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On the Radon Transform of the Rapidly Decreasing Functions on Symmetric Spaces II

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1. Introduction.

One of the problems which are proposed by S. Helgason for the Radon transform is to study the relations between the function spaces on a space X and on the dual space $\hat{X}$ by means of the Radon transform $f \to \hat{f}$. In [1], we considered the transform of the rapidly decreasing functions in $\mathcal{S}(S)$ on a Riemannian globally symmetric space S. But to construct a $\mathcal{S}$-theory for the Radon transform in a sense, it seems more favorable to study the Radon transform on the Schwartz space $\mathcal{C}(S)$, which is generalized by Harish-Chandra in [3], than on $\mathcal{S}(S)$, since we know that the Schwartz space is invariant under the left translations by G [3].

In this paper we shall study the Radon transform for the functions in the Schwartz space $\mathcal{C}(S)$ on a Riemannian globally symmetric space of the non-compact type. The main results are Theorems A, B, C and D.

2. Preliminaries.

As usual, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the fields of real and complex numbers respectively. If M and N are two topological spaces, φ a homeomorphism of M onto N and f a function on M, we put $f^{\varphi}=f\circ\varphi^{-1}$. If M is a C^{∞}-manifold, $C^{\infty}(M)$ (respectively, $C_c^{\infty}(M)$) denotes the space of differentiable functions (respectively, differentiable functions with compact support) on M. If G is a Lie group and K a closed subgroup of G, for $x \in G$, the left translation by x of the homogeneous space G/K of the left cosets onto itself will be denoted by $\tau(x)$.

$\boldsymbol{D}(G/K)$ denotes the algebra of differential operators on the homogeneous space G/K which are invariant under the left translations $\tau(x)$, $x \in G$. We write $\boldsymbol{D}(G)$ instead of $\boldsymbol{D}(G/e)$, where e is the identity element of G.

Let S be a Riemannian globally symmetric space of the noncompact type, and $G=I_0(S)$ denote the largest connected group of isometries of S in the compact open topology, then G is a semisimple Lie group and has no compact normal subgroup $\neq e$. Let any point o in S fix, K denote the isotropy subgroup of G at o, $\mathfrak{g}_0$ and $\mathfrak{k}_0$ denote the Lie algebras of G and K, respectively, and let $\mathfrak{p}_0$ denote the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing form B of $\mathfrak{g}_0$. Then G/K has a G-invariant Riemannian structure induced from B. Let θ be the involution of $\mathfrak{g}_0$ which associates with the Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$. Let $\mathfrak{h}_{\mathfrak{p}_0}$ denote a Cartan subalgebra for the space S and $A_{\mathfrak{p}}$ denote the analytic subgroup of G corresponding to $\mathfrak{h}_{\mathfrak{p}_0}$. Let C denote a Weyl chamber in $\mathfrak{h}_{\mathfrak{p}_0}$, then the dual space of $\mathfrak{h}_{\mathfrak{p}_0}$ can be ordered by calling a linear function λ on $\mathfrak{h}_{\mathfrak{p}_0}$ positive if $\lambda(H)>0$ for all $H \in C$. By this ordering we have an Iwasawa decomposition of G, $G=KA_{\mathfrak{p}}N$. For $g \in G$, let $H(g)$ denote the unique element in $\mathfrak{h}_{\mathfrak{p}_0}$ such that $g=k\exp H(g)n$ for $n \in N$ and $k \in K$.

Let M and M', respectively, denote the centralizer and normalizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in K. Let W denote the Weyl group M'/M. Let $\mathfrak{h}_0$ be any maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{h}_{\mathfrak{p}_0}$, let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$ and $\mathfrak{h}$ the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ denote the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let $\mathfrak{h}_{\mathfrak{k}_0}=\mathfrak{h}\cap\mathfrak{k}_0$, $\mathfrak{h}^*=\mathfrak{h}_{\mathfrak{p}_0}+i\mathfrak{h}_{\mathfrak{k}_0}$. All the roots in Δ are real on $\mathfrak{h}^*$. Let C^* be any Weyl chamber in $\mathfrak{h}^*$ whose closure contains the Weyl chamber C in $\mathfrak{h}_{\mathfrak{p}_0}$. We order the dual space of $\mathfrak{h}^*$ by means of the Weyl chamber C^*. Let $\bar{\alpha}$ denote the restriction to $\mathfrak{h}_{\mathfrak{p}_0}$ of a root $\alpha \in \Delta$. Then the set Δ^+ of positive roots in Δ is a disjoint union, $\Delta^+=P_+\cup P_-$, where α belongs to P_+ or P_- respectively according to whether $\bar{\alpha}>0$ or $\bar{\alpha}=0$. Let $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\alpha$. The adjoint representation of $\mathfrak{g}$ will be denoted by adX for $X \in \mathfrak{g}$. Let Σ denote the set of all linear functions on $\mathfrak{h}_{\mathfrak{p}_0}$ which are restrictions of the member of P_+. Let

$$\Sigma_0=\{\lambda \in \Sigma \mid \lambda/n \notin \Sigma \quad \textit{for all integers} \quad n\neq 1\},$$

$$\mathfrak{h}_{\mathfrak{p}_0}^+=\{H \in \mathfrak{h}_{\mathfrak{p}_0} \mid \alpha(H)>0 \quad \textit{for all} \quad \alpha \in \Sigma\},$$

and put $A_{\mathfrak{p}}^+= exp\ \mathfrak{h}_{\mathfrak{p}_0}^+$, where exp denotes the exponential mapping of $\mathfrak{g}_0$ into G. Let ${}^+\mathfrak{h}_{\mathfrak{p}_0}$ denote the set of all $H \in \mathfrak{h}_{\mathfrak{p}_0}$ such that $<H, H'>\geqq 0$ for every H' in $\mathfrak{h}_{\mathfrak{p}_0}^+$. Also let $Cl(A_{\mathfrak{p}}^+)$ denote the closure of $A_{\mathfrak{p}}^+$ in $A_{\mathfrak{p}}$.

The dual space of S is the space $\hat{S}$ of horocycles in S, that is, the set of all orbits of subgroups of the form gNg^{-1} for all elements g in G, with a differentiable structure in such a way that $\hat{S}$ is diffeomorphic to G/MN. We shall write $\boldsymbol{D}(S)$ for a $\boldsymbol{D}(G/K)$ and $\boldsymbol{D}(\hat{S})$ for $\boldsymbol{D}(G/MN)$ respectively.

Let ξ be any horocycle in S, ds_ξ the volume element on ξ in the Riemannian structure on ξ induced by S. For a good function f on S we put

$$\hat{f}(\xi)=\int_\xi f(s)ds_\xi, \qquad \xi \in \hat{S},$$

and call it the *Radon transform* of f. Let

$$\pi : G\to G/K, \quad \hat{\pi} : G\to G/MN$$

denote the projections, and let

$$F=f\circ\pi, \quad \hat{F}=\hat{f}\circ\hat{\pi}.$$

We can select a Haar measure dn on N such that the mapping $n\rightarrow n\cdot o$ of $\xi_0=\{MN\}$ onto itself is measure-preserving, and then

$$\hat{F}(g)=\int_N F(gn)dn.$$

or any continuous function φ on $\hat{S}$, we define the *dual Radon transform* by

$$\check{\varphi}(p)=\int_{\xi\ni p}\varphi(\xi)\,dm\,(\xi), \qquad p\in S,$$

where the integral on the right is the average of φ over the set of horocycles passing through p. If we select $g\in G$ such that $g\cdot o=p$. we have

$$\check{\varphi}(g\cdot o)=\int_K\varphi(gk\cdot\xi_0)\,dk,$$

where the Haar measure dk on K is so normalized that the total measure of K is 1.

Let $\boldsymbol{D}_0(G)$ denote the set of operators in $\boldsymbol{D}(G)$ which are invariant under all right translations from K. Let $S(\mathfrak{h}_{\mathfrak{p}_0})$ denote the symmetric algebra over $\mathfrak{h}_{\mathfrak{p}_0}$ and $I(\mathfrak{h}_{\mathfrak{p}_0})$ be the set of invariant polynomials in $S(\mathfrak{h}_{\mathfrak{p}_0})$ which are invariant under W. Then $\boldsymbol{D}(A_{\mathfrak{p}})$ is canonically isomorphic to $S(\mathfrak{h}_{\mathfrak{p}_0})$. Let ν be a linear function on $\mathfrak{h}_{\mathfrak{p}_0}$ then $e^{\nu}\in C^{\infty}(\mathfrak{h}_{\mathfrak{p}_0})$. For simplicity, the function $a\rightarrow e^{\nu(\log a)}$ on $A_{\mathfrak{p}}$ shall also be denoted by e^{ν}. A C^{∞}-function f on a manifold can be regarded as a differential operator $F\rightarrow fF$. As is well known [5],

(i) *for each $D\in\boldsymbol{D}(G)$ there exists a unique element $D_{\mathfrak{a}}\in\boldsymbol{D}(A_{\mathfrak{p}})$ such that*

$$D-D_{\mathfrak{a}}\in\mathfrak{n}_0\boldsymbol{D}(G)+\boldsymbol{D}(G)\mathfrak{k}_0,$$

(ii) *if $\phi\in C^{\infty}(G)$ such that $\phi(ngk)=\phi(g)$ for all $n\in N$, $g\in G$, $k\in K$ then*

$$(D\phi)^{-}=D_{\mathfrak{a}}\bar{\phi}, \qquad D\in\boldsymbol{D}(G),$$

where the bar denotes restriction to $A_{\mathfrak{p}}$,

(iii) *the mapping $D\rightarrow e^{-\rho}D\circ e^{\rho}$ is a homomorphism of $\boldsymbol{D}_0(G)$ onto $I(\mathfrak{h}_{\mathfrak{p}_0})$ and the kernel is $\boldsymbol{D}_0(G)\cap\boldsymbol{D}(G)\mathfrak{k}_0$,*

(iv) *the factor algebra $\boldsymbol{D}_0(G)/\boldsymbol{D}_0(G)\cap\boldsymbol{D}(G)\mathfrak{k}_0$ is canonically isomorphic to $\boldsymbol{D}(D/K)$*

Hence we have an isomorphism Γ of $\boldsymbol{D}(S)$ onto $I(\mathfrak{h}_{\mathfrak{p}_0})$. For each $D\in\boldsymbol{D}(S)$, let D_0 be any operator in $\boldsymbol{D}_0(G)$ which goes into D by the natural homomorphism μ of $\boldsymbol{D}_0(G)$ onto $\boldsymbol{D}(S)$. Making use of the canonical isomorphism

$D(A_\mathfrak{p})\cong S(\mathfrak{h}_{\mathfrak{p}_0})$, we obtain an isomorphism $\hat{\Gamma}$ of $\boldsymbol{D}(\hat{S})$ ont $S(\mathfrak{h}_{\mathfrak{p}_0})$ under the diffeomorphism ψ: $(kM, a)\to kaMN$, of the fibre bundle $K/M\times A_\mathfrak{p}$ onto $\hat{S}$ [5]. Also, under the canonical isomorphism $\boldsymbol{D}(A_\mathfrak{p})\cong S(\mathfrak{h}_{\mathfrak{p}_0})$, the unique automorphism $p\to{}'p$ of $S(\mathfrak{h}_{\mathfrak{p}_0})$ given by $'H=H-\rho(H)$ $(H\in\mathfrak{h}_{\mathfrak{p}_0})$ corresponds to the automorphism $D\to e^{\rho}D\circ e^{-\rho}$ of $\boldsymbol{D}(A_\mathfrak{p})$. If we define the mapping $\wedge: D\to\hat{D}$ by

$$'\hat{\Gamma}(\hat{D})=\Gamma(D),$$

it is an isomorphism of $\boldsymbol{D}(S)$ into $\boldsymbol{D}(\hat{S})$. The image of this mapping will be denoted by $\hat{\boldsymbol{D}}(\hat{S})$.

3. The functions ω, Ω, ξ, Ξ and σ.

For $x\in S=G/K$ and $g\in G$ such that $\pi(g)=x$, there exists a unique element $X\in\mathfrak{p}_0$ such that $x=\pi(\exp X)=\exp X\cdot K$. Put

$$\Omega(g)=\omega(x)=\{det(\sinh adX/adX)_{\mathfrak{p}_0}\}^{\frac{1}{2}},$$

where $(\sinh adX/adX)_{\mathfrak{p}_0}$ denotes the restriction on $\mathfrak{p}_0$ of the linear transformation

$$\sinh adX/adX=\sum_{q\geqq 0}(adX)^{2q}/(2q+1)!$$

of $\mathfrak{g}_0$ and $det(\ )$ denotes the determinant of ( ). Put

$$\sigma(g)=\sigma(x)=||X||,$$

where $||X||$ denotes the norm of X by means of the inner product which is induced from the Killing form B. Also put

$$\xi(x)=\int_K e^{-\rho(H(\exp X\cdot k))}\,dk,$$

and

$$\Xi(g)=\int_K e^{-\rho(H(gk))}\,dk.$$

If we write $h=\exp H\,(h\in A_\mathfrak{p}^+,\ H\in\mathfrak{h}_{\mathfrak{p}_0}^+)$ and $\pi(h)=\bar{h}$, since

$$\omega(\bar{h})^2=D(h),$$

where

$$D(h)=\prod_{\alpha\in\Sigma}(e^{\alpha(H)}-e^{-\alpha(H)}),$$

there exist a positive constant c_2 and a positive integer d such that

$$\omega(\bar{h}) \leqq c_2 \xi(\bar{h})^{-1}(1+\sigma(\bar{h}))^d, \quad h \in A_{\mathfrak{p}}^+ \tag{1}$$

[3].

4. The Schwartz spaces of S and $\hat{S}$.

After Harish-Chandra let us define the Schwartz space of S. For $f \in C^\infty(S)$, $D \in \boldsymbol{D}(G)$ and $d \geqq 0$, put

$$\nu_{D,d}(f) = \sup_G |D(f \circ \pi)| (1+\sigma)^d \xi^{-1},$$

$$\tau_{D,d}(f) = \sup_G |D(f \circ \pi)| (1+\sigma)^d \omega.$$

Let $\mathscr{C}(S)$ (respectively, $\mathscr{S}(S)$) denote the space of all $f \in C^\infty(S)$ such that $\nu_{D,d}(f) < +\infty$ (respectively, $\tau_{D,d}(f) < +\infty$) for all $D \in \boldsymbol{D}(G)$ and $d \geqq 0$. We topologize $\mathscr{C}(S)$ (respectively, $\mathscr{S}(S)$) by means of the system of the seminorms $\nu_{D,d}$ (respectively, $\tau_{D,d}$) ($D \in \boldsymbol{D}(G)$, $d \geqq 0$). Then $\mathscr{C}(S)$ and $\mathscr{S}(S)$ are Hausdorff, locally convex and complete spaces. And we call $\mathscr{C}(S)$ the Schwartz space of S.

Let $\mathscr{C}(\hat{S})$ denote the set of all functions $\varphi \in C^\infty(\hat{S})$ which satisfy the following condition: For every $E \in \boldsymbol{D}(A_{\mathfrak{p}})$, $u \in \boldsymbol{D}(K/M)$ and $r \geqq 0$

$$\mu_{E,u,r}(\varphi) = \sup_{(kM,a) \in (K/M) \times A_{\mathfrak{p}}} (1+\|\log a\|)^r |[Eu(\varphi \circ \psi)](kM, a)| < +\infty,$$

where ψ is the diffeomorphism $(kM, a) \to kaMN$ of $(K/M) \times A_{\mathfrak{p}}$ onto $\hat{S}$. By means of this system of the seminorms, we topologize $\mathscr{C}(\hat{S})$. Then $\mathscr{C}(\hat{S})$ is a locally convex space too, and we call it the *Schwartz space of* $\hat{S}$.

5. Proof of the theorems.

As a colollary of the theorem 1 in [1], we obtain by (1) the following

THEOREM A. *For any* $f \in \mathscr{C}(S)$ *and* $D \in \boldsymbol{D}(S)$

$$\widehat{Df} = \hat{D}\hat{f}.$$

Let us denote by e^ρ the function $(e^\rho)(kan) = e^{\rho(\log a)}$ defined on $G = KA_{\mathfrak{p}}N$ and put

$$F_f(xMN) = [e^\rho(\hat{f} \circ \hat{\pi})](x) = e^{\rho(H(x))} \int_N f(xn)\, dn, \qquad (x \in G).$$

THEOREM B. *The mapping* $f \to F_f$ *is a one-to-one continuous linear mapping of* $\mathscr{C}(S)$ *into* $\mathscr{C}(\hat{S})$.

PROOF. To prove $F_f \in \mathscr{C}(S)$ and the continuity of the mapping $f \to F_f$, we use the following

Lemma ([3], pp. 106). *Put* $\bar{n}=\theta(n^{-1})$. *Then there exist* $d \geqq 0$ *and* $c \geqq 1$ *such that*

$$1+\max(\sigma(a), \rho(H(\bar{n})) \leqq c(1+\sigma(an))$$

and

$$\Xi(an) \leqq c(1+\sigma(an))^d \exp\{-\rho(\log a)-\rho(H(\bar{n}))\}$$

for $a \in A_{\mathfrak{p}}$ *and* $n \in N$.

Now let $E \in \boldsymbol{D}(A_{\mathfrak{p}})$ and $u \in \boldsymbol{D}(K/M)$. Then we can regard E, u as $E, u \in \boldsymbol{D}(G)$ in a natural way and there exists an elment $\tilde{E} \in \boldsymbol{D}(A_{\mathfrak{p}})$, depending on E but independent of u, such that

$$(Eu)(e^{\rho}f)(kan)=e^{\rho(\log a)}[(\tilde{E}u)f](kan)$$

for $k \in K$, $a \in A_{\mathfrak{p}}$ and $n \in N$. Applying the above lemma, for every positive integers d and l, we can find a constant c_1 such that

$$(1+\sigma(a))^l \ [(Eu)(e^{\rho}f)](kan)$$
$$\leqq c_1(1+\sigma(kan))^{l+1+d}(\tilde{E}uf)(kan)e^{-\rho(H(\bar{n}))}(1+\rho(H(\bar{n})))^{-(1+d)}$$

for all $k \in K$, $a \in A_{\mathfrak{p}}$ and $n \in N$. Since there exists an integer d satisfying

$$\int_{\bar{N}} e^{-\rho(H(\bar{n}))}(1+\rho(H(\bar{n})))^{-(1+d)}d\bar{n}<+\infty$$

([2], pp. 289), for every integers l and every differential operators $E \in \boldsymbol{D}(A_{\mathfrak{p}})$, $u \in \boldsymbol{D}(K/M)$ we have

$$\sup_{(kM,a)\in(K/M)\times A_{\mathfrak{p}}} (1+\sigma(a))^l|[(Eu)(e^{\rho(\log a)}f(kM, a))]|$$
$$\leqq c_1 \sup_{g\in G}(1+\sigma(g))^{l+1+d}|(\tilde{E}uf)(g)|\Xi(g)^{-1}\int_{\bar{N}} e^{-\rho(H(\bar{n}))}(1+\rho(H(\bar{n})))^{-(1+d)}d\bar{n}$$
$$<+\infty,$$

which shows $F_f \in \mathscr{C}(S)$ and the mapping $f \to F_f$ is continuous.

From now on, we assume that G is a complex semisimple Lie group. Then there exists an explicit differential operator $\square \in \boldsymbol{D}(S)$ such that for all $f \in C_c^{\infty}(S)$,

$$\square((\hat{f})^{\vee})=cf,$$

where c is a constant $\neq 0$, independent of f [5]. Moreover we know that the inclusion mapping $C_c^{\infty}(S)$ into $\mathscr{C}(S)$ is continuous and the image is dense in $\mathscr{C}(S)$ and that $\mathscr{C}(S)$ is invariant under the left translations $\tau(x)$, $x \in G$ [3].

THEOREM C. *For any* $f \in \mathscr{C}(S)$,

$$\square((\hat{f})^{\vee})=cf.$$

PROOF. We prove this in the same way as in [5], by means of the density of $C_c^\infty(S)$ in $\mathscr{C}(S)$. Let $f_0 \in \mathscr{C}(S)$. Then there exists a sequence $\{f_m\}$ in $C_c^\infty(S)$ which converges to f_0 with respect to the topology in $\mathscr{C}(S)$. Put $F_m=f_m\circ\pi$ $(m=0, 1, 2,\cdots)$ and define F_{m1} by

$$F_{m1}(g)=\int_{K\times N}F_m(kng)\,dk\,dn, \qquad (m=0, 1, 2,\cdots).$$

Then we obtain

$$F_m(e)=c\lim_{\substack{b\to e\\ b\in A_\mathfrak{p}}}[\square_0F_{m1}](b), \qquad (m=1, 2,\cdots),$$

where c is a constant. We shall prove the same formula for F_0. Since for any $D \in \boldsymbol{D}(S)$

$$[D_0F_{01}](g)-[D_0F_{m1}](g)=\int_{K\times N}([D_0F_0](kng)-[D_0F_m](kng))dk\,dn,$$

$(m=1, 2,\cdots)$, in particular, for $b \in A_\mathfrak{p}$,

$$|[D_0F_{01}](b)-[D_0F_{m1}](b)|\leqq e^{2\rho(\log b)}\int_{K\times N}|[D_0F_0](kbn)-[D_0F_m](kbn)|\,dk\,dn$$

$$\leqq e^{2\rho(\log b)}c_1^{-1}c_2\sup_{g\in G}\Omega(g)(1+\sigma(g))^d\,|[D_0F_0](g)-[D_0F_m](g)|,$$

where c_1, c_2 are certain positive constants and d is a positive integer. Hence we have

$$\lim_{\substack{b\to e\\ b\in A_\mathfrak{p}}}|[D_0F_{01}](b)-[D_0F_{m1}](b)|\;=0.$$

And therefore

$$F_0(e)=c\lim_{\substack{b\to e\\ b\in A_\mathfrak{p}}}[\square_0F_{01}](b).$$

Now since $\mathscr{C}(S)$ is invariant under the left translations $\tau(x)$ $(x \in G)$, if we put $F_0^x=f_0^{(x^{-1})}\circ\pi$ and put

(3) $$F_{01}^x(g)=\int_{K\times N}F_0^x(kng)\,dk\,dn,$$

we have

(4) $$F_0(x)=c\lim_{\substack{b\to e\\ b\in A_\mathfrak{p}}}[\square_0F_{01}^x](b).$$

The rest of the proof is same as the one in [5].
From (3), we have

$$[\square_0 F_{01}^x](g)=\int_N\left(\int_K[\square_0 F_0](xkng)\ dk\right)dn.$$

Now let $g=b$ and $b\to e$ in $A_\mathfrak{p}$. Then by (4)

$$f_0(x\cdot o)=F_0(x)=c\int_N\left(\int_K[\square_0 F_0]\ (xkn)\ dk\right)dn,$$

which, by commutativity of the mean value operators with the differential operators [4], equals

$$c\int_N\square_x\left(\int_K F(xkn)\ dk\right)dn=c\square_{0x}\left(\int_{K\times N}F_0(xkn)\ dk\right)dn,$$

where the subscript x denotes the argument on which $\square$ and $\square_0$ act. Therefore we have

$$f_0(x\cdot o)=c[\square(\hat{f}_0)^\vee](x\cdot o)$$

and the theorem is proved.

the proof of this theorem suggests the following

THEOREM D. *Let $\check{E}\in \boldsymbol{D}(S)$ corresponds to $E\in\hat{\boldsymbol{D}}$ under the isomorphism $\boldsymbol{D}(S)\cong\hat{\boldsymbol{D}}$. For any function φ in the image of $\mathcal{C}(S)$ by the Radon transform, the following relation holds.*

$$(E\varphi)^\vee=\check{E}\check{\varphi}.$$

PROOF. Let $\hat{f}=\varphi$, $f\in\mathcal{C}(S)$, and $\hat{D}=E$, $D\in\boldsymbol{D}(S)$. And put $F=f\circ\pi$. Then

$$(\check{E}\check{\varphi})(x\cdot o)=D_0\int_K\int_N F(xkn)\ dn\ dk=D_0\int_N\int_K F(xkn)\ dk\ dn=$$

$$=\int_N\left(D\int_K F(xkn)\ dk\right)dn\cdot$$

Since, in the last integral, we can exchange the mean value operator for the differential operator, the last integral equals

$$\int_N\int_K[D_0F](xkn)\ dk\ dn=\int_K\widehat{DF}\ (xk)\ dk$$

$$=(E\varphi)^\vee(x\cdot o).$$

This proves the theorem.

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