better initial approximations will be obtained by use of the values of functions computed in the first step. Thus let

(7.4) $$\hat{l}_i = g(x_1 + a_i h,\ z_1 + a_i h k_1 + h^2 \sum_{j=1}^{i-1} b_{ij} \hat{l}_j + c_i u_2),$$

where

(7.5) $$u_2 = z_2 - z_1 - h k_1.$$

Then we construct the auxiliary formulas of the form

(7.6) $$w_2^{(s)} = u_1 + r_{s0} h (k_1 - k_0) + h^2 \sum_{j=1}^{r} r_{sj} l_j + h^2 \sum_{j=1}^{s} \hat{r}_{sj} \hat{l}_j$$

for approximating u_2.

7.1 Formulas for IA-3

(7.7) $$w_1^{(0)} = 0;$$

(7.8) $$r_{0,0} = 3,\ r_{0,1} = -3,$$

where

$$w_1^{(0)} = u_1 + \boldsymbol{O}(h^3),\ w_2^{(0)} = u_2 + \boldsymbol{O}(h^4).$$

7.2 Formulas for IA-4

(7.9) $$w_1^{(1)} = \frac{1}{2} h^2 l_1;$$

(7.10) $$r_{1,0} = -\ (1 + \sqrt{6}),\ r_{1,1} = \frac{3 + 7\sqrt{6}}{18},\ r_{1,2} = \frac{3 + 8\sqrt{6}}{18},\ \hat{r}_{1,1} = \frac{4 + \sqrt{6}}{6}$$

where

$$w_1^{(1)} = u_1 + \boldsymbol{O}(h^4),\ w_2^{(1)} = u_2 + \boldsymbol{O}(h^5).$$

7.3 Formulas for IA-5

(7.11) $$w_1^{(0)} = 0;\ w_1^{(1)} = \frac{1}{2} h^2 l_1;$$

(7.12) $$r_{0,0} = 13,\ r_{0,1} = -\frac{39 - 4\sqrt{6}}{6},\ r_{0,2} = -\frac{39 + 4\sqrt{6}}{6};$$

(7.13) $r_{1,0}=-(11+6\sqrt{6}),\ r_{1,1}=\frac{1}{114}(399+266\sqrt{6}),\ r_{1,2}=\frac{1}{114}(699+364\sqrt{6}),$

$$\hat{r}_{1,1}=\frac{26+9\sqrt{6}}{19},$$

where

$$w_2^{(0)}=u_2+\boldsymbol{O}(h^5),\ w_2^{(1)}=u_2+\boldsymbol{O}(h^5).$$

7.4 Formulas for IA-6

(7.14) $$q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{1-\sqrt{5}}{12},\ q_{2,2}=\frac{5+\sqrt{5}}{12};$$

(7.15) $$r_{1,0}=1,\ r_{1,1}=-1,\ r_{1,2}=\frac{5}{4}(\sqrt{5}-1),\ r_{1,3}=-\frac{5}{4}(\sqrt{5}+1);$$

(7.16) $$r_{2,0}=\frac{1}{11}(25+20\sqrt{5}),\ r_{2,1}=-\frac{1}{66}(13+6\sqrt{5}),$$

$$r_{2,2}=-\frac{1}{132}(235+111\sqrt{5}),\ r_{2,3}=-\frac{1}{44}(5+15\sqrt{5}),$$

$$\hat{r}_{2,1}=-\frac{1}{33}(31+27\sqrt{5}),\ \hat{r}_{2,2}=\frac{1}{33}(25+9\sqrt{5}),$$

where

$$w_1^{(2)}=u_1+\boldsymbol{O}(h^4),\ w_2^{(1)}=u_2+\boldsymbol{O}(h^6),\ w_2^{(2)}=u_2+\boldsymbol{O}(h^7).$$

7.5 Formulas for IA-7

(7.17) $$q_{1,1}=\frac{1}{2};\ q_{2,1}=-\frac{1+\sqrt{21}}{12},\ q_{2,2}=\frac{7+\sqrt{21}}{12};$$

(7.18) $$q_{3,1}=\frac{1}{6},\ q_{3,2}=0,\ q_{3,3}=\frac{1}{3};$$

(7.19) $$r_{1,0}=-129,\ r_{1,1}=\frac{53}{6},\ r_{1,2}=\frac{7}{36}(133+3\sqrt{21}),$$

$$r_{1,3}=\frac{496}{9},\ r_{1,4}=\frac{7}{36}(133-3\sqrt{21}),\ \hat{r}_{1,1}=\frac{40}{3};$$

(7.20) $$r_{2,0}=\frac{1}{5}(303+92\sqrt{21}),\ r_{2,1}=-\frac{1}{150}(517+158\sqrt{21}),$$

$$r_{2,2}=-\frac{7}{900}(2111+569\sqrt{21}),\ r_{2,3}=-\frac{16}{225}(341+104\sqrt{21}),$$

$$r_{2,4}=-\frac{7}{900}(1211+419\sqrt{21}),\ \hat{r}_{2,1}=-\frac{1}{75}(761+219\sqrt{21}),$$

$$\hat{r}_{2,2}=\frac{7}{75}(33+7\sqrt{21}),$$

where

$$w_1^{(2)}=u_1+O(h^4),\ w_1^{(3)}=u_1+O(h^5),\ w_2^{(i)}=u_2+O(h^7)\ (i=1,\ 2).$$

7.6 Formulas for IB-3

(7.21) $$w_1^{(1)}=\frac{1}{2}h^2 l_1;$$

(7.22) $$r_{1,0}=-2,\ r_{1,1}=\frac{1}{2},\ \hat{r}_{1,1}=\frac{3}{2},$$

where

$$w_2^{(1)}=u_2+O(h^4).$$

7.7 Formulas for IB-4-1

(7.23) $$w_1^{(0)}=0;\ w_1^{(1)}=\frac{1}{2}h^2 l_1;$$

(7.24) $$r_{0,0}=\sqrt{3},\ r_{0,1}=-\sqrt{3};\ r_{1,0}=-1,\ r_{1,1}=\frac{3-\sqrt{3}}{6},\ \hat{r}_{1,1}=\frac{3+\sqrt{3}}{6},$$

where

$$w_2^{(0)}=u_2+O(h^4),\ w_2^{(1)}=u_2+O(h^5).$$

7.8 Formulas for IB-4-2

(7.25) $$q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{1}{3},\ q_{2,2}=\frac{1}{6};$$

(7.26) $$r_{1,0}=-2,\ r_{1,1}=\frac{1}{2},\ r_{1,2}=\frac{3}{2};$$

(7.27) $$r_{2,0}=-1,\ r_{2,1}=\frac{1}{12},\ r_{2,2}=\frac{5}{6},\ \hat{r}_{2,2}=\frac{1}{12},$$

where

$$w_1^{(2)}=u_1+\boldsymbol{O}(h^4),\ w_2^{(1)}=u_2+\boldsymbol{O}(h^5),\ w_2^{(2)}=u_2+\boldsymbol{O}(h^5),\ \hat{l}_1=l_2.$$

7.9 Formulas for IB-5-1

$$w_1^{(0)}=0;\ q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{9+\sqrt{15}}{36},\ q_{2,2}=\frac{9-\sqrt{15}}{36}; \tag{7.28}$$

$$r_{0,0}=-5,\ r_{0,1}=\frac{15-\sqrt{15}}{6},\ r_{0,2}=\frac{15+\sqrt{15}}{6}; \tag{7.29}$$

$$r_{1,0}=\frac{5+9\sqrt{15}}{17},\ r_{1,1}=\frac{3-32\sqrt{15}}{102},\ r_{1,2}=-\frac{150+49\sqrt{15}}{102},\ \hat{r}_{1,1}=\frac{39+9\sqrt{15}}{34}; \tag{7.30}$$

$$r_{2,0}=0,\ r_{2,1}=-\frac{9-\sqrt{15}}{18},\ r_{2,2}=-\frac{9+\sqrt{15}}{18},\ \hat{r}_{2,1}=\frac{9+2\sqrt{15}}{18}, \tag{7.31}$$

$$\hat{r}_{2,2}=\frac{9-2\sqrt{15}}{18},$$

where

$$w_1^{(2)}=u_1+\boldsymbol{O}(h^4),\ w_2^{(i)}=u_2+\boldsymbol{O}(h^5)\ (i=0,\ 1),\ w_2^{(2)}=u_2+\boldsymbol{O}(h^6).$$

7.10 Formulas for IB-5-2

$$q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{3-\sqrt{6}}{18},\ q_{2,2}=\frac{6+\sqrt{6}}{18}; \tag{7.32}$$

$$r_{1,0}=-(5+3\sqrt{6}),\ r_{1,1}=\frac{\sqrt{6}}{3},\ r_{1,2}=\frac{3+8\sqrt{6}}{3},\ \hat{r}_{1,1}=4, \tag{7.33}$$

where

$$w_1^{(2)}=u_1+\boldsymbol{O}(h^4),\ w_2^{(1)}=u_2+\boldsymbol{O}(h^6).$$

7.11 Formulas for IB-6

$$q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{1}{3},\ q_{2,2}=\frac{1}{6}; \tag{7.34}$$

$$q_{3,1}=\frac{3-\sqrt{5}}{24},\ q_{3,2}=\frac{5}{12},\ q_{3,3}=\frac{\sqrt{5}-1}{24}; \tag{7.35}$$

$$r_{1,0}=-(2+3\sqrt{5}),\ r_{1,1}=\frac{\sqrt{5}-3}{4},\ r_{1,2}=\frac{11+\sqrt{5}}{4},\ r_{1,3}=\frac{5\sqrt{5}}{2}; \tag{7.36}$$

$$r_{2,0}=-\frac{14+9\sqrt{5}}{11},\ r_{2,1}=\frac{9\sqrt{5}-19}{132},\ r_{2,2}=\frac{179+9\sqrt{5}}{132}, \tag{7.37}$$

$$r_{2,3}=\frac{15\sqrt{5}}{22},\ \hat{r}_{2,2}=\frac{2}{33},$$

where

$$w_1^{(3)}=u_1+O(h^5),\ w_2^{(1)}=u_2+O(h^6),\ w_2^{(2)}=u_2+O(h^7),$$

$$\hat{l}_1=l_2.$$

7.12 Formulas for IB-7

(7.38) $$q_{1,1}=\frac{1}{2};\ q_{2,1}=\frac{1}{3},\ q_{2,2}=\frac{1}{6};$$

(7.39) $$q_{3,1}=\frac{1-\sqrt{21}}{24},\ q_{3,2}=\frac{\sqrt{21}-3}{24},\ q_{3,3}=\frac{7}{12};$$

(7.40) $$q_{4,1}=-\frac{1}{60},\ q_{4,2}=-\frac{1}{15},\ q_{4,3}=\frac{105+7\sqrt{21}}{360},\ q_{4,4}=\frac{105-7\sqrt{21}}{360};$$

(7.41) $$r_{1,0}=26,\ r_{1,1}=\frac{13}{12},\ r_{1,2}=\frac{67}{12},\ r_{1,3}=\frac{7}{12}(\sqrt{21}-28),$$

$$r_{1,4}=-\frac{7}{12}(28+\sqrt{21});$$

(7.42) $$r_{3,0}=\frac{1}{269}(46+48\sqrt{21}),\ r_{3,1}=\frac{1}{16140}(200\sqrt{21}-167),$$

$$r_{3,2}=-\frac{1}{16140}(2321+3264\sqrt{21}),\ r_{3,3}=-\frac{7}{16140}(1344+151\sqrt{21}),$$

$$r_{3,4}=-\frac{49}{5380}(16+5\sqrt{21}),\ \hat{r}_{3,2}=\frac{1}{4035}(46\sqrt{21}-68),$$

$$\hat{r}_{3,3}=\frac{28}{4035}(105+16\sqrt{21}),$$

where

$$w_1^{(2)}=u_1+O(h^4),\ w_1^{(3)}=u_1+O(h^5),\ w_1^{(4)}=u_1+O(h^6),$$

$$w_2^{(1)}=u_2+O(h^7),\ \ w_2^{(3)}=u_2+O(h^8),\ \hat{l}_1=l_2.$$

References

[1] Butcher, J. C.: *Implicit Runge-Kutta processes*, Math. Comp., **18** (1964), 50-64.
[2] Butcher, J. C.: *On the attainable order of Runge-Kutta methods*, Math. Comp., **19** (1965), 408-417.

[3] Cassity, C. R.: *Solutions of the fifth-order Runge-Kutta equations*, SIAM J. Numer. Anal., **3** (1966), 598-606.

[4] Cassity, C. R.: *The complete solution of the fifth order Runge-Kutta equations,* SIAM J. Numer. Anal., **6** (1969), 432-436.

[5] Ceschino, F. and J. Kuntzmann: *Numerical solution of initial value problems*, Prentice-Hall, 1966.

[6] Cooper, G. J.: *Interpolation and quadrature methods for ordinary differential equations*, Math. Comp., **22** (1968), 69-76.

[7] Costabile, F.: *Metodi pseudo Runge-Kutta di seconda specie*, Calcolo, **7** (1970), 305-322.

[8] Filippi, S. und W. Glasmacher: *Über neue Ergebnisse zu den Runge-Kutta-Verfahren mit Hilfe von nichtnumerischen programmen*, Elektronische Datenverarbeitung (1968), 16-23.

[9] Filippi, S. und D. Sommer: *Beiträge zu den impliziten Runge-Kutta-Verfahren*, Elektronische Datenverarbeitung (1968), 113-121.

[10] Gates, L. D.: *Numerical solution of differential equations by repeated quadratures*, SIAM Rev., **6** (1964), 134-147.

[11] Gruttke, W. B.: *Pseudo-Runge-Kutta methods of the fifth order*, J. Assoc. Comput. Mach., **17** (1970), 613-628.

[12] Guerra, S.: *Formule di Runge-Kutta di ordine elevato per i sistemi differenziali lineari del* 1^0 *ordine con applicazione al calcolo delle radici delle equazioni algebriche*, Calcolo, **3** (1966), 407-440.

[13] Konen, H. P. and H. A. Luther: *Some singular explicit fifth order Runge-Kutta solutions*, SIAM J. Numer. Anal., **4** (1967), 607-619.

[14] Lawson, J. D.: *An order five Runge-Kutta process with extended region of stability*, SIAM J. Numer. Anal., **3** (1966), 593-597.

[15] Lawson, J. D: *An order six Runge-Kutta process with extended region of stability*, SIAM J. Numer. Anal., **4** (1967), 620-625.

[16] Luther, H. A.: *Further explicit fifth-order Runge-Kutta formulas*, SIAM Rev., **8** (1966), 374-380.

[17] Luther, H. A.: *An explicit sixth-order Runge-Kutta formula*, Math. Comp., **22** (1968), 434-436.

[18] Luther, H. A. and H. G. Sierra: *On the optimal choice of fourth-order Runge-Kutta formulas*, Numer. Math., **15** (1970), 354-358.

[19] Rosser, J. B.: *A Runge-Kutta for all seasons*, SIAM Rev., **9** (1967), 417-452.

[20] Shampine, L. F. and H. A. Watts: *Block implicit one-step methods*, Math. Comp., **23** (1969), 731-740.

[21] Squier, D. P.: *One-step methods for ordinary differential equations*, Numer. Math., **13** (1969), 176-179.

[22] Urabe, M.: *An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations*, Numer. Math., **15** (1970), 151-164.

[23] Verner, J. H.: *The order of some implicit Runge-Kutta methods*, Numer. Math., **13** (1969), 14-23.

Department of Mathematics
Faculty of Science
Hiroshima University