HIROSHIMA MATH. J.
2 (1972), 483–533

Polarizations of Certain Homogeneous Spaces and Most Continuous Principal Series

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(Received September 20, 1972)

§ 1. Introduction

Our main purpose in this paper is to construct unitary representations of the most continuous principal series, using polarizations. As is stated in § 1 of [13], a polarization on a symplectic manifold was devised by Kostant with the aim of constructing unitary representations for an arbitrary Lie group. It is an extension of the nilpotent case given in Kirillov [8], and has enough effectiveness in solvable Lie groups of type I (Auslander-Kostant [2]). For semisimple Lie groups, however, the situation is slightly different from them. For example, it has been pointed out by many people that the discrete series representations of a non-compact semisimple Lie group of the non-Hermitian type can not be obtained by polarizations only, and some concepts, like cohomology spaces, seem to be required. However, we can show that the representations of the most continuous principal series can all be constructed by using polarizations (Theorem 6.6). This is partly because a polarization of any semisimple element in the Cartan subalgebra with maximal vector part can be chosen related with a minimal parabolic subalgebra by translating the element by the addition of a certain nilpotent element, and partly because the differential equations attached to the polarization can be replaced by the Borel-Weil theorem of a compact reductive Lie group. In this paper, we also make investigations in each simple Lie algebra, and prove that in case of (AI-AIII), $\mathfrak{so}(n, 1)$ or (EIV), every element has w-polarizations, while there exists an element with no polarizations in Lie algebras of any other type (Theorem 4.6). The proof is made by using a suitable TDS with high singularity.

The author should like to express his hearty thanks to Professor H. Ozeki for his kind advice and useful discussions.

§ 2. Real admissible polarizations

In this paper, except for § 5, we assume that G is a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_R$. (In § 3, $\mathfrak{g}_R$ is assumed to be simple.) Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_R$, and B the Killing form of $\mathfrak{g}$. Notations are due to [13].

LEMMA 2.1. *Let $\mathfrak{p}$ be a w-polarization (in the sense of Definition 7.1 [13]) of a nilpotent element e in $\mathfrak{g}$, and $\mathfrak{g}_j$ the j-eigenspace of $ad_{\mathfrak{g}}(x)$ where x is a mono-semisimple element corresponding to e. Then*

1) $\dim(Ad(g)\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}\cap\mathfrak{g}_j)$ *for every* $g\in(G^c)^e$,

2) $\dim(\sigma\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}\cap\mathfrak{g}_j)$ *if* $e\in\mathfrak{g}_R$.

(Note that in this case x does not necessarily belong to $\mathfrak{g}_R$.)

PROOF. 1) By Lemma 3.2 of [13], $(G^c)^e$ is the semi-direct product of $(G^c)_e$ and $(G^c)^e\cap(G^c)^x$. Since $(G^c)_e$ is connected, it stabilizes $\mathfrak{p}$, and so we need only to prove the relation 1) for $g\in(G^c)^e\cap(G^c)^x$. The space $\mathfrak{g}_j$ is stable under $Ad(g)(g\in(G^c)^e\cap(G^c)^x)$. So we have

$$\begin{aligned}Ad(g)\mathfrak{p}\cap\mathfrak{g}_j&=Ad(g)\mathfrak{p}\cap Ad(g)\mathfrak{g}_j\\&=Ad(g)(\mathfrak{p}\cap\mathfrak{g}_j).\end{aligned}$$

Hence

$$\begin{aligned}\dim(Ad(g)\mathfrak{p}\cap\mathfrak{g}_j)&=\dim Ad(g)(\mathfrak{p}\cap\mathfrak{g}_j)\\&=\dim(\mathfrak{p}\cap\mathfrak{g}_j).\end{aligned}$$

2) Let (x, e, f) be an S-triple containing e as the nilpositive element. Then $(\sigma x, e, \sigma f)$ is also an S-triple. Owing to the Kostant's results stated in §3 [13], we can find an element $g\in(G^c)_e$ such that $\sigma x=gx$. We shall show that $\sigma\mathfrak{g}_j$ coincides with $Ad(g)\mathfrak{g}_j$. Indeed, we have

$$\begin{aligned}\sigma\mathfrak{g}_j&=\{\sigma X;[x, X]=jX\}\\&=\{Y;[x,\sigma^{-1}Y]=j\sigma^{-1}Y\}\ (\text{where } Y=\sigma X)\\&=\{Y;[\sigma x, Y]=jY\}\\&=\{Y;[Ad(g)x, Y]=jY\}\\&=\{Y;Ad(g)[x, Ad(g^{-1})Y]=jY\}\\&=\{Ad(g)Z;Ad(g)[x, Z]=jAd(g)Z\}\\&\qquad\qquad(\text{where } Z=Ad(g^{-1})Y)\\&=\{Ad(g)X;[x, X]=jX\}\\&=Ad(g)\mathfrak{g}_j.\end{aligned}$$

Therefore

$$\begin{aligned}\dim(\sigma\mathfrak{p}\cap\mathfrak{g}_j)&=\dim\sigma(\mathfrak{p}\cap\sigma\mathfrak{g}_j)\\&=\dim(\mathfrak{p}\cap\sigma\mathfrak{g}_j)=\dim(\mathfrak{p}\cap Ad(g)\mathfrak{g}_j)\\&=\dim Ad(g)(Ad(g^{-1})\mathfrak{p}\cap\mathfrak{g}_j)\\&=\dim(Ad(g^{-1})\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}\cap\mathfrak{g}_j)\end{aligned}$$

since $Ad(g^{-1})\mathfrak{p}=\mathfrak{p}\quad(g\in(G^c)_e)$.

Q.E.D.

PROPOSITION 2.2. *Let e be a nilpotent element in $\mathfrak{g}_R$. Assume that the characteristic of a mono-semisimple element x of e consists only of integers. Then e has a real polarization.*

PROOF. We set $\mathfrak{p}=\sum_{j\geqq 0}\mathfrak{g}_j$. Then $\mathfrak{p}$ is a w-polarization of e (Proposition 5.1 of [13]). Further by Lemma 2.1, we have

$$Ad((G^c)^e)\mathfrak{p}=\mathfrak{p},$$

and

$$\sigma\mathfrak{p}=\mathfrak{p}.$$

Thus $\mathfrak{p}$ is a real polarization of e. Q.E.D.

PROPOSITION 2.3. *Let $\mathfrak{p}$ be a w-polarization of a nilpotent element e in $\mathfrak{g}_R$. Assume that e has not a w-polarization $\mathfrak{p}'$ of e other than $\mathfrak{p}$ such that* $\dim(\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}'\cap\mathfrak{g}_j)$ *for every j. Then $\mathfrak{p}$ is a real polarization of e.*

PROOF. $\sigma\mathfrak{p}$ and $Ad(g)\mathfrak{p}(g\in(G^c)^e)$ are w-polarizations of e satisfying

$$\dim(\sigma\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}\cap\mathfrak{g}_j),$$

and

$$\dim(Ad(g)\mathfrak{p}\cap\mathfrak{g}_j)=\dim(\mathfrak{p}\cap\mathfrak{g}_j),$$

by Lemma 2.1. And so, by our assumption, $\sigma\mathfrak{p}$ and $Ad(g)\mathfrak{p}$ must coincide with $\mathfrak{p}$. Thus $\mathfrak{p}$ is a real polarization of e. Q.E.D.

PROPOSITION 2.4. *Let $\mathfrak{p}$ be a w-polarization of an element X in $\mathfrak{g}_R$. Assume that any w-polarization of X except for $\mathfrak{p}$ is not conjugate to $\mathfrak{p}$ under the action of the automorphism group* $\mathrm{Aut}(\mathfrak{g})$ *of $\mathfrak{g}$. Then $\mathfrak{p}$ is a real polarization of X.*

PROOF. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then $\mathfrak{h}'=\sigma\mathfrak{h}$ is a Cartan subalgebra of $\sigma\mathfrak{p}$. Let Δ (resp. Δ') the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (resp. $\mathfrak{h}'$). For each $\alpha\in\Delta$, $\sigma\alpha\in\Delta'$ is defined by

$$(\sigma\alpha)(H)=\overline{\alpha(\sigma H)}\qquad\text{for every } H\in\mathfrak{h}',$$

and this correspondence becomes a bijection of Δ to Δ'. We define $H_\alpha \in \mathfrak{h}$ and $H'_\alpha \in \mathfrak{h}'(\alpha \in \Delta, \alpha' \in \Delta')$ by

$$B(H_\alpha, H)=\alpha(H) \qquad \text{for every } H \in \mathfrak{h},$$

$$B(H'_\alpha, H')=\alpha'(H') \qquad \text{for every } H' \in \mathfrak{h}',$$

and we set

$$\mathfrak{h}_R=\sum_{\alpha \in \Delta} \boldsymbol{R} H_\alpha,$$

and

$$\mathfrak{h}'_R=\sum_{\alpha' \in \Delta'} \boldsymbol{R} H'_\alpha,$$

where B denotes the Killing form of $\mathfrak{g}$. By Theorom 5.4 (Chap. III) of Helgason [7], there exists a Lie algebra automorphism φ of $\mathfrak{g}$, such that $\varphi=\sigma$ on $\mathfrak{h}_R$. Then we have

$$\varphi(\mathfrak{g}^\alpha)=\sigma(\mathfrak{g}^\alpha)=\mathfrak{g}^{\sigma\alpha}$$

for every $\alpha \in \Delta$, because, for $X \in \mathfrak{g}^\alpha$ and $H \in \mathfrak{h}_R$,

$$\begin{aligned}[H, \varphi X] &= \varphi[\varphi^{-1}H, X]=\varphi[\sigma^{-1}H, X] \\ &= \varphi[\sigma H, X]=\varphi(\alpha(\sigma H)X) \\ &= \alpha(\sigma H)\varphi(X)=\overline{\alpha(\sigma H)}\varphi(X) \\ &= (\sigma\alpha)(H)\varphi(X).\end{aligned}$$

So we have $\varphi(\mathfrak{p})=\sigma(\mathfrak{p})$, i.e., $\sigma\mathfrak{p}$ is a w-polarization of X which is conjugate to $\mathfrak{p}$ under Aut($\mathfrak{g}$). By our assumption, $\sigma\mathfrak{p}$ must coincide with $\mathfrak{p}$. It also follows from our assumption, that $\mathfrak{p}$ is $Ad((G^c)^e)$-stable, so $\mathfrak{p}$ is a real polarization of X. Q.E.D.

§ 3. Polarizations and cuspidal parabolic subalgebras

Let $\mathfrak{g}_R=\mathfrak{k}_0+\mathfrak{p}_0$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_R$, $\mathfrak{a}_+$ a maximal abelian subspace of $\mathfrak{p}_0$, and $\mathfrak{a}_0=\mathfrak{a}_-+\mathfrak{a}_+(\mathfrak{a}_-\subset\mathfrak{k}_0)$ be a Cartan subalgebra of $\mathfrak{g}_R$. Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_-^c$ and $\mathfrak{a}_+^c$ the complexification of $\mathfrak{g}_R$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_-$ and $\mathfrak{a}_+$ respectively. Fix a compatible order in the non-zero root system Δ of $(\mathfrak{g}, \mathfrak{a})$ with respect to $(\mathfrak{a}_R=\sqrt{-1}\mathfrak{a}_-+\mathfrak{a}_+, \mathfrak{a}_+)$, and we set

$$\Delta_+=\text{the set of all positive roots in } \Delta,$$

$$\Lambda_+=\{\alpha \in \Delta_+;\ \alpha \text{ does not vanish on } \mathfrak{a}_+\},$$

$$\Sigma \;=\{\alpha \in \Delta;\ \alpha \text{ vanishes on } \mathfrak{a}_+\},$$

$$\Sigma_+=\Sigma\cap\Delta_+,$$

$$\mathfrak{n} \;=\sum_{\alpha\in\Delta_+}\mathfrak{g}^\alpha,\ \mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_R,$$

$$\mathfrak{m} \;=\text{the centralizer of } \mathfrak{a}_+ \text{ in } \mathfrak{k},$$

$$\mathfrak{m}_0 =\mathfrak{m}\cap\mathfrak{g}_R=\text{the centralizer of } \mathfrak{a}_+ \text{ in } \mathfrak{k}_0,$$

$$\mathfrak{b}_0 =\mathfrak{m}_0+\mathfrak{a}_++\mathfrak{n}_0,$$

and

$$\mathfrak{b}=\mathfrak{b}_0^c=\mathfrak{m}+\mathfrak{a}_+^c+\mathfrak{n}.$$

For every $\alpha \in \Delta$, $H_\alpha \in \mathfrak{a}_R$ is defined by

$$B(H_\alpha, H)=\alpha(H) \qquad \text{for every } H\in\mathfrak{a},$$

where B denotes the Killing form of $\mathfrak{g}$. For simplicity, a root $\alpha\in\Delta$ is often identified with H_α.

Let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ be the fundamental root system, and $\{\varepsilon_1,\cdots,\varepsilon_l\}$ the basis of $\mathfrak{a}$ dual to $\{\alpha_1,\cdots,\alpha_l\}$.

THEOREM 3.1. *There exists a nilpotent element in* $\mathfrak{g}_R$ *with a real polarization* $\mathfrak{b}$.

PROOF. We set $\Phi=\Pi\cap\Delta_+$, and write $\alpha\sim\beta(\alpha,\ \beta\in\Phi)$ if $\alpha|\mathfrak{a}_+=\beta|\mathfrak{a}_+$. (In other words, $\alpha\sim\beta$ implies that $\alpha=\beta$ or they are combined with an arrow in the Satake diagram.) Let $\{\alpha_{\mu_i};\ 1\leqq i\leqq k\}$ be the subset of Φ consisting of all representatives of equivalence classes in Φ with respect to $\sim$. By a suitable arrangement of $\alpha_1,\cdots,\alpha_l$, we assume that $\Phi/\sim=\{\alpha_1,\cdots,\alpha_k\}$ and that

$$\sigma\alpha_i=\alpha_i \qquad \text{for } 1\leqq i\leqq p,$$

$$\sigma\alpha_i\neq\alpha_i \qquad \text{for } p+1\leqq i\leqq k.$$

We set

$$e=\sum_{i=1}^{p}e_{\alpha_i}+\sum_{i=p+1}^{k}(e_{\alpha_i}+\sigma e_{\alpha_i})$$

where e_α is a non-zero vector in $\mathfrak{g}^\alpha(\alpha\in\Delta)$ satisfying $B(e_\alpha, e_{-\alpha})=1$ and $\sigma e_\alpha=e_{\sigma\alpha}$. We set

$$f'=\sum_{i=1}^{p}r_ie_{-\alpha_i}+\sum_{i=p+1}^{k}(r_ie_{-\alpha_i}+s_ie_{-\sigma\alpha_i})$$

$$(r_i\in \boldsymbol{C}\,(1\leqq i\leqq k) \text{ and } s_i\in\boldsymbol{C}\,(p+1\leqq i\leqq k))$$

and

$$x=[e, f'].$$

Then we have

$$x=\sum_{j=1}^{p} r_j\alpha_j+\sum_{j=p+1}^{k}(r_j\alpha_j+s_j\sigma\alpha_j),$$

and $[x, e]$ is given as follows:

$$\begin{aligned}[x, e]&=\sum_{i=1}^{k}\alpha_i(\sum_{j=1}^{p} r_j\alpha_j+\sum_{j=p+1}^{k}(r_j\alpha_j+s_j\sigma\alpha_j))e_{\alpha_i}\\&\quad+\sum_{i=p+1}^{k}\sigma\alpha_i(\sum_{j=1}^{p} r_j\alpha_j+\sum_{j=p+1}^{k}(r_j\alpha_j+s_j\sigma\alpha_j))e_{\sigma\alpha_i}\\&=\sum_{i=1}^{k}\{\sum_{j=1}^{k} r_j(\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\alpha_i, \sigma\alpha_j)\}e_{\alpha_i}\\&\quad+\sum_{i=p+1}^{k}\{\sum_{j=1}^{k} r_j(\sigma\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\sigma\alpha_i, \sigma\alpha_j)\}e_{\sigma\alpha_i}\\&=\sum_{i=1}^{k}\{\sum_{j=1}^{k} r_j(\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\alpha_i, \sigma\alpha_j)\}e_{\alpha_i}\\&\quad+\sum_{i=p+1}^{k}\{\sum_{j=1}^{k} r_j(\sigma\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\alpha_i, \alpha_j)\}e_{\sigma\alpha_i}.\end{aligned}$$

From the relation $[x, e]=e$ (this is a necessary and sufficient condition in order that x may be a mono-semisimple element corresponding to e), we have a system of linear equations:

$$\text{(1)}\qquad\begin{cases}\sum_{j=1}^{k} r_j(\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\alpha_i, \sigma\alpha_j)=1 & (1\leqq i\leqq k),\\ \sum_{j=1}^{k} r_j(\sigma\alpha_i, \alpha_j)+\sum_{j=p+1}^{k} s_j(\alpha_i, \alpha_j)=1 & (p+1\leqq i\leqq k).\end{cases}$$

Now we set

$$A=\begin{pmatrix}(\alpha_1, \alpha_1) & \cdots (\alpha_1, \alpha_k) & \cdots (\alpha_1, \sigma\alpha_{p+1}) & \cdots (\alpha_1, \sigma\alpha_k)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha_k, \alpha_1) & \cdots (\alpha_k, \alpha_k) & \cdots (\alpha_k, \sigma\alpha_{p+1}) & \cdots (\alpha_k, \sigma\alpha_k)\\ (\sigma\alpha_{p+1}, \alpha_1) & \cdots (\sigma\alpha_{p+1}, \alpha_k) & \cdots (\alpha_{p+1}, \alpha_{p+1}) & \cdots (\alpha_{p+1}, \alpha_k)\\ \vdots & \vdots & \vdots & \vdots\\ (\sigma\alpha_k, \alpha_1) & \cdots (\sigma\alpha_k, \alpha_k) & \cdots (\alpha_k, \alpha_{p+1}) & \cdots (\alpha_k, \alpha_k)\end{pmatrix}$$

The matrix A is a positive definite real matrix, since the Killing form B is strictly positive definite on $\mathfrak{a}_R$, and $\{\alpha_1,\cdots, \alpha_k, \sigma\alpha_1,\cdots, \sigma\alpha_k\}$ is linearly independent. The equations (1) are written in the matrix form:

$$(2) \qquad A\cdot{}^t(r_1,\cdots, r_k, s_{p+1},\cdots, s_k)={}^t(1,\cdots, 1)$$

The linear equation (2) (or (1)) has a unique solution, and $r_1,\cdots, r_k, s_{p+1},\cdots, s_k$ are determined as real numbers. Now we shall show that $r_i=s_i$ (for $p+1\leqq i\leqq k$). Noting that

$$(\alpha_i, \alpha_j)=(\sigma\alpha_i, \sigma\alpha_j)=(\alpha_i, \sigma\alpha_j) \qquad \text{for } 1\leqq i\leqq p \text{ and } 1\leqq j\leqq k,$$

we have from (1)

$$\begin{cases} \sum_{j=1}^{p}(\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \sigma\alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)s_j=1 & (\text{for } 1\leqq i\leqq p), \\ \sum_{j=1}^{p}(\sigma\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\sigma\alpha_i, \alpha_j)s_j=1 & (\text{for } p+1\leqq i\leqq k), \\ \sum_{j=1}^{p}(\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \sigma\alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)s_j=1 & (\text{for } p+1\leqq i\leqq k). \end{cases}$$

Changing the second equations with the third, and the second terms with the third, we have

$$(1)' \quad \begin{cases} \sum_{j=1}^{p}(\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)s_j+\sum_{j=p+1}^{k}(\alpha_i, \sigma\alpha_j)r_j=1 & (1\leqq i\leqq p), \\ \sum_{j=1}^{p}(\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)s_j+\sum_{j=p+1}^{k}(\alpha_i, \sigma\alpha_j)r_j=1 & (p+1\leqq i\leqq k), \\ \sum_{j=1}^{p}(\sigma\alpha_i, \alpha_j)r_j+\sum_{j=p+1}^{k}(\sigma\alpha_i, \alpha_j)s_j+\sum_{j=p+1}^{k}(\alpha_i, \alpha_j)r_j=1 & (p+1\leqq i\leqq k). \end{cases}$$

Using the matrix A, these equations become as follow:

$$(2)' \qquad A\cdot{}^t(r_1,\cdots, r_p, s_{p+1},\cdots, s_k, r_{p+1},\cdots, r_k)={}^t(1,\cdots, 1).$$

By the uniqueness of solutions of equations (2) and (2)′, we have

$$s_i=r_i \qquad (\text{for } p+1\leqq i\leqq k).$$

Thus we have proved that the element

$$x=\sum_{i=1}^{p} r_i H_{\alpha_i}+\sum_{i=p+1}^{k} r_i(H_{\alpha_i}+\sigma H_{\alpha_i}) \in \mathfrak{a}_R$$

(where $r_1,\cdots, r_k$ are solutions of equations (1)) satisfies $[x, e]=e$ and $x \in [e, \mathfrak{g}]$, so x is a mono-semisimple element corresponding to e. Now we expand the above x by the basis $\{\varepsilon_1,\cdots, \varepsilon_l\}$:

$$x=\sum_{i=1}^{l} c_i\varepsilon_i,$$

where $c_i=\alpha_i(x) \in \boldsymbol{R}$ for $1\leqq i\leqq l$.

We shall make an investigation into the characteristic $(c_1, \cdots, c_l)$ of x. We shall show that

$$c_i = \begin{cases} 0 \text{ (if } \alpha_i \text{ is a purely-imaginary root)} \\ 1 \text{ (otherwise).} \end{cases}$$

Indeed, if α_i is a purely-imaginary simple root (i.e., $\sigma\alpha_i = -\alpha_i$), we have

$$(\alpha_j, \alpha_i) = 0 \qquad \text{for } 1 \leqq j \leqq p$$

(because $(\alpha_j, \alpha_i) = (\sigma\alpha_j, \sigma\alpha_i) = -(\alpha_j, \alpha_i)$) and

$$(\alpha_j, \alpha_i) + (\sigma\alpha_j, \alpha_i) = 0 \qquad \text{for } p+1 \leqq j \leqq k.$$

Thus we have

$$\begin{aligned} c_i &= \alpha_i(x) \\ &= \sum_{j=1}^{p} r_j(\alpha_j, \alpha_i) + \sum_{j=p+1}^{k} r_j\{(\alpha_j, \alpha_i) + (\sigma\alpha_j, \alpha_i)\} \\ &= 0. \end{aligned}$$

Next we consider the case when $\alpha_i \in \Phi$. From $x = \sum_{i=1}^{l} c_i \varepsilon_i$, we have

$$[x, e] = \sum_{i=1}^{p} c_i e_{\alpha_i} + \sum_{i=p+1}^{k} (c_i e_{\alpha_i} + (x, \sigma\alpha_i) e_{\sigma\alpha_i}).$$

Comparing the coefficients of e_α in the both-hand sides of $[x, e] = e$, we have

$$\begin{cases} c_i = 1 & (1 \leqq i \leqq k) \\ (x, \sigma\alpha_i) = 1 & (p+1 \leqq i \leqq k). \end{cases}$$

For each $i = p+1, \cdots, k$, we can find $\beta_i \in \Pi$ and $\gamma_i \in \mathfrak{a}_R^* = \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{a}_R, \boldsymbol{R})$ such that

$$\begin{cases} \gamma_i | \mathfrak{a}_+ = 0 \\ \sigma\alpha_i = \beta_i + \gamma_i. \end{cases}$$

The root β_i is either equal to α_i or combined with α_i by an arrow in Satake diagram. Then, due to $(x, \sigma\alpha_i) = 1$ and $(x, \gamma_i) = 0$ (this is because $x \in \mathfrak{a}_+$), we have

$$\beta_i(x) = 1 \qquad (p+1 \leqq i \leqq k).$$

By the definition of the equivalence relation "$\sim$" in Φ, Φ is exhausted by

$$\{\alpha_1, \cdots, \alpha_k, \beta_{p+1}, \cdots, \beta_k\},$$

where the expression of this set permits repetition. Thus we have proved that

$$c_i=1 \qquad \text{if } \alpha_i \in \Phi.$$

Since the characteristic of x consists only of integers, $\mathfrak{p}=\sum_{j\geq 0}\mathfrak{g}_j$ is a w-polarization of e by Proposition 5.1 of [13]. Further $\mathfrak{p}$ is a real polarization of e by Proposition 2.2. It is easily seen from the characteristic of x ($c_i=0$ if α_i is purely-imaginary, and $c_i=1$ if $\alpha_i \in \Phi$), that

$$\mathfrak{g}_0=\mathfrak{a}+\sum_{\alpha \in \Sigma}\mathfrak{g}^{\alpha}=\mathfrak{a}_+^c+\mathfrak{m},$$

$$\sum_{j>0}\mathfrak{g}_j=\sum_{\alpha \in \Delta_+}\mathfrak{g}^{\alpha}=\mathfrak{n}.$$

So we have $\mathfrak{p}=\mathfrak{b}$. Q.E.D.

From the proof of the above theorem, we have:

COROLLARY 3.2. *The element in $\mathfrak{a}_+$, whose characteristic is equal to 0 at purely-imaginary roots and to 1 elsewhere, is a mono-semisimple element corresponding to a certain nilpotent element in $\mathfrak{g}_R$.*

Now we introduce the notion of a principal nilpotent element of a real semisimple Lie algebra:

DEFINITION 3.1. A nilpotent element e in $\mathfrak{g}_R$ is called *a principal nilpotent element of* $\mathfrak{g}_R$ if $\dim\mathfrak{g}^e \leqq \dim\mathfrak{g}^X$ for any nilpotent element X in $\mathfrak{g}_R$.

PROPOSITION 3.3 1) *The nilpotent element in Theorem 3.1 is a principal nilpotent element of $\mathfrak{g}_R$.*

2) *Principal nilpotent elements are all conjugate to each other under the action of G^c.*

PROOF. 1) Choose x and e as in the proof of Theorem 3.1. Let e' be a nilpotent element in $\mathfrak{g}_R$, and x' a mono-semisimple element corresponding to e'. The element x' may be assumed to be contained in the closure of the positive Weyl chamber in $\mathfrak{a}_+$ (Corollary 4.2 of [13]). Let $\mathfrak{g}'_j$ be the j-eigenspace of $ad_{\mathfrak{g}}x'$. Since $x' \in \mathfrak{a}_+$, we have

$$\mathfrak{g}'_0=\mathfrak{a}+\sum_{\alpha(x')=0}\mathfrak{g}^{\alpha}\supset\mathfrak{a}+\sum_{\substack{\alpha \in \Delta \\ \alpha|\mathfrak{a}_+=0}}\mathfrak{g}^{\alpha}$$

$$=\mathfrak{a}+\mathfrak{m}=\mathfrak{a}_+^c+\mathfrak{m}.$$

So we have

$$\dim\mathfrak{g}^{e'}=\dim\mathfrak{g}'_0+\dim\mathfrak{g}'_{\frac{1}{2}}\geqq\dim(\mathfrak{a}_+^c+\mathfrak{m}).$$

As we have shown in the proof of Theorem 3.1, the characteristic $(c_1,\cdots, c_l)$

of x has the property:

$$c_i=\begin{cases} 0 & \text{if } \alpha_i \text{ is purely-imaginary,} \\ 1 & \text{otherwise,} \end{cases}$$

and so

$$\dim \mathfrak{g}^e=\dim \mathfrak{g}_0=\dim(\mathfrak{a}_+^c+\mathfrak{m}).$$

Hence $\dim \mathfrak{g}^e \leqq \dim \mathfrak{g}^{e'}$, so e is a principal nilpotent element of $\mathfrak{g}_R$.

2) Let e' be a principal nilpotent element of $\mathfrak{g}_R$, and x' a mono-semisimple element corresponding to e'. By Corollary 4.2 of [13], x' may be assumed to be contained in the closure of the positive Weyl chamber in $\mathfrak{a}_+$. In order to prove this proposition, it is enough to show that $x=x'$. We consider the characteristic $(c_1',\cdots, c_l')=(\alpha_1(x'),\cdots, \alpha_l(x'))$ of x'. From $x' \in \mathfrak{a}_+$, we have $c_i'=0$ for each purely-imaginary root α_i. Now we shall prove that $c_i'=1$ for $\alpha_i \in \Phi=\Pi-\{\text{purely-imaginary simple roots}\}$. By Lemma 3.1 of [13], each c_i' is equal to 0, $\frac{1}{2}$ or 1. Suppose that $c_i'=0$ or $\frac{1}{2}$ for some $\alpha_i \in \Phi$. If $c_i'=0$ for some $\alpha_i \in \Phi$, we have

$$\mathfrak{g}_0' \supset \mathfrak{a}+\mathfrak{m}+\mathfrak{g}^{\alpha_i}+\mathfrak{g}^{-\alpha_i},$$

so we have

$$\dim \mathfrak{g}^{e'} \geqq \dim \mathfrak{g}_0' > \dim(\mathfrak{a}+\mathfrak{m})=\dim \mathfrak{g}^e.$$

This contradicts the fact that e' is principal nilpotent. If $c_i'=\frac{1}{2}$ for some $\alpha_i \in \Phi$, we have

$$\mathfrak{g}_0' \supset \mathfrak{a}+\mathfrak{m} \quad \text{and} \quad \mathfrak{g}_{\frac{1}{2}}' \supset \mathfrak{g}^{\alpha_i},$$

so we have

$$\begin{aligned}\dim \mathfrak{g}^{e'}&=\dim \mathfrak{g}_0'+\dim \mathfrak{g}_{\frac{1}{2}}' > \dim \mathfrak{g}_0' \\ &\geqq \dim(\mathfrak{a}+\mathfrak{m}).\end{aligned}$$

This also contradicts the principality of e'. Thus we have proved $c_i'=1$ for $\alpha_i \in \Phi$, and so we have $x=x'$. Therefore e' is G^C-conjugate to e.

Q.E.D.

Note: Any two principal nilpotent elements in $\mathfrak{g}_R$ are not necessarily conjugate to one another under the action of G. For example, the set of all principal nilpotent elements in $\mathfrak{sl}(2, \boldsymbol{R})$ separates into two $SL(2, \boldsymbol{R})$-orbits; the one through $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the other through $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

From the proof of Proposition 3.3, we have:

COROLLARY 3.4. *Let e be a nilpotent element in $\mathfrak{g}_R$. Then e is a principal nilpotent element of $\mathfrak{g}_R$ if and only if* $\dim \mathfrak{g}^e = \dim(\mathfrak{a}+\mathfrak{m})$.

COROLLARY 3.5. *$\mathfrak{g}_R$ contains a principal nilpotent element of $\mathfrak{g}$, if and only if there exists no purely-imaginary root in Δ (i.e., Σ is empty).*

PROOF. Let e be a principal nilpotent element of $\mathfrak{g}_R$. Then, by Corollary 3.4,

$$\dim \mathfrak{g}^e = \dim(\mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{g}^{\alpha}).$$

So the condition that e is a principal nilpotent element of $\mathfrak{g}$ is equivalent to

$$\dim(\mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{g}^{\alpha}) = \operatorname{rank} \mathfrak{g} = \dim \mathfrak{a},$$

which is equivalent to $\Sigma = \phi$ (the empty set). Q.E.D.

The following theorem is concerned with cuspidal parabolic subalgebras and polarizations, and plays an important role in the construction of principal series representations.

THEOREM 3.6. *Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra, and notations are the same as in §4 of [13]. Let $H_0 = H_1 + H_2 (H_1 \in \mathfrak{h}_-$ and $H_2 \in \mathfrak{h}_+)$ be an element in $\mathfrak{h}_0$ such that $\alpha(H_0) \neq 0$ for every $\alpha \in \Sigma_{\mathfrak{p}}$. Then there exists a nilpotent element e in $\mathfrak{n}_0$ such that*

$$\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}^{\alpha} + \sum_{\substack{\alpha \in \Sigma \\ \alpha(\sqrt{-1} H_1) \geqq 0}} \mathfrak{g}^{\alpha}$$

is an admissible w-polarization of $H_0 + e$.

PROOF. The centralizer $(\mathfrak{g}_R)^{H_0}$ of H_0 in $\mathfrak{g}_R$ is a reductive Lie algebra with the center

$$\mathfrak{z}_0 = \Big(\sum_{\substack{\alpha(H_0)=0 \\ \alpha \in \Delta}} \boldsymbol{R} H_{\alpha} \Big) \cap \mathfrak{g}_R$$

and the semisimple part

$$\mathfrak{l}_0 = \mathfrak{h}_0' + (\sum_{\alpha \in \Delta'} \mathfrak{g}^{\alpha}) \cap \mathfrak{g}_R,$$

where

$$\Delta' = \{\alpha \in \Delta ;\, \alpha(H_0) = 0\} \subset \Sigma_{\mathfrak{k}} \cup \Lambda,$$

$$\mathfrak{h}' = \sum_{\alpha \in \Delta'} \boldsymbol{C} H_{\alpha},$$

and

$$\mathfrak{h}_0' = \mathfrak{h}' \cap \mathfrak{g}_R.$$

As is shown in the proof of Proposition 4.5 of [13], $\mathfrak{h}_0'$ is a Cartan subalgebra of $\mathfrak{l}_0$ with maximal vector part, and $\mathfrak{h}_-' = \mathfrak{h}_0' \cap \mathfrak{k}_0$ (resp. $\mathfrak{h}_+' = \mathfrak{h}_0' \cap \mathfrak{p}_0$) is the toroidal (resp. vector) part of $\mathfrak{h}_0'$. And a lexicographic linear order in the non-zero root system R (which may be identified with $\{\alpha|\mathfrak{h}'; \alpha \in \Delta'\}$) of $(\mathfrak{l}, \mathfrak{h}')$ compatible to $(\mathfrak{h}_0', \mathfrak{h}_+')$ can be chosen so that the subset R_+ of all positive roots in R coincides with $\{\alpha|\mathfrak{h}'; \alpha \in \Delta' \cap \Delta_+\}$. By Theorem 3.1, we can find a principal nilpotent element e of $\mathfrak{l}_0$ with a real polarization (in $\mathfrak{l}$)

$$\mathfrak{q}' = \mathfrak{h}' + \sum_{\substack{\alpha \in \Delta' \\ \alpha|\mathfrak{h}_+' = 0}} \mathfrak{g}^\alpha + \sum_{\substack{\alpha \in \Delta_+' \\ \alpha|\mathfrak{h}_+' \neq 0}} \mathfrak{g}^\alpha.$$

Now we put $X = H_0 + e$, and we shall prove that $\mathfrak{q}$ is an admissible w-polarization of X.

0) $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ since the linear order in Δ is compatible.

i) By definition of $\mathfrak{q}$, we have

$$\dim \mathfrak{g} - \dim(\mathfrak{l} + \mathfrak{z}) = 2(\dim \mathfrak{q} - \dim \mathfrak{q}' - \dim \mathfrak{z}).$$

Since $\mathfrak{q}'$ is a polarization of e in $\mathfrak{l}$, we have

$$\dim \mathfrak{l} - \dim \mathfrak{q}' = \dim \mathfrak{q}' - \dim Z_{\mathfrak{l}}(e).$$

And, as is proved in the above,

$$\dim \mathfrak{g}^X = \dim \mathfrak{z} + \dim Z_{\mathfrak{l}}(e).$$

So we have

$$\dim \mathfrak{q} - \dim \mathfrak{g}^X = \dim \mathfrak{g} - \dim \mathfrak{q}.$$

ii) By definition of $\mathfrak{q}$, we have

$$[\mathfrak{q}, \mathfrak{q}] = \sum_{\substack{\alpha \in \Sigma \\ \alpha(H_0) = 0}} \boldsymbol{C} H_\alpha + \sum_{\alpha \in \Lambda_+} \mathfrak{g}^\alpha + \sum_{\substack{\alpha \in \Sigma \\ \alpha(\sqrt{-1} H_1) \geqq 0}} \mathfrak{g}^\alpha.$$

On the other hand, $X \in \boldsymbol{C} H_0 + \sum_{\alpha \in \Lambda_+} \mathfrak{g}^\alpha$.

Thus we have $B(X, [\mathfrak{q}, \mathfrak{q}]) = \{0\}$.

iv) Since $\sum_{\alpha \in \Lambda_+} \mathfrak{g}^\alpha$ is σ-stable and $\sigma\alpha = -\alpha$ for $\alpha \in \Sigma$, we have

$$\mathfrak{q} + \sigma\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in \Lambda_+} \mathfrak{g}^\alpha + \sum_{\alpha \in \Sigma} \mathfrak{g}^\alpha,$$

which is a subalgebra of $\mathfrak{g}$ because the linear order in Δ is compatible to $(\mathfrak{h}_R, \mathfrak{h}_+)$.

Thus the statement of Theorem 3.6 is proved. Q.E.D.

Note: In the above Theorem 3.6, $(\mathfrak{q}+\sigma\mathfrak{q})\cap\mathfrak{g}_R$ is a cuspidal parabolic subalgebra of $\mathfrak{g}_R$ corresponding to $\mathfrak{h}_0$.

Note: As to the assumption in Theorem 3.6, we remark here that any (non-zero) semisimple element H in $\mathfrak{g}_R$ is G-conjugate to an element H' in some θ-stable Cartan subalgebra $\mathfrak{h}_0$, and which can be chosen so that $\alpha(H')\neq 0$ for every $\alpha\in\Sigma_{\mathfrak{p}}$ (Lemma 4.4 of [13]).

§4. A discussion in simple cases

Let $\mathfrak{g}_R$ be a non-compact real simple Lie algebra, and $\mathfrak{g}_R=\mathfrak{k}_0+\mathfrak{p}_0$ its Cartan decomposition. Choose a Cartan subalgebra $\mathfrak{a}_0=\mathfrak{a}_-+\mathfrak{a}_+$ ($\mathfrak{a}_-\subset\mathfrak{k}_0$, $\mathfrak{a}_+\subset\mathfrak{p}_0$) with maximal vector part, and we set $l=\dim\mathfrak{a}_0(=\operatorname{rank}(\mathfrak{g}))$ and $r=\dim\mathfrak{a}_+(=$ rank of the symmetric space $G/K)$. Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_-^C$ and $\mathfrak{a}_+^C$ the complexification of $\mathfrak{g}_R$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_-$ and $\mathfrak{a}_+$, respectively. Let Δ be the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. A lexicographic order in $\mathfrak{a}_R=\sqrt{-1}\mathfrak{a}_-+\mathfrak{a}_+$ compatible to $\mathfrak{a}_+$ induces a linear order in Δ, and we denote by Δ^+ the set of all positive roots. Let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ be the system of simple roots arranged according to the Dynkin diagram, and $\{\varepsilon_1,\cdots,\varepsilon_l\}$ the basis of $\mathfrak{a}_R$ dual to $\{\alpha_1,\cdots,\alpha_l\}$. A root $\alpha\in\Delta$ will be called *a purely-imaginary root* if $\alpha|\mathfrak{a}_+=0$, *a real root* if $\alpha|\mathfrak{a}_-=0$, and *a mixed root* otherwise. A positive root $\alpha=\sum_{i=1}^{l}a_i\alpha_i$ is expressed simply by $(a_1,\cdots,a_l)$. (In case of type (D) or (E), α is expressed also by $\begin{pmatrix}a_1\cdots a_{l-3}a_{l-2}a_{l-1}\\ a_l\end{pmatrix}$ or $\begin{pmatrix}a_1a_2a_3a_4\cdots a_{l-1}\\ a_l\end{pmatrix}$.) For $\alpha\in\Delta$, we set

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g};\ ad(H)X=\alpha(H)X\quad \text{for every } H\in\mathfrak{a}\}.$$

We choose $e_\alpha\in\mathfrak{g}^\alpha(\alpha\in\Delta)$ such that

$$B(e_\alpha, e_{-\alpha})=1 \text{ and } \sigma e_\alpha=e_{\sigma\alpha},$$

where B denotes the Killing form of $\mathfrak{g}$. And we set

$$H_\alpha=[e_\alpha, e_{-\alpha}].$$

It is well-known that $H_\alpha\in\mathfrak{a}_R$ and $B(H_\alpha, H)=\alpha(H)$ for every $H\in\mathfrak{h}$. We set $|\alpha|^2=\alpha(H_\alpha)$ for $\alpha\in\Delta$.

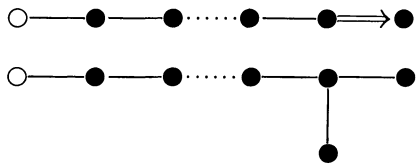
LEMMA 4.1. *In case of* $\mathfrak{g}_R=\mathfrak{so}(n, 1)$,

1) *every (non-zero) nilpotent element in* $\mathfrak{g}_R$ *is a principal nilpotent element of* $\mathfrak{g}_R$, *and*

2) *every nilpotent element in* $\mathfrak{g}_R$ *has a real polarization.*

PROOF. 1) A (non-zero) nilpotent element in $\mathfrak{g}_R$ is embedded into an S-triple in $\mathfrak{g}_R$ as the nilpositive element, which is G-conjugate to a standard

S-triple (x, e, f) (Lemma 3.3. [13] and Corollary 4.2 [13]). The characteristic of $x (x \in \mathfrak{a}_+)$ is zero at purely-imaginary roots. The Satake diagram of $\mathfrak{g}_R$ is as follows:



(if $n=2l$)

(if $n=2l+1$)

So the characteristic of x is

i) $(1, 0, \cdots, 0)$ $\quad (x=\varepsilon_1)$, or

ii) $\left(\frac{1}{2}, 0, \cdots, 0\right)$ $\quad \left(x=\frac{1}{2}\varepsilon_1\right)$.

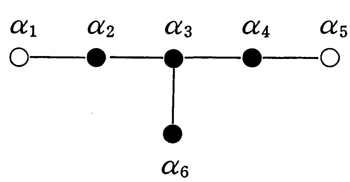
The case ii) does not occur since ii) is inconsistent with $\mathfrak{g}_1 \neq \{0\}$. So the only possible case is i), which is the characteristic corresponding to a principal nilpotent element of $\mathfrak{g}_R$.

The statement 2) follows from 1), Theorem 3.1 and Proposition 3.3.

Q.E.D.

LEMMA 4.2. *Every nilpotent element in the simple Lie algebra of type* (E IV) *has a real w-polarization.*

PROOF. The Satake diagram of (E IV) is



From the table of Dynkin ([5] p. 178), the characteristic of a standard S-triple (x, e, f) is

i) $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ & & 0 & & \end{pmatrix}$ $\quad (x=\varepsilon_1+\varepsilon_5)$,

or

ii) $\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ & & 0 & & \end{pmatrix}$ $\quad \left(x=\frac{1}{2}(\varepsilon_1+\varepsilon_5)\right)$.

In case i), e is a principal nilpotent element of $\mathfrak{g}_R$, and has a real polarization (Theorem 3.1 and Proposition 3.3). So we shall consider the case ii). We set

$$V^1=\sum_{\substack{\alpha(x)=-\frac{1}{2}\\ \alpha(\varepsilon_1)=1}} \mathfrak{g}^\alpha, \qquad V^2=\sum_{\substack{\alpha(x)=-\frac{1}{2}\\ \alpha(\varepsilon_5)=1}} \mathfrak{g}^\alpha.$$

Then one can see from an easy calculation of roots that

$$\dim V^i=\frac{1}{2}\dim \mathfrak{g}_{-\frac{1}{2}},$$

V^i is an abelian subalgebra of $\mathfrak{g}$, and

V^i is stable under the adjoint action of $\mathfrak{g}_0$

for $i=1, 2$. Therefore

$$\mathfrak{p}_i=\sum_{j\geqq 0}\mathfrak{g}_j+V^i \qquad (i=1, 2)$$

is a w-polarization of e (Proposition 5.2 of [13]). Moreover, since $V^i(i=1, 2)$ is σ-stable, $\mathfrak{p}_i$ is a real w-polarization. Q.E.D.

Lemma 4.3. *Let $\mathfrak{g}_R$ be a real simple Lie algebra not of type (A), and μ the highest root. Then e_μ has not a w-polarization in the sense of Definition* 7.1. *of* [13]. (*Note that e_μ is not necessarily in $\mathfrak{g}_R$.*)

Proof. Consider an S-triple

$$(x, e, f)=\left(\frac{1}{|\mu|^2}H_\mu,\ e_\mu,\ \frac{1}{|\mu|^2}e_{-\mu}\right).$$

We can see from the root table of each case that there exists uniquely the simple root α_i such that $\mu-\alpha_i \in \Delta$ (so x is a scalar multiple of ε_i), and that the coefficient of μ at α_i is equal to 2 (i.e., $\mu(\varepsilon_i)=2$). From $[x, e]=e$, we have $\mu(x)=1$, so we have $x=\frac{1}{2}\varepsilon_i$. Thus the characteristic of x is

$$\Big(0,\cdots, 0, \overset{i}{\overset{\smile}{\frac{1}{2}}}, 0,\cdots, 0)\Big).$$

We set

$$\Delta_0=\{\alpha \in \Delta;\ \alpha(x)=0\},$$

$$\Delta_0^+=\Delta_0\cap\Delta^+$$

For $\alpha \in \Delta_0$, let $\mu-p\alpha, \mu-(p-1)\alpha,\cdots, \mu+q\alpha (p, q\geqq 0)$ be an α-series containing μ. Then

$$q-p=-2\frac{\mu(H_\alpha)}{|\alpha|^2}=0.$$

And either p or q is equal to zero, since μ is the highest root. (If $\alpha \in \Delta^+$, then $q=0$; and if $-\alpha\in\Delta^{+}$, then $p=0$.) So we have $p=q=0$, and $[e, \mathfrak{g}^{\alpha}]=\{0\}$ for $\alpha\in\Delta_{0}$. Thus we have

$$\mathfrak{g}_{0}\cap\mathfrak{g}^{e}=\sum_{j\neq i}\boldsymbol{C}H_{\alpha_{j}}+\sum_{\alpha\in\Delta_{0}}\mathfrak{g}^{\alpha}.$$

Assume that e has a w-polarization $\mathfrak{p}$. By Proposition 5.3 of [13] $(x\in\mathfrak{p})$ and the above, we have $\mathfrak{g}_{0}\subset\mathfrak{p}$, so we have $\sum_{j\geqq 0}\mathfrak{g}_{j}\subset\mathfrak{p}$. Since $\mathfrak{g}_{-1}=\boldsymbol{C}f$ and $\mathfrak{g}_{j}=\{0\}$ $\left(j\leqq-\frac{3}{2}\right)$, we have

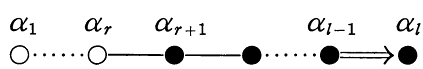
$$\dim(\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}})=\frac{1}{2}\dim\mathfrak{g}_{-\frac{1}{2}}$$

by the condition ii) of polarizations. But this cannot happen because, as one can see from an easy calculation of roots, $\mathfrak{g}_{-\frac{1}{2}}$ is an irreducible $\mathfrak{g}_{0}$-module. Therefore e has no w-polarizations. Q.E.D.

PROPOSITION 4.4. *In case that $\mathfrak{g}_{R}$ is a non-compact real form of type (B), (D) or (E), except for $\mathfrak{so}(n, 1)$ and $(E$ IV$)$, there exists a nilpotent element with no w-polarizations.*

PROOF. It suffices to show that in each case the highest root μ is a real root.

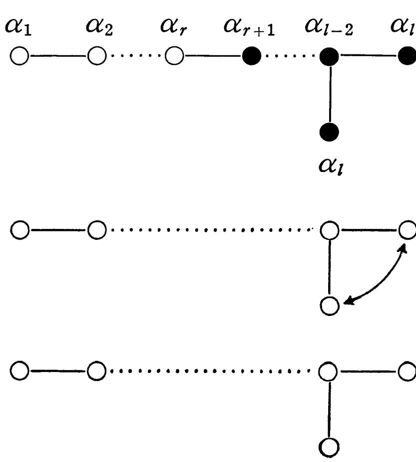
1) The Satake diagram of type (B) (except for $\mathfrak{so}(2l, 1)$) is as follows:



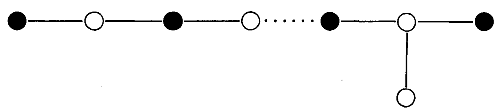
$(2\leqq r\leqq l)$.

The highest root $\mu=(12\cdots 2)$ is real since it is orthogonal to purely-imaginary simple roots.

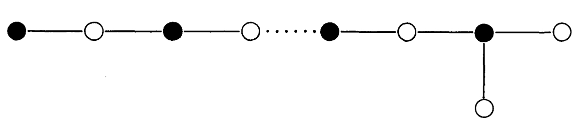
2) The Satake diagram of type $(D_{l})(l\geqq 4)$ (except for $\mathfrak{so}(2l+1, 1)$) is as follows:



$(2\leqq r\leqq l-2)$



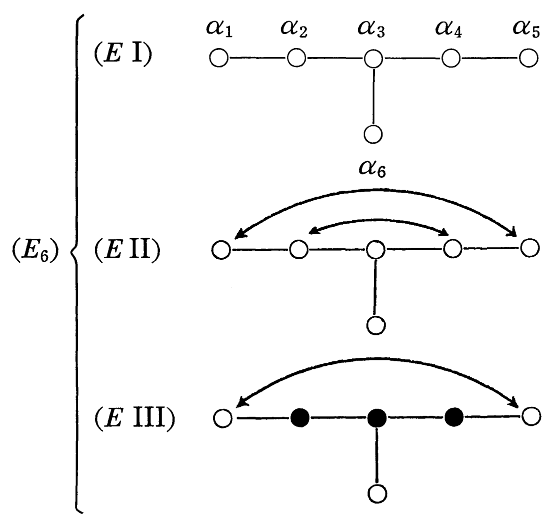
$(l=2m\geqq 6)$

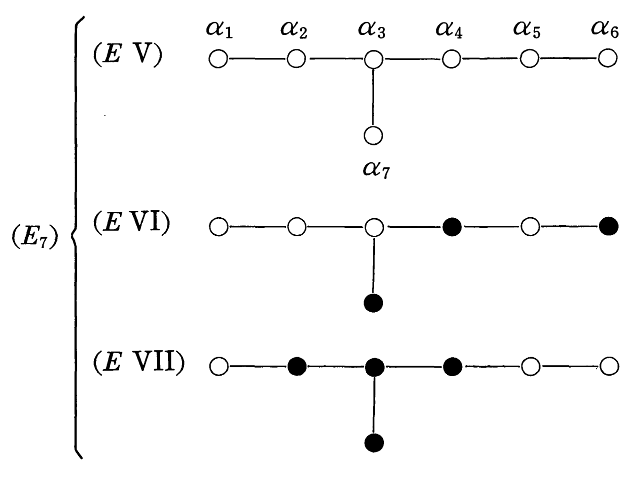


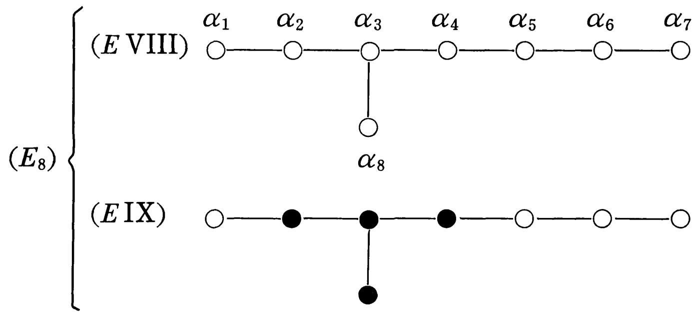
$(l=2m+1\geqq 5)$

In each case, the highest root $\mu=\begin{pmatrix}1 & 2 & 2 & \cdots & 2 & 1\\ & & & & 1 & \end{pmatrix}$ is orthogonal to purely-imaginary roots and simple roots with an arrow, and so μ is real.

3) The Satake diagram of type (E) (except for $(E\,\mathrm{IV})$) is as follows:







The highest root μ is

$$\begin{pmatrix} 1\ 2\ \overset{3}{2}\ 2\ 1 \end{pmatrix} \qquad \text{if } \mathfrak{g}_R \text{ is of type } (E_6),$$

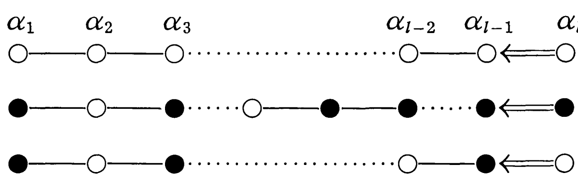
$$\begin{pmatrix} 2\ 3\ \overset{4}{2}\ 3\ 2\ 1 \end{pmatrix} \qquad \text{if } \mathfrak{g}_R \text{ is of type } (E_7),$$

$$\begin{pmatrix} 2\ 4\ \overset{6}{3}\ 5\ 4\ 3\ 2 \end{pmatrix} \qquad \text{if } \mathfrak{g}_R \text{ is of type } (E_8).$$

We have $(\mu, \alpha_i)=0$ for $i \neq 6$ if $\mathfrak{g}_R$ is (E_6), for $i \neq 1$ if $\mathfrak{g}_R$ is (E_7), for $i \neq 7$ if $\mathfrak{g}_R$ is (E_8), and so in each case, μ is orthogonal to purely-imaginary roots and simple roots with arrows. Thus μ is a real root. Q.E.D.

LEMMA 4.5. *In a non-compact real form of type (C), there exists a nilpotent element with no w-polarizations.*

PROOF. The Satake diagram of (C_l) is



The root $\mu'=(1\ 2\ 2 \cdots 2\ 1)$ is real since $(\mu', \alpha_i)=0$ for $i \neq 2$. We consider an S-triple (in $\mathfrak{g}_R$)

$$(x, e, f)=\left(\frac{1}{|\mu'|^2} H_{\mu'},\ e_{\mu'},\ \frac{1}{|\mu'|^2} e_{-\mu'}\right).$$

Since $(\mu', \alpha_i)=0$ (so x is a scalar multiple of ε_2) and $[x, e]=e$ (i.e., $\mu'(x)=1$) and the coefficient of μ' at α_2 is equal to 2 (i.e., $\mu'(\varepsilon_2)=2$), we have $x=\frac{1}{2}\varepsilon_2$. So the characteristic of x is

$$\left(0, \frac{1}{2}, 0, \cdots, 0\right). \quad \text{We set}$$

$$\Delta_1=\{\alpha\in\Delta;\ \alpha(x)=1\}$$
$$=\{(1\ 2\ 2\cdots 2\ 1),\ (2\ 2\ 2\cdots 2\ 1),\ (0\ 2\ 2\cdots 2\ 1)\},$$
$$\Delta^1_{\frac{1}{2}}=\left\{\alpha\in\Delta;\ \alpha(x)=\frac{1}{2},\ \alpha(\varepsilon_1)=1\right\},$$
$$\Delta^2_{\frac{1}{2}}=\left\{\alpha\in\Delta;\ \alpha(x)=\frac{1}{2},\ \alpha(\varepsilon_1)=0\right\},$$
$$\Delta_{\frac{1}{2}}=\left\{\alpha\in\Delta;\ \alpha(x)=\frac{1}{2}\right\}=\Delta^1_{\frac{1}{2}}\cup\Delta^2_{\frac{1}{2}},$$
$$\Delta_{0e}=\{\alpha\in\Delta;\ \alpha(x)=0,\ \alpha(\varepsilon_1)=0\},$$
$$\Delta_0=\{\alpha\in\Delta;\ \alpha(x)=0\}=\Delta_{0e}\cup\{\pm(1\ 0\cdots 0)\},$$
$$\Delta_{-j}=\{-\alpha;\ \alpha\in\Delta_j\}\qquad \left(j=\frac{1}{2},1\right).$$

Then we have

$$\mathfrak{g}_j=\{0\}\qquad \left(\text{if } |j|\geqq\frac{3}{2}\right),$$
$$\mathfrak{g}_j=\sum_{\alpha\in\Delta_j}\mathfrak{g}^\alpha\qquad \left(\text{if } |j|=\frac{1}{2},1\right),$$
$$\mathfrak{g}_0\cap\mathfrak{g}^e=\sum_{i\neq 2}\boldsymbol{C}H_{\alpha_i}+\sum_{\alpha\in\Delta_{0e}}\mathfrak{g}^\alpha,$$
$$\mathfrak{g}_0=\boldsymbol{C}x+(\mathfrak{g}_0\cap\mathfrak{g}^e)+\mathfrak{g}^{(10\ldots0)}+\mathfrak{g}^{-(10\ldots0)},$$
$$=\mathfrak{l}+\mathfrak{g}^{(10\ldots0)}+\mathfrak{g}^{-(10\ldots0)},$$
$$\mathfrak{g}^e=(\mathfrak{g}_0\cap\mathfrak{g}^e)+\mathfrak{g}_{\frac{1}{2}}+\mathfrak{g}_1,$$
$$\mathfrak{g}_{-\frac{1}{2}}=V^1+V^2,$$

where

$$\mathfrak{l}=\boldsymbol{C}x+(\mathfrak{g}_0\cap\mathfrak{g}^e),$$
$$V^i=\sum_{\alpha\in\Delta^i_{\frac{1}{2}}}\mathfrak{g}^{-\alpha}\qquad (i=1,2).$$

By a simple calculation of roots, we see;

V^i is $\mathfrak{l}$-irreducible,

$[V^1, V^1]=\mathfrak{g}^{-(22\ldots21)}$,

$$[V^2, V^2]=\mathfrak{g}^{-(02\ldots21)}.$$

Now we assume that e has a w-polarization $\mathfrak{p}$. We have $\mathfrak{l}+\mathfrak{g}_{\frac{1}{2}}+\mathfrak{g}_1\subset\mathfrak{p}$, by Proposition 2.1 and Proposition 5.3 of [13].

First we shall prove that $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}=\{0\}$. If $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}\neq\{0\}$, $\mathfrak{p}$ includes V^1 or V^2. Suppose that $\mathfrak{p}\supset V^1$. Then

$$\mathfrak{g}^{-(22\ldots21)}=[V^1, V^1]\subset\mathfrak{p}.$$

Since $e\in\mathfrak{p}$, we have

$$\mathfrak{g}^{-(10\ldots0)}=[e, \mathfrak{g}^{-(22\ldots21)}]\subset\mathfrak{p}.$$

So we have

$$V^2=[\mathfrak{g}^{-(10\ldots0)}, V^1]\subset\mathfrak{p}.$$

Hence

$$\mathfrak{g}_{-\frac{1}{2}}\subset\mathfrak{p}, \text{ and } \mathfrak{g}_{-1}=[\mathfrak{g}_{-\frac{1}{2}}, \mathfrak{g}_{-\frac{1}{2}}]\subset\mathfrak{p}.$$

Thus we have $\mathfrak{p}=\mathfrak{g}$, which contradicts the condition ii) of polarizations. The supposition $\mathfrak{p}\supset V^2$ leads us to the same contradiction. Thus we have proved that $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}=\{0\}$,

Next we shall prove that $\mathfrak{p}\cap\mathfrak{g}_{-1}=\{0\}$. If $\mathfrak{p}\cap\mathfrak{g}_{-1}\neq\{0\}$, $\mathfrak{p}$ includes $\mathfrak{g}^{-(22\ldots21)}$ or $\mathfrak{g}^{-(022\ldots21)}$. ($f\in\mathfrak{p}$ does not occur because $f\in\mathfrak{p}$, with $\mathfrak{g}^e\subset\mathfrak{p}$, implies $\mathfrak{p}=\mathfrak{g}$.) Suppose that $\mathfrak{p}\supset\mathfrak{g}^{-(22\ldots21)}$. Since $\mathfrak{g}^{(110\ldots0)}\subset\mathfrak{g}_{\frac{1}{2}}\subset\mathfrak{p}$, we have

$$\mathfrak{g}^{-(112\ldots21)}=[\mathfrak{g}^{-(22\ldots21)}, \mathfrak{g}^{(110\ldots0)}]\subset\mathfrak{p},$$

which contradicts the fact that $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}=\{0\}$. If we suppose that $\mathfrak{p}\supset\mathfrak{g}^{-(022\ldots21)}$, we have

$$\mathfrak{g}^{-(012\ldots21)}=[\mathfrak{g}^{-(022\ldots21)}, \mathfrak{g}^{(010\ldots0)}]\subset\mathfrak{p},$$

since $\mathfrak{g}^{-(010\ldots0)}\subset\mathfrak{g}_{\frac{1}{2}}\subset\mathfrak{p}$. This is also inconsistent with $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}=\{0\}$.

So we have $\mathfrak{p}\subset\sum_{j\geqq0}\mathfrak{g}_j$. Hence

$$\begin{aligned}\dim\mathfrak{p}&\leqq\dim\sum_{j\geqq0}\mathfrak{g}_j=\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}_0)\\&<\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}_0+\dim\mathfrak{g}_{\frac{1}{2}})\\&=\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^e).\end{aligned}$$

This is contradictory to the condition ii) of polarizations. So e has not a w-polarization. Q.E.D.

Summing up Corollaries 6.2–6.3 ([13]), Examples 6.3–6.4 ([13]) and the above lemmata and propositions, we have:

THEOREM 4.6. 1) *In case that* $\mathfrak{g}_R$ *is a real simple Lie algebra of type* $(A\ \mathrm{I})$ $(A\ \mathrm{II})(E\ \mathrm{IV})$ *or* $\mathfrak{so}(n, 1)$, *every nilpotent element in* $\mathfrak{g}_R$ *has a real w-polarization.*

2) *In case that* $\mathfrak{g}_R$ *is a real simple Lie algebra of type* $(A)(E\ \mathrm{IV})$ *or* $\mathfrak{so}(n, 1)$, *every element in* $\mathfrak{g}_R$ *has a w-polarization.*

3) *If* $\mathfrak{g}_R$ *is a non-compact real simple Lie algebra of other type, there exists a nilpotent element in* $\mathfrak{g}_R$ *with no w-polarizations.*

The following is an immediate consequence of Proposition 2.6 [13] and the above theorem:

COROLLARY 4.7. 1) *In case that* $\mathfrak{g}_R$ *is a real semisimple Lie algebra consisting only of simple ideals of type* $(A\ \mathrm{I})$ $(A\ \mathrm{II})$ $(E\ \mathrm{IV})$ *or* $\mathfrak{so}(n, 1)$, *every nilpotent element in* $\mathfrak{g}_R$ *has a real w-polarization.*

2) *In case that* $\mathfrak{g}_R$ *is a real semisimple Lie algebra constisting only of simple ideals of type* $(A)(E\ \mathrm{IV})$ *or* $\mathfrak{so}(n, 1)$, *every element in* $\mathfrak{g}_R$ *has a w-polarization.*

3) *If* $\mathfrak{g}_R$ *is a non-compact real semisimple Lie algebra of other type, there exists a nilpotent element in* $\mathfrak{g}_R$ *with no w-polarizations.*

§ 5. Orbits and unitary representations

5.1. In this section we shall give a sketch of the Kostant's method (Kostant [11], [12] and Kirillov [9]) from the viewpoint of induced representations.

Let $\mathfrak{g}_R$ be a Lie algebra of a connected Lie group G, and $\mathfrak{g}$ its complexification. The group G acts on the dual space $\mathfrak{g}_R^* = \mathrm{Hom}_R(\mathfrak{g}_R, \boldsymbol{R})$ as the contragredient representation of $(Ad, \mathfrak{g}_R)$. Namely, for every $g \in G$ and $\lambda \in \mathfrak{g}_R^*$, $g\lambda$ is defined by

$$(g\lambda)(X) = \lambda(Ad(g^{-1})X) \qquad \text{for every } X \in \mathfrak{g}_R.$$

The G-orbit $O = G/G^\lambda$ in $\mathfrak{g}_R^*$ through λ admits the canonical G-invariant symplectic structure ω defined as follows (a non-degenerate closed 2-form on an even dimensional C^∞-differentiable manifold is called *a symplectic structure*):

$$\omega_p(\sigma(X)_p, \sigma(Y)_p) = -p([X, Y])$$

for every $X, Y \in \mathfrak{g}$ and $p \in O$, where $\sigma(X)$ $(X \in \mathfrak{g}_R)$ denotes the vector field on O generated by the 1-parameter subgroup $\{\exp tX\}_{-\infty<t<\infty}$ of G, i.e.,

$$\sigma(X)_p f = (\sigma(X)f)(p) = \left[\frac{d}{dt} f(\exp - tX \cdot p)\right]_{t=0}$$

(for $f \in C^\infty(O)$ and $p \in O$), and $\sigma(X)(X \in \mathfrak{g})$ is its canonical extension. It is proved in [12] that ω is well-defined as above and that ω is a G-invariant symplectic form on O.

Let $\mathfrak{p}$ be an admissible polarization of λ, and define a linear mapping χ_*^λ of $\mathfrak{p}$ to $\boldsymbol{C}$ by

$$\chi_*^\lambda(X) = 2\pi\sqrt{-1}\,\lambda(X) \qquad \text{for every } X \in \mathfrak{p}.$$

Then by the condition i) of a polarization, χ_*^λ is a Lie algebra homomorphism. We set

$$\mathfrak{d} = \mathfrak{p} \cap \sigma\mathfrak{p}, \qquad \mathfrak{d}_0 = \mathfrak{d} \cap \mathfrak{g}_R = \mathfrak{p} \cap \mathfrak{g}_R,$$

$$\mathfrak{e} = \mathfrak{p} + \sigma\mathfrak{p}, \qquad \mathfrak{e}_0 = \mathfrak{e} \cap \mathfrak{g}_R,$$

and denote by D_0 (resp. E_0) the analytic subgroup of G generated by $\mathfrak{d}_0$ (resp. $\mathfrak{e}_0$). We assume that D_0 and E_0 are closed subgroups of G. Let D (resp. E) be the subgroup of G generated by D_0 (resp. E_0) and G^λ. By the condition iii) of a polarization,

$$D = G^\lambda D_0 = \{xy;\ x \in G^\lambda,\ y \in D_0\},$$

$$E = G^\lambda E_0 = \{xy;\ x \in G^\lambda\ y \in E_0\},$$

and D_0 (resp. E_0) is a normal subgroup of D (resp. E).

REMARK 5.1. When G is semisimple, the above definition of D and E seems to need some modifications, as will be pointed out in 5.6.

5.2. The symplectic form ω on O determines the de Rham cohomology class $[\omega]$.

LEMMA 5.2.1 (Kostant [12].) *When G is simply connected, the following conditions are equivaleut:*

1) *There exists a character χ^λ of G^λ, whose differential coincides with χ_*^λ on $\mathfrak{g}_R^\lambda$.*

2) *$[\omega]$ is integral.*

Let G_0^λ be the connected component of G^λ containing the unit. Then we have

LEMMA 5.2.2. 1) *The manifold D/G^λ is (canonically) diffeomorphic to $D_0/(D_0 \cap G^\lambda)$.*

2) *If D/G^λ is simply connected, then*

i) *$D_0 \cap G^\lambda = G_0^\lambda$, and*

ii) *There exists a canonical* 1–1 *correspondence between the set of the connected components of* D *and that of* G^λ.

PROOF. 1) Define a mapping $\varphi: D_0/(D_0 \cap G^\lambda) \to D/G^\lambda$ by

$$\varphi(\bar{g}) = gG^\lambda \qquad \text{for every } g \in D_0,$$

where $\bar{g}$ denotes the element in $D_0/(D_0 \cap G^\lambda)$ corresponding to g. This mapping φ is injective, since $gG^\lambda = g'G^\lambda$ implies that $g^{-1}g' \in D_0 \cap G^\lambda$, for g, $g' \in D_0$. Each element g in D can be decomposed as $g = g'h$ ($g' \in D_0$, $h \in G^\lambda$). Then

$$\varphi(\bar{g}') = g'G^\lambda = g'hG^\lambda = gG^\lambda.$$

Hence φ is surjective

2) i) It suffices to show that $D_0 \cap G^\lambda$ is connected. There exists the following homotopy exact sequence of the fibre space $(D_0, p, D_0/D_0 \cap G^\lambda)$:

$$\pi_1(D_0/D_0 \cap G^\lambda, p(e)) \to \pi_0(D_0 \cap G^\lambda, e) \to \pi_0(D_0, e)$$

where p denotes the canonical projection of D_0 onto $D_0/D_0 \cap G^\lambda$, and e the unit of D_0. Here, we have

$$\pi_1(D_0/D_0 \cap G^\lambda, p(e)) = \{0\}$$

by 1) and the assumption on D/G^λ, and

$$\pi_0(D_0, e) = \{0\}$$

since D_0 is connected. Hence

$$\pi_0(D_0 \cap G^\lambda, e) = \{0\},$$

and so $D_0 \cap G^\lambda$ is connected.

ii) Let $x = hy$ and $x' = h'y'$ be elements in D (y, $y' \in D_0$, h, $h' \in G^\lambda$). Since $x^{-1}x' = y^{-1}h^{-1}h'y'$, the relation $x^{-1}x' \in D_0$ is equivalent to $h^{-1}h' \in D_0$, which is also equivalent to $h^{-1}h' \in G_0^\lambda$ because $G_0^\lambda = D_0 \cap G^\lambda$. So we can assign the connected component of G^λ containing h to the connected component of D containing x, and this assignment gives a 1–1 correspondence between the set of connected components of D and that of G^λ. Q.E.D.

We set

$$R^\lambda = \left\{ \chi; \begin{array}{ll} \text{i)} & \chi \text{ is a character of } G^\lambda, \\ \text{ii)} & \text{the derivative of } \chi \text{ coincides} \\ & \text{with } \chi_*^\lambda \text{ on } \mathfrak{g}_R^\lambda. \end{array} \right\},$$

and

$$R_0^\lambda = \left\{ \begin{array}{ll} \chi; & \chi \text{ is a character of } D \\ & \text{with the infinitesimal representation} \\ & \chi_*^\lambda. \end{array} \right\}.$$

Then, by the restriction on G^λ, R_0^λ is naturally included in R^λ; i.e., $R_0^\lambda \hookrightarrow R^\lambda$.

LEMMA 5.2.3. *When D/G^λ is simply connected, the natural inclusion of R_0^λ to R^λ is a bijection.*

PROOF. It suffices to show that a character χ of G^λ whose infinitesimal representation coincides with χ_*^λ on $\mathfrak{g}_R^\lambda$ is extendible to a unitary character of D.

Let $\tilde{D}_0$ be the universal covering group of D_0, and Z the subgroup of the center of $\tilde{D}_0$ such that $\tilde{D}_0/Z \cong D_0$, and p the canonical homomorphism of $\tilde{D}_0$ onto D_0. The analytic subgroup H of $\tilde{D}_0$ generated by $\mathfrak{g}_R^\lambda$ coincides with $p^{-1}(G_0^\lambda)$, so H is closed, and we have

$$H/(H \cap Z) \cong G_0^\lambda \qquad \text{(isomorphic as Lie groups).}$$

The mapping $\bar{p}$ of $\tilde{D}_0/H$ to D_0/G_0^λ is well-defined by

$$\bar{p}(\bar{g}) = p(g)G_0^\lambda \in D_0/G_0^\lambda,$$

where $\bar{g} = gH$ denotes the element in $\tilde{D}_0/H$ corresponding to $g \in \tilde{D}_0$. It is easily seen that $\bar{p}$ is surjective and locally diffeomorphic, so $\bar{p}$ is a covering mapping of $\tilde{D}_0/H$ onto D_0/G_0^λ. By Lemma 5.2.2. the manifold D_0/G_0^λ is simply connected, so we have $\tilde{D}_0/H \cong D_0/G_0^\lambda$. Hence

$$Z \subset H$$

and

$$G_0^\lambda \cong H/(H \cap Z) \cong H/Z.$$

(The proof of $Z \subset H$ is as follows: each element $z \in Z$ satisfies $p(z) = e$, so we have $\bar{p}(\bar{z}) = eG_0^\lambda$, which implies $z \in H$, since $\tilde{D}_0/H \cong D_0/G_0^\lambda$.)

Since $\tilde{D}$ is simply connected, the Lie algebra homomorphism χ_*^λ can be lifted uniquely to the character $\tilde{\chi}$ of $\tilde{D}_0$. The representation of $\tilde{D}_0$ does not necessarily, in general, induce the representation of $G_0^\lambda = H/Z$. In our case, however, we discuss under the assumption that there exists a character χ of G_0^λ with the infinitesimal representation $\chi_*^\lambda | \mathfrak{g}_R^\lambda$ (i.e., R^λ is not empty). Therefore $\tilde{\chi}$ induces the character χ of G_0^λ, so we have $\tilde{\chi}(Z) = \{1\}$. Thus $\tilde{\chi}$ induces a character χ_1 of D_0, since $D_0 \cong \tilde{D}_0/Z$. In particular, we have

$$\chi_1 = \chi \qquad \text{on } G_0^\lambda.$$

Each element h in G^λ induces an automorphism I_h of D_0 by

$$I_h(g) = hgh^{-1} \qquad \text{for every } g \in D_0.$$

We set $\chi_1'=\chi_1\circ I_h$. The infinitesimal representation $(\chi_1')_*$ of χ_1 is given by

$$(\chi_1')_*=(\chi_1)_*\circ Ad(h),$$

and we have

$$\begin{aligned}(\chi_1')_*(X)&=(\chi_1)_*(Ad(h)X)\\&=2\pi\sqrt{-1}\,\lambda(Ad(h)X)\\&=2\pi\sqrt{-1}\,(h^{-1}\lambda)(X)\\&=2\pi\sqrt{-1}\,\lambda(X)=(\chi_1)_*(X),\end{aligned}$$

for every $X\in\mathfrak{d}_0$. So we have $(\chi_1')_*=(\chi_1)_*$, and $\chi_1'=\chi_1$. Thus we have proved that

$$\chi_1(hgh^{-1})=\chi_1(g)$$

for every $g\in D_0$ and $h\in G^\lambda$.

Let $x=yh=y'h'$ be two expressions of an element x in D, where $y, y'\in D_0$ and $h, h'\in G^\lambda$. Since $y^{-1}y'=hh'^{-1}\in D_0\cap G^\lambda=G_0^\lambda$ and $\chi_1=\chi$ on G_0^λ, we have

$$\chi_1(y^{-1}y')=\chi(hh'^{-1}).$$

So we can define a mapping χ_0 of D to $\boldsymbol{C}^*$ by

$$\chi_0(yh)=\chi_1(y)\chi(h)$$

where $y\in D_0$ and $h\in G^\lambda$.

By the definition of χ_0, in order to prove that $\chi_0\in R_0^\lambda$, it is enough to show that χ_0 is a group homomorphism. For $x=yh$, $x'=y'h'\in D(y, y'\in D_0$ and $h, h'\in G^\lambda)$, we have

$$xx'=yhy'h'=y(hy'h^{-1})hh'.$$

So we have

$$\begin{aligned}\chi_0(xx')&=\chi_1(y\cdot hy'h^{-1})\chi(hh')\\&=\chi_1(y)\chi_1(hy'h^{-1})\chi(h)\chi(h')\\&=\chi_1(y)\chi_1(y')\chi(h)\chi(h')\\&=\chi_1(y)\chi(h)\cdot\chi_1(y')\chi(h')\\&=\chi_0(x)\chi_0(x').\end{aligned}$$

And

$$\chi_0(e)=\chi_1(e)\chi(e)=1.$$

Thus we have $\chi_0 \in R_0^\lambda$, and the restriction of χ_0 on G^λ coincides with χ.
Q.E.D.

Hereafter we assume that χ_*^λ can be lifted to a unitary character χ^λ of D.

5.3. In this section, we introduce G-quasi-invariant measures on G/D and G/E.

LEMMA 5.3.1. $\mathfrak{d}=\{X \in \mathfrak{e};\ \lambda([e, X])=\{0\}\}$.

PROOF. From the conditions of polarizations and the non-singularity of the symplectic structure ω on O, we have

$$\mathfrak{p}=\{X \in \mathfrak{g};\ \lambda([\mathfrak{p}, X])=\{0\}\}.$$

We set

$$V=\{X \in \mathfrak{e};\ \lambda([\mathfrak{e}, X])=\{0\}\}.$$

Since $\mathfrak{e}=\mathfrak{p}+\sigma\mathfrak{p}$ and $\sigma\lambda=\lambda$, we have

$$\begin{aligned} V&=\{X \in \mathfrak{e};\ \lambda([\mathfrak{p}, X])=\{0\}\}\cap\{X \in \mathfrak{e};\ \lambda([\sigma\mathfrak{p}, X])=\{0\}\} \\ &=\mathfrak{p}\cap\{X \in \mathfrak{e};\ \lambda([\mathfrak{p}, \sigma X])=\{0\}\} \\ &=\mathfrak{p}\cap\{\sigma X;\ X \in \mathfrak{e} \text{ and } \lambda([\mathfrak{p}, X])=\{0\}\} \\ &=\mathfrak{p}\cap\sigma\mathfrak{p}=\mathfrak{d}. \end{aligned}$$

Q.E.D.

LEMMA 5.3.2. *$\det Ad_D(x)=\det Ad_E(x)$ for every $x \in D$.*

PROOF. The statement of this lemma is shown using the theory of symplectic structures. We set $\lambda_0=\varphi(\lambda)$, where φ is the canonical projection of $\mathfrak{g}_R^*$ onto the dual space $\mathfrak{e}_0^*=\mathrm{Hom}_R(\mathfrak{e}_0, \boldsymbol{R})$ of $\mathfrak{e}_0$. The E-orbit Ω in $\mathfrak{e}_0^*$ through λ_0 admits a canonical E-invariant symplectic structure ω_0 (ω_0 is defined in the same way as in 5.1). Let E^{λ_0} denote the isotropy subgroup of E with respect to λ_0 and $\mathfrak{e}_0^{\lambda_0}$ its Lie algebra, i.e.,

$$\begin{aligned} \mathfrak{e}_0^{\lambda_0}&=\{X \in \mathfrak{e}_0;\ \lambda_0([\mathfrak{e}, X])=\{0\}\} \\ &=\{X \in \mathfrak{e}_0;\ \lambda([\mathfrak{e}, X])=\{0\}\} \\ &=\mathfrak{d}_0, \end{aligned}$$

by Lemma 5.3.1. Then D and E^{λ_0} are Lie subgroups of G with the same Lie algebra $\mathfrak{d}_0$. Since D_0 is the connected component of D containing the unit, $D_0\subset E^{\lambda_0}$. We have $G^\lambda\subset E^{\lambda_0}$, since G^λ is the stabilizer of λ in G and G^λ is included in E.
So we have

$$D=G^{\lambda}D_0\subset E^{\lambda_0}.$$

Now the orbit $\Omega=E/E^{\lambda_0}$ has the B-invariant volume element induced from the symplectic structure ω_0, and this volume element is realized as a differential form. So by Proposition 1.6 Chap. X of Helgason [7], we have

$$\det Ad_{E^{\lambda_0}}(x)=\det Ad_E(x) \qquad \text{for every } x\in E^{\lambda_0}.$$

Since E^{λ_0} and D are Lie subgroups of E with the same Lie algebra $\mathfrak{d}_0$, we have

$$\det Ad_{E^{\lambda_0}}(x)=\det Ad_D(x) \qquad \text{for every } x\in D.$$

So we have

$$\det Ad_D(x)=\det Ad_E(x) \qquad \text{for every } x\in D.$$

Q.E.D.

Let μ_G (resp. μ_D or μ_E) denote a left-invariant measure, and Δ_G (resp. Δ_D or Δ_E) the modular function on G (resp. D or E); i.e.,

$$d\mu_G(yx)=\Delta_G(x^{-1})d\mu_G(y), \text{ etc..}$$

μ_G, μ_D and μ_E are determined uniquely up to constant factors. Modular functions are given explicitly by

$$\Delta_G(x)=\det Ad_G(x) \qquad \text{for every } x\in G,$$

$$\Delta_D(x)=\det Ad_D(x) \qquad \text{for every } x\in D,$$

$$\Delta_E(x)=\det Ad_E(x) \qquad \text{for every } x\in E.$$

This is due to Corollary 1.3 Chap. X of Helgason [7]. It is known from the invariant measure theory that there exists a C^∞-function ρ on G satisfying

1) $\rho(g)>0$ for every $g\in G$,

2) $\rho(gh)=\dfrac{\Delta_E(h)}{\Delta_G(h)}\rho(g)$ for $g\in G$ and $h\in E$,

and that there exists such G-quasi-invariant measures ν_D and ν_E on G/D and G/E that

$$\int_G f(g)\rho(g)d\mu_G(g)=\int_{G/D}d\nu_D(gD)\int_D f(gh)d\mu_D(h)$$

$$=\int_{G/E}d\nu_E(gE)\int_E f(gh)d\mu_E(h)$$

for every continuous function f on G with compact support, where gD (resp. gE) denotes the element in G/D (resp. G/E) corresponding to $g\in G$.

We denote by $C_c(G)$ (resp. $C_c(G/D)$ or $C_c(G/E)$) the space of all continuous functions on G (resp. G/D or G/E) with compact support. We shall often use the following lemma:

LEMMA 5.3.3. (Helgason [7] Lemma 1.8 Chap. X). *Let G be a Lie group and H a closed subgroup. Let dh be a left invariant measure >0 on H and put*

$$\bar{f}(gH)=\int_H f(gh)dh,\ f\in C_c(G).$$

Then the mapping $f\to\bar{f}$ is a linear mapping of $C_c(G)$ onto $C_c(G/H)$.

For each element $g\in G$, we define a C^∞-function ξ_g on G by

$$\xi_g(x)=\frac{\rho(gx)}{\rho(x)}.$$

LEMMA 5.3.4. *The function ξ_g has the following property:*

$$\xi_g(xh)=\xi_g(x) \qquad \text{for every } x\in G \text{ and } h\in E.$$

PROOF. This is shown by an easy calculation:

$$\xi_g(xh)=\frac{\rho(gxh)}{\rho(xh)}=\frac{\frac{\Delta_E(h)}{\Delta_G(h)}\rho(gx)}{\frac{\Delta_E(h)}{\Delta_G(h)}\rho(x)}=\frac{\rho(gx)}{\rho(x)}=\xi_g(x).$$

Q.E.D.

So we can define a C^∞-function ξ_g^D(resp. ξ_g^E) on G/D (resp. G/E) by

$$\xi_g^D(xD)=\xi_g(x) \qquad \text{for every } xD\in G/D,$$

$$\xi_g^E(xE)=\xi_g(x) \qquad \text{for every } xE\in G/E.$$

For $g\in G$, let $\gamma(g)\nu_D$ denote the quasi-invariant measure on G/D defined by

$$(\gamma(g)\nu_D)(S)=\nu_D(g^{-1}S),$$

where S is a ν-measurable subset of G/D and $g^{-1}S=\{g^{-1}x;\ x\in S\}$. With the usual notation, $\gamma(g)\nu_D$ is expressed by

$$d(\gamma(g)\nu_D)(x)=d\nu_D(g^{-1}x)$$

where $g\in G$ and $x\in G/D$. The left-translation $\gamma(g)\nu_E$ of the measure ν_E is also defined in the same way as above.

LEMMA 5.3.5. *For every $g\in G$, we have*

$$d\nu_D(gx)=\xi_g^D(x)d\nu_D(x),$$

and

$$d\nu_E(gx)=\xi_g^E(x)d\nu_E(x).$$

PROOF. By Lemma 5.3.3, for each $\bar{f} \in C_c(G/D)$, we can find $f \in C_c(G)$ such that

$$\bar{f}(yD)=\int_D f(yh)d\mu_D(h) \qquad \text{for } y \in G.$$

We set $x=yD \in G/D$, then

$$\int_{G/D}\bar{f}(x)d\nu_D(gx)=\int_{G/D}\bar{f}(g^{-1}x)d\nu_D(x)$$

$$=\int_{G/D}d\nu_D(x)\int_D f(g^{-1}yh)d\mu_D(h)$$

$$=\int_G f(g^{-1}u)\rho(u)d\mu_G(u)$$

$$=\int_G f(u)\rho(gu)d\mu_G(u)$$

$$=\int_G\left[f(u)\frac{\rho(gu)}{\rho(u)}\right]\rho(u)d\mu_G(u)$$

$$=\int_{G/D}d\nu_D(x)\int_D f(yh)\frac{\rho(gyh)}{\rho(yh)}d\mu_D(h)$$

$$=\int_{G/D}\bar{f}(x)\xi_g^D(x)d\nu_D(x).$$

Thus we have $d\nu_D(gx)=\xi_g^D(x)d\nu_D(x)$. The discussion as to ν_E is the same as above. Q.E.D.

Let ρ_D (resp. ρ_E) be the canonical projection of G onto G/D (resp. G/E), and ρ_{DE} that of G/D onto G/E. The following lemma is an easy consequence of the definition of ξ_g and Lemma 5.3.4.

LEMMA 5.3.6. *For every* g, $g' \in G$, *we have*

1) $\xi_{gg'}(x)=\xi_g(g'x)\xi_{g'}(x)$ *for every* $x \in G$,

$\xi_{gg'}^D(x)=\xi_g^D(g'x)\xi_{g'}^D(x)$ *for every* $x \in G/D$,

$\xi_{gg'}^E(x)=\xi_g^E(g'x)\xi_{g'}^E(x)$ *for every* $x \in G/E$.

2) ξ_g (*resp.* ξ_g^D) *is constant on each fibre of* ρ_E (*resp.* ρ_{DE}).

5.4. Let L_λ denote the Hermitian G-homogeneous line bundle over G/D associated to the unitary character χ^λ of D, and we set

$\Gamma(L_\lambda)$ = the space of all C^∞-sections of L_λ,

$\Gamma_2(L_\lambda)$ = the pre-Hilbert space of all square-integrable C^∞-sections of L_λ,

$C^\infty(G)^\lambda$ = the space of all C^∞-functions f on G such that $f(gh)=\chi^\lambda(h^{-1})f(g)$ for every $g\in G$ and $h\in D$.

Notations: 1) For $x\in G/D$, $|\ \ |_x$ (or simply $|\ \ |$) denotes the Hermitian norm on the fibre over x of the line bundle L_λ.

2) For $s\in\Gamma(L_\lambda)$, $\|s\|(0\leqq\|s\|\leqq\infty)$ denotes the square-integral-norm of s:

$$\|s\|^2=\int_{G/D}|s(x)|_x^2 d\nu_D(x).$$

The group G acts on $\Gamma(L_\lambda)$ by

$$(gs)(x)=g(s(g^{-1}x))\qquad \text{for } s\in\Gamma(L_\lambda),\ g\in G \text{ and } x\in G/D,$$

and acts on $C^\infty(G)^\lambda$ by left-translations, and $\Gamma_2(L_\lambda)$ is a G-invariant subspace of $\Gamma(L_\lambda)$. There exists the canonical G-isomorphism between $\Gamma(L_\lambda)$ and $C^\infty(G)^\lambda$, which we shall denote by

$$\begin{array}{ccc} \Gamma(L_\lambda) & \longrightarrow & C^\infty(G)^\lambda \\ \cup & & \cup \\ s & \longrightarrow & \psi_s \end{array}\quad\text{and}\quad \begin{array}{ccc} \Gamma(L_\lambda) & \longleftarrow & C^\infty(G)^\lambda \\ \cup & & \cup \\ s_\psi & \longleftarrow & \psi . \end{array}$$

Each element X in $\mathfrak{g}_R$ acts on $C^\infty(G)$ as a left-invariant vector field $\tilde{X}$:

$$(\tilde{X}f)(g)=\left[\frac{d}{dt}f(g\exp tX)\right]_{t=0}\qquad \text{for every } f\in C^\infty(G) \text{ and } g\in G,$$

and, by the canonical extension, $\tilde{X}$ is defined for every $X\in\mathfrak{g}$. We set

$$\mathfrak{H}'_\lambda=\{s\in\Gamma_2(L_\lambda);\ \tilde{X}\psi_s=2\pi\sqrt{-1}\,\lambda(X)\psi_s \text{ for every } X\in\mathfrak{p}\},$$

and for every $g\in G$ and $s\in\mathfrak{H}'_\lambda$, we define a section $\pi'_\lambda(g)s$ by

$$(\pi'_\lambda(g)s)(x)=\sqrt{\xi^D_{g^{-1}}(x)}\cdot(gs)(x)\qquad \text{for } x\in G/D.$$

Then we have

LEMMA 5.4.1. 1) *$\mathfrak{H}'_\lambda$ is $\pi'_\lambda(G)$-stable.*

2) $\pi'_\lambda(gg')=\pi'_\lambda(g)\pi'_\lambda(g')$ *for every* $g, g'\in G$.

3) *$\pi'_\lambda(g)$ is norm-preserving.*

PROOF. 3) For $s\in\mathfrak{H}'_\lambda$ and $g\in G$, we have

$$\begin{aligned}||\pi'_\lambda(g)s||^2 &= \int_{G/D} |(\pi'(g)s)(x)|^2 d\nu_D(x) \\ &= \int_{G/D} |g(s(g^{-1}x))|^2 \xi^D_{g^{-1}}(x) d\nu_D(x) \\ &= \int_{G/D} |g(s(x))|^2 \xi^D_{g^{-1}}(gx) d\nu_D(gx) \\ &= \int_{G/D} |s(x)|^2 \xi^D_{g^{-1}}(gx) \xi^D_g(x) d\nu_D(x) \\ &= \int_{G/D} |s(x)|^2 \xi^D_e(x) d\nu_D(x) \\ &= \int_{G/D} |s(x)|^2 d\nu_D(x) = ||s||^2,\end{aligned}$$

where we have used Lemma 5.3.5 and Lemma 5.3.6. So $\pi'_\lambda(g)$ is norm-preserving.

1) Fix $g \in G$ and $s \in \mathfrak{H}'_\lambda$. Since $\pi'_\lambda(g)$ is norm-preserving, $\pi'_\lambda(g)s$ belongs to $\Gamma_2(L_\lambda)$. So we need only to show that

$$\tilde{X}\psi_{\pi'_\lambda(g)s} = 2\pi\sqrt{-1}\lambda(X)\psi_{\pi'_\lambda(g)s} \qquad \text{for every } X \in \mathfrak{p}.$$

Now it is easily seen that

$$\psi_{\pi'_\lambda(g)s} = \sqrt{\xi_{g^{-1}}} \cdot g\psi_s$$

Since ξ_g is constant on each fibre of ρ_E (Lemma 5.3.6), we have $\tilde{X}\xi_g = 0$ for every $X \in \mathfrak{p}$, and by the left-invariantness of $\tilde{X}$,

$$\tilde{X}(g\psi_s) = g(\tilde{X}\psi_s) \qquad \text{for every } X \in \mathfrak{g}.$$

Therefore, for each $X \in \mathfrak{p}$, we have

$$\begin{aligned}\tilde{X}\psi_{\pi'_\lambda(g)s} &= \sqrt{\xi_{g^{-1}}} \cdot g(\tilde{X}\psi_s) \\ &= \sqrt{\xi_{g^{-1}}} \cdot g(2\pi\sqrt{-1}\lambda(X)\psi_s) \\ &= 2\pi\sqrt{-1}\lambda(X) \cdot \sqrt{\xi_{g^{-1}}} \cdot g\psi_s \\ &= 2\pi\sqrt{-1}\lambda(X)\psi_{\pi'_\lambda(g)s}.\end{aligned}$$

Thus we have proved that $\pi'_\lambda(g)s \in \mathfrak{H}'_\lambda$ for every $g \in G$ and $s \in \mathfrak{H}'_\lambda$.

2) The statement 1) of Lemma 5.3.6 implies that $\xi_{gg'} = (g'^{-1}\xi_g) \cdot \xi_{g'}$. So we have,

$$\begin{aligned}\pi'_\lambda(gg')s &= \sqrt{\xi_{g'^{-1}g^{-1}}} \cdot g(g's) \\ &= \sqrt{g\xi_{g'^{-1}}}\sqrt{\xi_{g^{-1}}} \cdot g(g's)\end{aligned}$$

$$=\sqrt{\xi_{g^{-1}}}g(\sqrt{\xi_{g'^{-1}}}g's)$$
$$=\pi'_\lambda(g)(\pi'_\lambda(g')s),$$

for every $s \in \mathfrak{H}'_\lambda$. Q.E.D.

Let $(\pi_\lambda, \mathfrak{H}_\lambda)$ be the completion of $(\pi'_\lambda, \mathfrak{H}'_\lambda)$ with respect to the norm $\| \ \|$. Then, by the above lemma, we have:

THEOREM 5.4.2. *$(\pi_\lambda, \mathfrak{H}_\lambda)$ is a unitary representation of G.*

5.5. We shall give here an example when G is a non-compact simple Lie group.

Example 5.5. $G=SL(3, \boldsymbol{R})$ $(\mathfrak{g}_R=\mathfrak{sl}(3, \boldsymbol{R}))$.
We set

$$K=SO(3, \boldsymbol{R})=\{g \in G; {}^t g=g^{-1}\},$$

$$A_+=\left\{\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & (rs)^{-1} \end{pmatrix}; r, s>0\right\},$$

$$A=\left\{\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & (rs)^{-1} \end{pmatrix}; r, s \text{ are non-zero real numbers}\right\},$$

$$A_-=A\cap K=\left\{\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}; \text{the number of ``minus''-signs is even}\right\},$$

$$N=\left\{\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}; u, v, w \in \boldsymbol{R}\right\}.$$

Then $G=KA_+N$ is an Iwasawa decomposition of G, and A is a Cartan subgroup of G with maximal vector part. The centralizer M of A_+ in K coincides with A_-, and

$$B=MA_+N=AN=\left\{\begin{pmatrix} r & u & w \\ 0 & s & v \\ 0 & 0 & (rs)^{-1} \end{pmatrix} \in G\right\}$$

is a minimal parabolic subgroup of G. We set

$$e=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_R.$$

Since e is a principal nilpotent element of $\mathfrak{g}$, e has a unipue w-polarization $\mathfrak{p}$, and it is at the same time a real polarization (Corollary 5.6 of [13]). $\mathfrak{p}$ is given by

$$\mathfrak{p}=\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix};\ \begin{matrix} a_{ij} \in \boldsymbol{C} \\ \sum_{i=1}^{3} a_{ii}=0 \end{matrix}\right\},$$

and subgroups D_0 and E_0 in 5.1 are given by

$$D_0=E_0=\left\{\begin{pmatrix} r & u & w \\ 0 & s & v \\ 0 & 0 & (rs)^{-1} \end{pmatrix};\ \begin{matrix} u,\ v,\ w \in \boldsymbol{R} \\ r, s>0 \end{matrix}\right\}$$

$$=A_+N.$$

The subgroup G^e is obtained by a simple calculation:

$$G^e=\left\{\begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix};\ b,\ c \in \boldsymbol{R}\right\}.$$

So G^e is connected, and we have

$$D=G^eD_0=A_+N.$$

Then the unitary representation of G constructed on the G-orbit through e is equivalent to $\operatorname{ind}_{A_+N\uparrow G}(1_{A_+N})$, where 1_{A_+N} denotes the trivial character of A_+N. This representation is reducible, and the direct sum of 4-numbers of irreducible components:

$$\operatorname{ind}_{A_+N\uparrow G}(1_{A_+N})\sim\sum_{i=0}^{3}\operatorname{ind}_{AN\uparrow G}(\varepsilon_i),$$

where $\varepsilon_i(0\leqq i\leqq 3)$ is a unitary character of AN defined by

$$\varepsilon_0(x)=1,$$

and

$$\varepsilon_i(x)=\mathrm{sgn}(x_{ii}) \qquad (i=1, 2, 3),$$

where

$$x=\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \in AN,$$

and $\mathrm{sgn}(u)$ designates the sign of a non-zero real number u.

5.6. In order to avoid the inconvenience as in Example 5.5, we make a modification on the definition of D and E, when G is a connected semisimple Lie group. Since a polarization $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ (Theorem 2.2 of [13]), $\mathfrak{p}\cap\mathfrak{g}_R$ contains a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$. The Cartan subgroup H of G corresponding to $\mathfrak{h}_0$ is, by definition, the centralizer of $\mathfrak{h}_0$ in G, and let D_0 and E_0 be the same as in 5.1. We set $D=AD_0$ and $E=AE_0$. Since A stabilizes $\mathfrak{d}_0$ and $\mathfrak{e}_0$, D and E are subgroups of G. The argument in 5.3–5.4 is still valid for such D and E.

5.7. We shall give another expression of the G-quasi-invariant measure ν_D (or ν_E) in 5.3.

DEFINITION 5.7. 1) A linear mapping ν of $C_c(G/D)$ to $\boldsymbol{C}$ with the following property is called *a Radon measure on* G/D: for each compact subset K of G/D, there exists such a non-negative constant M_K that

$$|\nu(f)|\leqq M_K \sup_{x\in G/D}|f(x)|$$

for all $f\in C_c(G/D)$ whose support is contained in K.

2) A linear mapping ν of $C_c(G/D)$ to $\boldsymbol{R}$ which satisfies $\nu(f)\geqq 0$ for every $f\geqq 0$ is called *a positive Radon measure on* G/D. (It is a well-known fact that a positive Radon measure is a Radon measure.)

Each element ψ in $C^\infty(G)^\lambda$ defines a linear mapping ν'_ψ of $C_c(G)$ to $\boldsymbol{C}$ by

$$\nu'_\psi(f)=\int_G|\psi(g)|^2 f(g)\rho(g)d\mu_G(g)$$

for $f\in C_c(G)$.

LEMMA 5.7.1. *Let $\psi\in C^\infty(G)^\lambda$ be fixed, then*

$$\nu'_\psi(f)=\int_{G/D}|s_\psi(x)|^2\tilde{f}(x)d\nu_D(x)$$

for every $f\in C_c(G)$, where $f\to\tilde{f}$ is a linear mapping in Lemma 5.3.3.

PROOF. We have

$$\begin{aligned}\nu'_\psi(f)&=\int_G|\psi(g)|^2 f(g)\rho(g)d\mu_G(g)\\&=\int_{G/D}d\nu_D(\bar{g})\int_D|\psi(gh)|^2 f(gh)d\mu_D(h)\\&=\int_{G/D}|s_\psi(x)|^2\bar{f}(x)d\nu_D(x),\end{aligned}$$

since

$$|\psi(gx)|=|\psi(g)|=|s_\psi(\bar{g})| \qquad (\bar{g}=gD\in G/D).$$

Q.E.D.

By the above lemma, $\nu'_\psi(f)$ does not depend on the choice of a representative f of $\bar{f}$, but depends only on ψ and $\bar{f}$. So a linear mapping ν_ψ of $C_c(G/D)$ to $\boldsymbol{C}$ is well-defined by

$$\nu_\psi(\bar{f})=\nu'_\psi(f),$$

and ν_ψ is a positive Radon measure on G/D. Let $\|\psi\|^2(0\leqq\|\psi\|^2\leqq\infty)$ be the total volume of G/D with respect to this measure ν_ψ:

$$\|\psi\|=\mathrm{vol}_{\nu_\psi}(G/D).$$

We set

$$C_2^\infty(G)^\lambda=\{\psi\in C^\infty(G)^\lambda;\ \|\psi\|<\infty\}.$$

For ψ and $\psi'\in C_2^\infty(G)^\lambda$, the Radon measure $\nu_{(\psi,\psi')}$ on G/D is defined by using $\psi(g)\overline{\psi'(g)}$, and we set

$$(\psi,\psi')=\mathrm{vol}_{\nu_{(\psi,\psi')}}(G/D).$$

The space $C_2^\infty(G)^\lambda$ becomes a pre-Hilbert space with this Hermitian inner product, and at the same time it is a G-submodule of $C^\infty(G)^\lambda$. We set

$$\tilde{\mathfrak{H}}'_\lambda=\{\psi\in C_2^\infty(G)^\lambda;\ \tilde{X}\psi=2\pi\sqrt{-1}\lambda(X)\psi \qquad \text{for every } X\in\mathfrak{p}\},$$

and for every $g\in G$ and $\psi\in\tilde{\mathfrak{H}}'_\lambda$, we define a C^∞-function $\pi'_\lambda(g)\psi$ on G by

$$\tilde{\pi}'_\lambda(g)\psi=\sqrt{\xi_{g^{-1}}}\cdot g\psi.$$

LEMMA 5.7.2. 1) *$\tilde{\mathfrak{H}}'_\lambda$ is $\tilde{\pi}'_\lambda(G)$-stable.*

2) $\tilde{\pi}'_\lambda(gg')=\tilde{\pi}'_\lambda(g)\tilde{\pi}'_\lambda(g')$ *for every* $g, g'\in G$.

3) $\tilde{\pi}'_\lambda(g)$ *is norm-preserving.*

The proof of this lemma is the same as that of Lemma 5.4.1. And the completion $(\tilde{\pi}_\lambda, \tilde{\mathfrak{H}}_\lambda)$ of $(\tilde{\pi}'_\lambda, \tilde{\mathfrak{H}}'_\lambda)$ with respect to the norm $\|\ \ \|$ is a unitary representation of G.

LEMMA 5.7.3. *The mapping $s \rightarrow \psi_s$ is an isometry $\Gamma_2(L_\lambda)$ onto $C_2^\infty(G)^\lambda$.*

PROOF. It suffices to show that $\|\psi_s\| = \|s\|$ for every $s \in \Gamma(L_\lambda)$. Let K_1, $K_2, \cdots$ be a sequence of compact sets in G/D such that

$$K_n \subset K_{n+1} \qquad \text{for every } n \in \boldsymbol{N},$$

and

$$G/D = \bigcup_{n=1}^{\infty} K_n,$$

where $\boldsymbol{N}$ is the set of all positive integers. Let $\varphi_n \in C_c(G)$ $(n \in \boldsymbol{N})$ be a function such that

i) $\bar{\varphi}_n = 1$ on K_n and $\bar{\varphi}_n \geqq 0$ on G/D,

ii) $\bar{\varphi}_n \leqq \bar{\varphi}_{n+1}$ for every $n \in \boldsymbol{N}$.

Then we have, by Lemma 5.7.1,

$$\nu_{\psi_s}(\bar{\varphi}_n) = \int_G |\psi_s(g)|^2 \varphi_n(g) \rho(g) d\mu_G(g)$$

$$= \int_{G/D} |s(x)|^2 \bar{\varphi}_n(x) d\nu_D(x),$$

therefore

$$\|\psi_s\|^2 = \mathrm{vol}_{\nu_{\psi_s}}(G/D) = \lim_{n\to\infty} \nu_{\psi_s}(\bar{\varphi}_n)$$

$$= \lim_{n\to\infty} \int_{G/D} |s(x)|^2 \bar{\varphi}_n(x) d\nu_D(x).$$

Then, by the Lebesgue's integral theorem for a sequence of monotonously increasing non-negative integrable functions, we have

$$\|\psi_s\|^2 = \int_{G/D} \lim_{n\to\infty} |s(x)|^2 \bar{\varphi}_n(x) d\nu_D(x)$$

$$= \int_{G/D} |s(x)|^2 d\nu_D(x) = \|s\|^2.$$

Q.E.D.

This lemma, combined with the fact that $\psi_{\pi'_\lambda(g)s} = \sqrt{\xi_{g^{-1}}} \cdot g\psi_s = \tilde{\pi}'_\lambda(g)\psi_s$, leads us to:

THEOREM 5.7.4. *$(\tilde{\pi}_\lambda, \tilde{\mathfrak{H}}_\lambda)$ is a unitary representation of G equivalent to $(\pi_\lambda, \mathfrak{H}_\lambda)$, and $s \to \psi_s$ induces an isometric intertwining operator between them.*

§ 6. Polarizations and most continuous principal series

In this section, we construct representations of most continuous principal series using orbits and polarizations. First of all, we shall state the Borel-Weil theorem for a (non-connected in general) reductive compact Lie group.

6.1. Let G be a connected semisimple compact Lie group with Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and Δ the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and Δ_+ the set of all positive roots with respect to an arbitrarily fixed lexicographic linear order in Δ. For $\alpha \in \Delta$, we set

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g};\ ad(H)X = \alpha(H)X \qquad \text{for every } H \in \mathfrak{h}\}.$$

Let (σ_ν, V_ν) be a finite-dimensional irreducible representation of G with highest weight ν, and we set

$$\Delta' = \{\alpha \in \Delta;\ \langle\alpha, \nu\rangle = 0\},$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathfrak{h}^* = \mathrm{Hom}_{\mathbf{C}}(\mathfrak{h}, \boldsymbol{C})$ induced from the Killing form of $\mathfrak{g}$. Let L be a subgroup of G generated by $\mathfrak{l}_0 = \mathfrak{h}_0 + (\sum_{\alpha \in \Delta'} \mathfrak{g}^\alpha) \cap \mathfrak{g}_0$, and ε_ν the unitary character of L defined by ν. We set

$$\mathfrak{H}_\nu = \left\{ \begin{array}{l} f \in C^\infty(G);\ f(gl) = \varepsilon_\nu(l^{-1}) f(g) \qquad \text{for every } g \in G \text{ and } l \in L, \\ \qquad\qquad \tilde{X} f = 0 \qquad \text{for every } X \in \mathfrak{g}_+ \end{array} \right\},$$

where $\tilde{X}(X \in \mathfrak{g}_0)$ denotes the left-invariant vector field on G defined by

$$(\tilde{X} f)(g) = \left[\frac{d}{dt} f(g \exp tX)\right]_{t=0}$$

for every $f \in C^\infty(G)$ and $g \in G$, and $\tilde{X}(X \in \mathfrak{g})$ is its canonical extension and $\mathfrak{g}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}^\alpha$. The group G acts on $\mathfrak{H}_\nu$ by the left-translation:

$$(\pi_\nu(g) f)(x) = f(g^{-1} x)$$

for every $f \in \mathfrak{H}_\nu$ and $g, x \in G$. Then the well-known Borel-Weil theorem is stated in the following form:

LEMMA 6.1. *$(\pi_\nu, \mathfrak{H}_\nu)$ is a finite-dimensional irreducible representation of G equivalent to (σ_ν, V_ν).*

6.2. Let G be a connected reductive compact Lie group and $\mathfrak{g}_0$ its Lie algebra. Then $\mathfrak{g}_0$ admits the direct sum decomposition (as Lie algebras):

$$\mathfrak{g}_0=\mathfrak{z}_0+\mathfrak{g}_0^s,$$

where $\mathfrak{z}_0$ (resp. $\mathfrak{g}_0^s$) is the center (resp. the semisimple part) of $\mathfrak{g}_0$ (i.e., $\mathfrak{g}_0^s=[\mathfrak{g}_0, \mathfrak{g}_0]$). Let Z (resp. G^s) be the analytic subgroup of G generated by $\mathfrak{z}_0$ (resp. $\mathfrak{g}_0^s$), then we have

$$G=ZG^s=\{zg;\ z\in Z,\ g\in G^s\},$$

since G is connected. Let $\mathfrak{h}_0^s$ be a Cartan subalgebra of $\mathfrak{g}_0^s$, Δ the non-zero root system of $(\mathfrak{g}^s, \mathfrak{h}^s)$, and Δ_+ and $\mathfrak{g}^\alpha$ be the same as in 6.1. For a finite-dimensional irreducible representation (σ, V) of G, $\sigma(z)$ $(z\in Z)$ is a scalar operator on V and the restriction $(\sigma|G^s, V)$ of (σ, V) to G^s is an irreducible representation of G^s. So the representation (σ, V) of G is characterized by the character μ of Z and the highest weight ν of G^s with respect to Δ_+. (If necessary, we write $(\sigma_{\mu\nu}, V_{\mu\nu})$ instead of (σ, V).) We set

$$\Delta'=\{\alpha\in\Delta;\ <\alpha, \nu>=0\},$$

$$\mathfrak{h}_0=\mathfrak{z}_0+\mathfrak{h}_0^s,$$

and

$$\mathfrak{l}_0=\mathfrak{h}_0+\sum_{\alpha\in\Delta'}\mathfrak{g}^\alpha,$$

where $<\ ,\ >$ denotes the inner product in $(\mathfrak{h}^s)^*=\mathrm{Hom}_c(\mathfrak{h}^s, \boldsymbol{C})$ induced from the Killing form of $\mathfrak{g}^s$. Let L be the analytic subgroup of G generated by $\mathfrak{l}_0$, and ε the character of L defined by μ and ν. We set

$$\mathfrak{H}=\left\{\begin{array}{ll} f\in C^\infty(G);\ f(gl)=\varepsilon(l^{-1})f(g) & \text{for every } g\in G \text{ and } l\in L, \\ \tilde{X}f=0 \text{ for every } X\in g_+ & \end{array}\right\},$$

and denote by π the left-translation of G on $\mathfrak{H}$. Then

LEMMA 6.2. *$(\pi, \mathfrak{H})$ is a finite-dimensional irreducible representation of G equivalent to (σ, V).*

PROOF The subgroup $L^s=L\cap G^s$ is the analytic subgroup of G^s generated by

$$\mathfrak{l}_0^s=\mathfrak{l}_0\cap\mathfrak{g}_0^s=\mathfrak{h}_0^s+\sum_{\alpha\in\Delta'}\mathfrak{g}^\alpha.$$

We put

$$\varepsilon_\nu=\varepsilon|L^s,$$

and

$$\mathfrak{H}_\nu = \left\{ \begin{array}{ll} f \in C^\infty(G^s);\, f(gl) = \varepsilon_\nu(l^{-1})f(g) & \text{for } g \in G^s \text{ and } l \in L^s, \\ \tilde{X}f = 0 & \text{for every } X \in \mathfrak{g}_+ \end{array} \right\},$$

and let π_ν denote the representation of G^s on $\mathfrak{H}_\nu$ defined by the left-translation. By Lemma 6.1, $(\pi_\nu, \mathfrak{H}_\nu)$ is the irreducible representation of G^s equivalent to $(\sigma | G^s, V)$. Define a linear mapping φ of $\mathfrak{H}$ to $\mathfrak{H}_\nu$ by

$$\varphi(f) = f | G^s \qquad \text{for } f \in \mathfrak{H}.$$

Then φ is an injective G^s-homomorphism, and the image of φ is a non-zero G^s-submodule of $\mathfrak{H}_\nu$, which must coincide with $\mathfrak{H}_\nu$ by the G^s-irreducibility of $\mathfrak{H}_\nu$. Then φ is a G^s-isomorphism of $\mathfrak{H}$ onto $\mathfrak{H}_\nu$. Therefore $\mathfrak{H}$ is G^s-irreducible (equivalent to $(\sigma | G^s, V)$), and so G-irreducible and equivalent to (σ, V), since $\pi(z)$ $(z \in Z)$ is a scalar operator on $\mathfrak{H}$ which is equal to $\mu(z)$.

Q.E.D.

6.3. Let G be a (non-connected) reductive compact Lie group, and $\mathfrak{g}_0 = \mathfrak{z}_0 + \mathfrak{g}_0^s$ its Lie algebra. ($\mathfrak{z}_0$(resp. $\mathfrak{g}_0^s$) is the center (resp. the semisimple part) of $\mathfrak{g}_0$.) Let $\mathfrak{h}_0^s$ be a Cartan subalgebra of $\mathfrak{g}_0^s$, and we define Δ, Δ_+ and $\mathfrak{g}^\alpha$ in the same way as in 6.2. Let H be the centralizer of $\mathfrak{h}_0 = \mathfrak{z}_0 + \mathfrak{h}_0^s$ in G (i.e., H is the Cartan subgroup of G corresponding to $\mathfrak{h}_0$). We assume that H is an abelian subgroup of G and that $G = HG_0$, where G_0 is the connected component of G containing the unit. For a finite-dimensional irreducible representation (σ, V) of G, we define a subspace V^+ (the subspace of highest weight vectors) of V by

$$V^+ = \{v \in V;\, \sigma_*(X)v = 0 \qquad \text{for every } X \in \mathfrak{g}_+\}.$$

Then V^+ is an H-submodule of V.

Note: V_+ is 1-dimensional and $(\sigma | G_0, V)$ is an irreducible representation of G_0. In fact, we put $k = \dim V_+$. Then V^+ can be decomposed directly as

$$V^+ = \sum_{i=1}^{k} V_i^+,$$

where $V_i^+ (1 \leqq i \leqq k)$ is a 1-dimensional H-submodule. We set

$$\mathfrak{U}(\mathfrak{g}_-) = \text{the universal enveloping algebra}$$

$$\text{over } \mathfrak{g}_- = \sum_{\alpha \in \Delta_+} \mathfrak{g}^{-\alpha},$$

and

$$V_i = \sigma_*(\mathfrak{U}(\mathfrak{g}_-))V_i^+ \qquad (1 \leqq i \leqq k).$$

Then $V_i (1 \leqq i \leqq k)$ is an irreducible G-submodule, and $V = \sum_{i=1}^{k} V_i$ (direct sum as G-modules). So we have $k = 1$ by the irreducibility of V.

Using $(\sigma|G_0, V)$, we define Δ' as in 6.2, and we set

$$\mathfrak{l}_0=\mathfrak{h}_0+\sum_{\alpha\in\Delta'}\mathfrak{g}^{\alpha},$$

$$L_0=\text{the analytic subgroup of } G \text{ generated by } \mathfrak{l}_0,$$
$$L=HL_0.$$

And let ε be the character of H defined by

$$\sigma(h)v=\varepsilon(h)v \qquad \text{for } h\in H \text{ and } v\in V^{+}.$$

We set

$$\mathfrak{H}=\left\{\begin{array}{ll} f\in C^{\infty}(G);\, f(gl)=\varepsilon(l^{-1})f(g) & \text{for } g\in G \text{ and } l\in L, \\ \tilde{X}f=0 & \text{for every } X\in\mathfrak{g}_{+}\end{array}\right\},$$

and let π denote the representation of G on $\mathfrak{H}$ defined by left-translations.

LEMMA 6.3. *$(\pi, \mathfrak{H})$ is a finite-dimensional irreducible representation of G equivalent to (σ, V).*

PROOF We set

$$\varepsilon_0=\varepsilon|L_0,$$

$$\mathfrak{H}_0=\left\{\begin{array}{ll} f\in C^{\infty}(G_0);\, f(gl)=\varepsilon_0(l^{-1})f(g) & \text{for } g\in G_0 \text{ and } l\in L_0, \\ \tilde{X}f=0 & \text{for every } X\in\mathfrak{g}_{+}\end{array}\right\},$$

and let π_0 denote the left-translation of G_0 on $\mathfrak{H}_0$. Then, by Lemma 6.2, $(\pi_0, \mathfrak{H}_0)$ is the irreducible representation of G_0 equivalent to $(\sigma|G_0, V)$. Define a linear mapping φ of $\mathfrak{H}$ to $\mathfrak{H}_0$ by

$$\varphi(f)=f|G_0 \qquad \text{for } f\in\mathfrak{H}.$$

Then, by the assumption that $G=G_0H$, φ is an injective G_0-homomorphism. So the image of φ is a non-zero G_0-submodule of $\mathfrak{H}_0$, which must coincide with $\mathfrak{H}_0$ by the irreducibility of $\mathfrak{H}_0$. Thus φ is a bijective G_0-isomorphism. Therefore $\mathfrak{H}$ is G_0-irreducible (equivalent to $(\sigma|G_0, V)$) and so G-irreducible.

Denote by V_+(resp. $\mathfrak{H}_+$) the space of all highest weight vectors in V (resp. $\mathfrak{H}$), regarding them as the representation spaces of G_0. In order to prove the G-equivalence of σ and π, it suffices to show that the action of H on $\mathfrak{H}_+$ is equivalent with that on V_+: i.e., $\pi(a)f=\varepsilon(a)f$ for every $a\in H$ and $f\in\mathfrak{H}_+$. Now we consider a linear mapping T of $\mathfrak{H}$ onto $\boldsymbol{C}$ defined by $Tf=f(e)$ $(f\in\mathfrak{H})$. If we regard the space $\boldsymbol{C}$ as an H-module by $ac=\varepsilon(a)c$ for $a\in H$ and $c\in\boldsymbol{C}$, then T is an H-intertwining operator since

$$T(af)=(af)(e)=f(a^{-1})=\varepsilon(a)f(e)$$

$$=\varepsilon(a)(Tf)$$

for $a \in H$ and $f \in \mathfrak{H}$. Let W be the sum of all H-submodules of $\mathfrak{H}$ which are isomorphic to ε. Then W is non-trivial and must coincide with $\mathfrak{H}_+$ because $\mathfrak{H}_+$ is the only subspace of $\mathfrak{H}$ whose H_0-module structure is isomorphic to $\varepsilon | H_0$. Thus we have proved that $\mathfrak{H}_+$ is the H-submodule equivalent to ε.

Q.E.D.

6.4. Henceforward we fix a connected semisimple Lie group G with Lie algebra $\mathfrak{g}_0$. Let θ be a Cartan involution of $\mathfrak{g}_0$, and $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ associated to θ, where $\mathfrak{k}_0$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{a}_0=\mathfrak{a}_-+\mathfrak{a}_+(\mathfrak{a}_-\subset\mathfrak{k}_0, \mathfrak{a}_+\subset\mathfrak{p}_0)$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ with maximal vector part. We set $\mathfrak{g}=\mathfrak{g}_0^c$, $\mathfrak{a}=\mathfrak{a}_0^c$ and $\mathfrak{a}_R=\sqrt{-1}\mathfrak{a}_- +\mathfrak{a}_+$. The non-zero root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ admits a direct sum decomposition $\Delta=\Sigma\cup\Lambda$, where

$$\Sigma=\{\alpha \in \Delta;\ \alpha|\mathfrak{a}_+=0\}=\{\alpha\in\Delta;\ \mathfrak{g}^\alpha\subset\mathfrak{k}\},$$

$$\Lambda=\{\alpha\in\Delta;\ \alpha|\mathfrak{a}_+\neq 0\}.$$

A lexicographic order in $\mathfrak{a}_R$ compatible to $\mathfrak{a}_+$ induces a linear order in Δ and determines positive subsystems Δ_+, Σ_+ and Λ_+. We set

$$\mathfrak{n}_0=(\sum_{\alpha\in\Lambda_+}\mathfrak{g}^\alpha)\cap\mathfrak{g}_0,$$

$$\mathfrak{m}_0=\mathfrak{a}_-+(\sum_{\alpha\in\Sigma}\mathfrak{g}^\alpha)\cap\mathfrak{g}_0,$$

$$\mathfrak{m}_+=\sum_{\alpha\in\Sigma_+}\mathfrak{g}^\alpha,$$

$$M=Z_K(\mathfrak{a}_+)=\text{the centralizer of } \mathfrak{a}_+ \text{ in } K,$$

$$A=\text{the Cartan subgroup of } G \text{ corresponding to } \mathfrak{a}_0,$$

$$A_-=A\cap K,$$

$$A_+=A\cap\exp\mathfrak{p}_0=\exp\mathfrak{a}_+$$

$$=\text{the analytic subgroup of } G \text{ generated by } \mathfrak{a}_+,$$

$$N=\text{the analytic subgroup of } G \text{ generated by } \mathfrak{n}_0.$$

Then we have an Iwasawa decomposition $G=KA_+N$, and $B=MA_+N$ is a minimal parabolic subgroup of G, and A_- is a Cartan subgroup of the (non-connected in general) reductive compact Lie group M. Let M_0 (resp. $(A_-)_0$ or A_0) be the connected component of M (resp. A_- or A) containing the identity element, then $M=A_-M_0$. Hereafter we assume that A is abelian. This condition is always satisfied if G admits the complexification.

6.5. Let (σ, V) be a finite-dimensional irreducible unitary representation of M, and λ the unitary character of A_+. Then the irreducible unitary representation (σ, λ) of B is well-defined by

$$(\sigma, \lambda)(\mathrm{man})=\sigma(m)\lambda(a) \qquad \text{for } m \in M,\ a \in A_+ \text{ and } n \in N.$$

We define the unitary character ε of A_- as in 6.3. Then ε and λ determine elements $H_1 \in \mathfrak{a}_-$ and $H_2 \in \mathfrak{a}_+$ by

$$\varepsilon_*(H)=2\pi\sqrt{-1}<H_1, H> \qquad \text{for every } H \in \mathfrak{a}_-,$$

$$\lambda_*(H)=2\pi\sqrt{-1}<H_2, H> \qquad \text{for every } H \in \mathfrak{a}_+,$$

where $<\ ,\ >$ denotes the inner product in $\mathfrak{a}_0$ defined by the Killing form B. We set

$$H_0=H_1+H_2 \in \mathfrak{a}_0.$$

Then by Theorem 3.6, there exists a nilpotent element e in $\mathfrak{n}_0$ such that

$$[H_0, e]=0$$

and

$$\mathfrak{q}=\mathfrak{a}+\sum_{\alpha \in \Delta_+} \mathfrak{g}^\alpha+\sum_{\substack{\alpha \in \Sigma \\ \alpha(\sqrt{-1}H_1)\geqq 0}} \mathfrak{g}^\alpha \quad \text{is an admissible polarization of } X=H_0+e.$$

We set

$$\mathfrak{l}_0=\mathfrak{a}_-+\Big(\sum_{\substack{\alpha \in \Sigma \\ \alpha(H_1)=0}} \mathfrak{g}^\alpha\Big)\cap \mathfrak{g}_0,$$

$$L_0=\text{the analytic subgroup of } G \text{ with Lie algebra } \mathfrak{l}_0,$$

and

$$L=A_-L_0.$$

Then ε can be extended uniquely to the character of L, which is also denoted by ε. In this case, we have

$$\mathfrak{d}_0=\mathfrak{l}_0+\mathfrak{a}_++\mathfrak{n}_0,$$

$$\mathfrak{e}_0=\mathfrak{m}_0+\mathfrak{a}_++\mathfrak{n}_0,$$

$$D_0=L_0A_+N,$$

$$E_0=M_0A_+N.$$

As we have noted in 5.6, we define subgroups D and E of G by

$$D=A_-D_0=LA_+N,$$

and

$$E=A_{-}E_0=MA_{+}N=B.$$

By Lemma 5.3.2, there exists a B-invariant volume element $\nu_{B/D}$ on B/D, which can be normalized by

$$\int_B f(b)d\mu_B(b)=\int_{B/D} d\nu_{B/D}(bD)\int_D f(bh)d\mu_D(h)$$

for every $f \in C_c(B)$. Since B/D ($\cong M/L$) is compact, we can normalize μ_D so that the total volume of B/D with respect to $\nu_{B/D}$ may be equal to 1. The following lemma is useful for calculation of measures:

LEMMA 6.5.1 (Helgason [7] Lemma 1.10 (Chap X)). Let U be a Lie group with Lie algebra $\mathfrak{u}$. Suppose $\mathfrak{u}$ is a direct sum $\mathfrak{u}=\mathfrak{m}+\mathfrak{h}$ where $\mathfrak{m}$ and $\mathfrak{h}$ are subalgebras of $\mathfrak{u}$. Let M and H denote the analytic subgroups of U with Lie algebras $\mathfrak{m}$ and $\mathfrak{h}$, respectively. Suppose the mapping α: $(m, h)\rightarrow mh$ is a 1-1 mapping of $M\times H$ onto U. Then the positive left invariant measures dh, dm, du can be normalized in such a way that

$$\int_U f(u)du=\int_{M\times H} f(mh)\frac{\det Ad_H(h)}{\det Ad_U(h)}dmdh$$

for all $f \in C_c(U)$.

As a simple application of this lemma, we have

LEMMA 6.5.2. *The left invariant measures dm, dl, da, dn on M, L, A_+, N can be normalized by*

$$\int_B f(b)d\mu_B(b)=\int_{M\times A_+\times N} f(man)\ dmdadn$$

for every $f \in C_c(B)$, and

$$\int_D f(x)d\mu_D(x)=\int_{L\times A_+\times N} f(lan)dldadn$$

for every $f \in C_c(D)$.

PROOF. Fix positive left-invariant measures da and dn arbitrarily. Since

$$\det Ad_N(n)=\det Ad_{A_+N}(n)=1$$

for every $n \in N$, the positive left invariant measure $d(an)$ on A_+N can be normalized by $d(an)=dadn$. We set

$$\rho_0=\frac{1}{2}\sum_{\alpha\in A_+}\alpha \in \mathfrak{a}_+^*=\mathrm{Hom}_R(\mathfrak{a}_+, \boldsymbol{R}).$$

Since

$$\det Ad_{A_+N}(an)=\det Ad_B(an)$$
$$=\det Ad_D(an)=e^{2\rho_0(\log(a))}$$

for every $a \in A_+$ and $n \in N$, the positive left invariant measures dm, dl on M, L can be normalized as

$$d(man)=dm\ d(an),$$

and

$$d(lan)=dl\ d(an).$$

Thus we have proved

$$d(man)=dmdadn,$$

and

$$d(lan)=dldadn.$$

Q.E.D.

Let $\nu_{M/L}$ be the M--invariant volume element on M/L such that

$$\int_M f(m)dm=\int_{M/L} d\nu_{M/L}(mL)\int_L f(ml)dl$$

for every $f \in C_c(M)$. Then we have

LEMMA 6.5.3 $\nu_{M/L}=\nu_{B/D}$, *under the canonical diffeomorphism* $M/L\cong B/D$.

PROOF. For $f \in C_c(B)$, we have

$$\int_B f(b)d\mu_B(b)=\int_{M\times A_+\times N} f(man)dmdadn$$
$$=\int_{M/L} d\nu_{M/L}(mL)\int_{L\times A_+\times N} f(mlan)dldadn$$
$$=\int_{M/L} d\nu_{M/L}(mL)\int_D f(mx)d\mu_D(x).$$

Comparing with the definition of $\nu_{B/D}$, we have

$$d\nu_{M/L}(mL)=d\nu_{B/D}(mD).$$

Thus we have proved that $\nu_{M/L}=\nu_{B/D}$. Q.E.D.

6.6. The unitary representation $(\tilde{\pi}_{\sigma\lambda}, \tilde{\mathfrak{H}}_{\sigma\lambda})$ constructed in 5.7 is the completion of $(\tilde{\pi}'_{\sigma\lambda}, \tilde{\mathfrak{H}}'_{\sigma\lambda})$:

$$\tilde{\mathfrak{H}}'_{\sigma\lambda}=\left\{\begin{array}{ll} f\in C^{\infty}(G);\ 1) & f(xlan)=\varepsilon(l^{-1})\lambda(a^{-1})f(x) \\ & \quad \text{for } x\in G,\ l\in L,\ a\in A_+ \text{ and } n\in N, \\ \quad 2) & \tilde{X}f=0 \quad \text{for } X\in \sum_{\alpha\in\Sigma_+}\mathfrak{g}^{\alpha}, \\ \quad 3) & \|f\|_D<\infty \end{array}\right\},$$

$$\tilde{\pi}'_{\sigma\lambda}(g)f=\sqrt{\xi_{g^{-1}}}\cdot gf.$$

We set

$$W'_{\sigma\lambda}=\left\{\begin{array}{ll} f;\ 0) & f \text{ is a } V\text{-valued } C^{\infty}\text{-function on } G, \\ \quad 1) & f(xb)=(\sigma,\lambda)(b^{-1})f(x) \quad \text{for } x\in G \text{ and } b\in B, \\ \quad 2) & \|f\|_B<\infty \end{array}\right\},$$

$$\eta'_{\sigma\lambda}(g)f=\sqrt{\xi_{g^{-1}}}\cdot gf \qquad \text{for } g\in G \text{ and } f\in W'_{\sigma\lambda},$$

where $\|f\|_B$ is the norm of f defined in the same way as in 5.7, and gf denotes the left translation of f by g. The completion $(\eta_{\sigma\lambda}, W_{\sigma\lambda})$ of $(\eta'_{\sigma\lambda}, W'_{\sigma\lambda})$ is called *a representation of most continuous principal series*, and sometimes denoted by $\operatorname{ind}_{B\uparrow G}(\sigma,\lambda)$.

By Lemma 6.3, the representation (σ, V) of M is equivalent to (σ', V'), where

$$V'=\left\{\begin{array}{l} f\in C^{\infty}(M);\ f(ml)=\varepsilon(l^{-1})f(m) \qquad \text{for } m\in M \text{ and } l\in L, \\ \qquad \tilde{X}f=0 \qquad \text{for every } X\in \sum_{\alpha\in\Sigma_+}\mathfrak{g}^{\alpha} \end{array}\right\},$$

and σ' is left-translation of M on V'. We introduce a Hermitian inner product $(\ ,\ )$ in V' as follows: for $f, f'\in V'$, the C^{∞}-function φ on M/L is well-defined by $\varphi(xL)=f(x)\overline{f'(x)}$ $(x\in M)$ since ε and λ are unitary, and so we put

$$(f, f')=\int_{M/L}\varphi(y)d\nu_{M/L}(y).$$

By the M-invariantness of $\nu_{M/L}$, (σ', V') is a unitary representation of M with respect to this Hermitian inner product. Let S be an isometric intertwining operator of V onto V'. And we define a linear mapping T' of $W'_{\sigma\lambda}$ to $\tilde{\mathfrak{H}}'_{\sigma\lambda}$ by

$$(T'f)(x)=[S(f(x))](e)$$

for $f\in W'_{\sigma\lambda}$ and $x\in G$, where e denotes the unit of G.

THEOREM 6.6. *T' is an isometry of $W'_{\sigma\lambda}$ onto $\tilde{\mathfrak{H}}'_{\sigma\lambda}$, which commutes with*

G-actions. So T' can be extended to an isometric intertwining operator of $(\eta_{\sigma\lambda}, W_{\sigma\lambda})$ onto $(\tilde{\pi}_{\sigma\lambda}\ \tilde{\mathfrak{H}}_{\sigma\lambda})$, and the representation $(\tilde{\pi}_{\sigma\lambda}, \tilde{\mathfrak{H}}_{\sigma\lambda})$ is a unitary representation of G of the most continuous principal series.

We shall give a proof of this theorem step-wisely.

6.7. We set $\varphi(x)=[S(f(x))](e)$ for $f \in W'_{\sigma\lambda}$ and $x \in G$.

LEMMA 6.7.1. 1) $\varphi(xm)=[S(f(x))](m)$ *for* $x \in G$ *and* $m \in M$.

PROOF. Since S is an intertwining operator, we have

$$\begin{aligned}\varphi(xm)&=[S(f(xm))](e)=[S(\sigma(m^{-1})f(x))](e)\\&=[\sigma'(m^{-1})S(f(x))](e)\\&=[S(f(x))](m).\end{aligned}$$

Q.E.D.

LEMMA 6.7.2. 1) $\tilde{X}\varphi=0$ *for* $X \in \mathfrak{m}_+$.

2) $\varphi(xlan)=\varepsilon(l^{-1})\lambda(a^{-1})\varphi(x)$ *for* $x \in G$, $l \in L$, $a \in A_+$ *and* $n \in N$.

PROOF. 1) For $f \in W'_{\sigma\lambda}$, we define a C^∞-function f_x on M by $f_x(m)=[S(f(x))](m)$. Then, by Lemma 6.7.1, we have

$$(\tilde{X}\varphi)(x)=(\tilde{X}f_x)(e)$$

for every $x \in G$ and $X \in \mathfrak{m}$. So we have

$$\tilde{X}\varphi=0 \qquad \text{for } X \in \mathfrak{m}_+$$

since $f_x \in V'$.

2) Since $f \in W'_{\sigma\lambda}$, and $S(f(x)) \in V'$, we have

$$\begin{aligned}\varphi(xlan)&=[S(f(xlan))](e)\\&=\lambda(a^{-1})[S(f(xl))](e)\\&=\lambda(a^{-1})[S(f(x))](l)\\&=\lambda(a^{-1})\varepsilon(l^{-1})[S(f(x))](e)\\&=\lambda(a^{-1})\varepsilon(l^{-1})\varphi(x).\end{aligned}$$

Q.E.D.

LEMMA 6.7.3. $\|T'f\|=\|f\|$ *for* $f \in W'_{\sigma\lambda}$.

PROOF. We denote by $\|\ \|_B$, $\|\ \|_D$, $\|\ \|_V$ and $\|\ \|_{V'}$ the norm of $W_{\sigma\lambda}$, $\tilde{\mathfrak{H}}_{\sigma\lambda}$, V and V' respectively. Let $K_1, K_2, \cdots$ be a sequence of compact sets in G/B such that

$$K_n \subset K_{n+1} \qquad \text{for every } n \in \boldsymbol{N},$$

and

$$G/B = \bigcup_{n=1}^{\infty} K_n,$$

and let $\psi_n \in C_c(G/B)\ (n \in \boldsymbol{N})$ be a function such that

i) $\psi_n = 1$ on K_n and $\psi_n \geqq 0$ on G/B,

ii) $\psi_n \leqq \psi_{n+1}$ for every $n \in \boldsymbol{N}$.

We set

$$\psi'_n = \psi_n \circ \rho_{DB} \qquad (n \in \boldsymbol{N})$$

where ρ_{DB} is the canonical fibration defined in 5.3. There exists a sequence $\varphi_1, \varphi_2, \cdots$ in $C_c(G)$ such that

i) $\varphi_n \geqq 0$,

ii) $\varphi_n \leqq \varphi_{n+1}$,

iii) $\bar{\varphi}_n = \psi'_n$,

where $\bar{\varphi}_n(xD) = \int_D \varphi_n(xh)\, d\mu_D(h)$.

We set

$$\tilde{\varphi}_n(xB) = \int_B \varphi_n(xb)\, d\mu_B(b).$$

Then $\tilde{\varphi}_n \in C_c(G/D)$, and we have

$$\begin{aligned}
\tilde{\varphi}_n(xB) &= \int_{B/D} d\nu_{B/D}(bD) \int_D \varphi_n(xbh)\, d\mu_D(h) \\
&= \int_{B/D} \bar{\varphi}_n(xbD)\, d\nu_{B/D}(bD) \\
&= \int_{B/D} \psi'_n(xbD)\, d\nu_{B/D}(bD).
\end{aligned}$$

Since $\psi'_n(xbD) = \psi_n(xE)$ and $\nu_{B/D}(B/D) = 1$, we have $\tilde{\varphi}_n = \psi_n$.

Now, for any $f \in W'_{\sigma\lambda}$, we shall calculate $\|f\|_B$ and $\|Tf\|_D$:

$$\begin{aligned}
\|f\|_B^2 &= \lim_{n\to\infty} \int_G \|f(x)\|_V^2 \varphi_n(x) \rho(x)\, d\mu_G(x) \\
&= \lim_{n\to\infty} \int_G d\nu_B(xB) \int_B |f(xb)|^2 \varphi_n(xb)\, d\mu_B(b)
\end{aligned}$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_B |f(x)|^2\varphi_n(xb)d\mu_B(b)$$

$$=\lim_{n\to\infty}\int_{G/B} |f(xB)|^2\tilde{\varphi}_n(xB)d\nu_B(xB)$$

$$=\int_{G/B} \|f(y)\|^2 d\nu_B(y),$$

by the Lebesgue's integral theorem for a sequence of monotonously increasing non-negative integrable functions.

$$\|T'f\|_D^2=\lim_{n\to\infty}\int_G |(T'f)(x)|^2\varphi_n(x)\rho(x)d\mu_G(x)$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_B |(T'f)(xb)|^2\varphi_n(xb)d\mu_B(b)$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_{M\times A_+\times N} |(T'f)(xman)|^2\varphi_n(xman)dmdadn$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_{M/L} d\nu_{M/L}(mL)\int_{L\times A_+\times N}|(T'f)(xmlan)|^2$$

$$\times\varphi_n(xmlan)dldadn.$$

For each $x\in G$, we put $f_x=S(f(x))\in V'$. Then

$$|(T'f)(xmlan)|=|\lambda(a^{-1})\varepsilon(l^{-1})(T'f)(xm)|$$

$$=|(T'f)(xm)|=|f_x(m)|$$

by Lemma 6.7.1. So we have

$$\|T'f\|_D^2=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_{M/L} d\nu_{M/L}(mL)\int_{L\times A_+\times N}|f_x(m)|^2\varphi_n(xmlan)dldadn$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_{M/L} |f_x(m)|^2\bar{\varphi}_n(xmD)d\nu_{M/L}(mL)$$

$$=\lim_{n\to\infty}\int_{G/B} d\nu_B(xB)\int_{M/L} |f_x(m)|^2\bar{\varphi}_n(xD)d\nu_{M/L}(mL),$$

since $\bar{\varphi}_n=\tilde{\varphi}_n\circ\rho_{DE}$ is constant on each fibre of ρ_{DB}. By the definition of $\|\ \ \|_{V'}$ in 6.6,

$$\|f_x\|_{V'}^2=\int_{M/L} |f_x(m)|^2 d\nu_{M/L}(mL),$$

and, since S is unitary,

$$\|f_x\|_{V'}=\|f(x)\|_V=\|f(xB)\|_V.$$

So we have

$$\|T'f\|_D^2=\lim_{n\to\infty}\int_{G/B}\|f_x\|_{V'}^2\tilde{\varphi}_n(xB)\,d\nu_B(xB)$$
$$=\int_{G/B}\|f_x\|_{V'}^2\,d\nu_B(xB)$$
$$=\int_{G/B}\|f(xB)\|_V^2\,d\nu_B(xB)$$
$$=\int_{G/B}\|f(y)\|_V^2\,d\nu_B(y),$$

where we used the Lebesgue's theorem. Thus we have proved that $\|T'f\|_D=\|f\|_B$. Q.E.D.

LEMMA 6.7.4. *$\pi'_{\sigma\lambda}(g)T'f=T'\eta'_{\sigma\lambda}(g)f$ for every $f\in W'_{\sigma\lambda}$ and $g\in G$.*

PROOF. This is shown by an easy calculaton:

$$(\tilde{\pi}'_{\sigma\lambda}(g)T'f)(x)=\sqrt{\xi_{g^{-1}}(x)}\,(T'f)(g^{-1}x)$$
$$=\sqrt{\xi_{g^{-1}}(x)}\,[S(f(g^{-1}x))](e),$$
$$(T'(\eta'_{\sigma\lambda}(g)f))(x)=[S((\eta'_{\sigma\lambda}(g)f)(x))](e)$$
$$=[S(\sqrt{\xi_{g^{-1}}(x)}\,f(g^{-1}x))](e),$$

for all $x\in G$. Q.E.D.

6.8. We shall prove the bijectiveness of T'. For $\varphi\in\tilde{\mathfrak{H}}'_{\sigma\lambda}$ and $x\in G$, we define $\varphi_x\in V'$ by

$$\varphi_x(m)=\varphi(xm),$$

and we set $f(x)=S^{-1}\varphi_x \qquad (\in V)$.

LEMMA 6.8.1. $\varphi_{xman}=\sigma'(m^{-1})\lambda(a^{-1})\varphi_x$
for every $x\in G$, $m\in M$, $a\in A_+$ and $n\in N$.

PROOF. For $m'\in M$, we have

$$\varphi_{xman}(m')=\varphi(xmanm')$$
$$=\varphi(xmm'\cdot a\cdot m'^{-1}nm')$$
$$=\lambda(a^{-1})\varphi(xmm')$$
$$=\lambda(a^{-1})\varphi_x(mm')$$
$$=\lambda(a^{-1})(\sigma'(m^{-1})\varphi_x)(m').$$

Q.E.D.

LEMMA 6.8.2. $f \in W'_{\sigma\lambda}$.

PROOF. For $x \in G$, $m \in M$, $a \in A_+$ and $n \in N$, we have

$$f(xman)=S^{-1}\varphi_{xman}=S^{-1}(\sigma'(m^{-1})\lambda(a^{-1})\varphi_x)$$
$$=\sigma(m^{-1})\lambda(a^{-1})S^{-1}\varphi_x$$
$$=\sigma(m^{-1})\lambda(a^{-1})f(x).$$

Q.E.D.

Thus $\varphi \to f$ determines a linear mapping U of $\tilde{\mathfrak{H}}'_{\sigma\lambda}$ to $W'_{\sigma\lambda}$: $(U\varphi)(x)=S^{-1}\varphi_x$, for $\varphi \in \mathfrak{H}'_{\sigma\lambda}$ and $x \in G$.

LEMMA 6.8.3. 1) *UT' is the identity of $W'_{\sigma\lambda}$,*

2) *$T'U$ is the identity of $\tilde{\mathfrak{H}}'_{\sigma\lambda}$.*

(And so T is a linear isomorphism of $W'_{\sigma\lambda}$ onto $\tilde{\mathfrak{H}}'_{\sigma\lambda}$.)

PROOF. 1) For $f \in W'_{\sigma\lambda}$ and $x \in G$, we have

$$(T'f)_x=S(f(x)),$$

since

$$(T'f)_x(m)=(T'f)(xm)=[S(f(xm))](e)$$
$$=[S(f(x))](m)$$

for every $m \in M$. So we have

$$(UT'f)(x)=S^{-1}(T'f)_x=S^{-1}S(f(x))=f(x).$$

2) For $\varphi \in \tilde{\mathfrak{H}}'_{\sigma\lambda}$ and $x \in G$, we have

$$(T'U\varphi)(x)=[S((U\varphi)(x))](e)=[S(S^{-1}\varphi_x)](e)$$
$$=\varphi_x(e)=\varphi(x).$$

Q.E.D.

Theorem 6.6 follows from Lemma 6.7.3, Lemma 6.7.4 and Lemma 6.8.3

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