HIROSHIMA MATH. J.
2 (1972) 345–352

Immersions and Embeddings of Lens Spaces

Teiichi KOBAYASHI

(Received September 20, 1972)

§1. Introduction

Throughout this note q will denote an odd integer >1, and n a positive integer. Let $L^n(q)$ be the $(2n+1)$-dimensional standard lens space mod q, and let $L^\infty(q)$ be $\bigcup_n L^n(q)$, which is the Eilenberg-MacLane space $K(Z_q, 1)$, where Z_q is a cyclic group of order q. Denote by $\iota(\in H^1(K(Z_q, 1); Z_q)\cong Z_q)$ the fundamental class of $K(Z_q, 1)$. The element $y_n(\in H^1(L^n(q); Z_q)\cong Z_q)$ is called the distinguished generator if $y_n = i^*\iota$, where $i^*: H^1(K(Z_q, 1); Z_q)\to H^1(L^n(q); Z_q)$ is the isomorphism induced by the natural inclusion $i: L^n(q)\to L^\infty(q)=K(Z_q, 1)$.

For a given $d\in Z_q$, a continuous map $f: L^n(q)\to L^m(q)$ is said to have degree $d(=\deg(f))$, if $f^*y_m = dy_n$, where $f^*: H^1(L^m(q); Z_q)\to H^1(L^n(q); Z_q)$ is the homomorphism induced by f, and where y_n and y_m are the distinguished generators. If $n<m$, the set of homotopy classes of maps of $L^n(q)$ in $L^m(q)$ is in one-to-one correspondence with $H^1(L^n(q); Z_q)(\cong Z_q)$. Thus the homotopy class of a map $f: L^n(q)\to L^m(q)$, $n<m$, is completely characterized by $\deg(f)$.

The first purpose of this paper is to consider the question: "Which homotopy classes of continuous maps $L^n(q)\to L^m(q)$ contain a differentiable immersion (or a differentiable embedding)?"

S. Feder has investigated in [2] the question on complex projective spaces and H. Suzuki has studied in [10] and [11] the question in the case of higher order non-singular immersions for projective spaces. The problem for general manifolds is treated by E. Thomas [13], [14] and M. Adachi [1].

By the work of M.W. Hirsch [3] and D. Sjerve [8] we see that any map $L^n(q)\to L^m(q)$ is homotopic to an immersion for $m\geqq n+[n/2]+1$ if q is odd (cf. §2). For $m\leqq n+[n/2]$, we have the following results.

THEOREM A. *Let q be an odd integer >1, and let k be an integer with $0<k\leqq[n/2]$. If a map $f: L^n(q)\to L^{n+k}(q)$ with degree d is homotopic to an immersion, then*

$$\sum_{i+j=k}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}$$

is a quadratic residue mod q.

COROLLARY A. *If* $(-1)^{[n/2]}\binom{n+[n/2]}{[n/2]}$ *is not a quadratic residue* mod q, *then a map* $L^n(q)\to L^{n+[n/2]}(q)$ *with degree* 0 *is not homotopic to an immersion.*

THEOREM B. *Let p be an odd prime, and let k be an integer such that* $0<k<[n/2]$. *Assume that* $\binom{n+l}{j}\not\equiv 0$ mod p *for some l and j with* $k<l\leqq j\leqq [n/2]$. *Then a map* $f\colon L^n(p)\to L^{n+k}(p)$ *is homotopic to an immersion if and only if* $\deg(f)=\pm 1$.

COROLLARY B. *Suppose p is an odd prime and* $\binom{n+[n/2]}{[n/2]}\not\equiv 0$ mod p. *Then a map* $f\colon L^n(p)\to L^{n+[n/2]-1}(p)$ *is homotopic to an immersion if and only if* $\deg(f)=\pm 1$.

According to D. Sjerve [9, Th. (4.7)], for an odd prime p there are examples of immersions of $L^n(p)$ in $L^{n+[n/2]-1}(p)$ with degree 0, if $\binom{n+[n/2]}{[n/2]}\equiv 0$ mod p.

As for embeddings, we get the following

THEOREM C. *Let q be an odd integer* >1, *and let k be an integer with* $0<k\leqq[n/2]$. *If a map* $f\colon L^n(q)\to L^{n+k}(q)$ *with degree d is homotopic to an embedding, then the following congruence holds*:

$$\sum_{i+j=k}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}\equiv d^{2n+2k+2} \qquad \text{mod } q.$$

COROLLARY C. *If* $\binom{n+[n/2]}{[n/2]}\not\equiv 0$ mod q, *then a map* $L^n(q)\to L^{n+[n/2]}(q)$ *with degree* 0 *is not homotopic to an embedding.*

It is well-known that for an odd integer q there is no embedding of $L^n(q)$ in Euclidean $2n+2[n/2]+1$ space $R^{2n+2[n/2]+1}$ if $\binom{n+[n/2]}{[n/2]}\not\equiv 0$ mod q.

The second object of this note is to discuss the embeddability of $L^n(5)$ in Euclidean space. We have

THEOREM D. *If* $n=3\cdot 5^{t+1}+5^t$, *then* $L^n(5)$ *cannot be embedded in* R^{3n+2}.

Recently, E. Rees has proved in [7, Th.1] that a smooth odd torsion manifold M of dimension m (i.e., a closed, smooth m-manifold such that $H_i(M;Z)\otimes Z_2=0$ for $0<i<m$) can be embedded in R^r for every $r\geqq 3(m+1)/2$. The result of E. Rees is seen to be the best possible general result for odd dimensional manifolds by the above theorem, because the mod 5 lens space is an odd torsion manifold.

In §2 some preliminaries are given. The proofs of Theorems A and B are carried out in §3, and Theorem C is proved in §4, by the technique of

S. Feder [2, Part I]. In §5 we verify Theorem D by making use of W.S. Massey's subalgebra (cf. [4], [5] and [6]).

§2. Preliminaries

Let X be a finite CW-complex. It is said that a stable bundle $\alpha \in \widetilde{KO}(X)$ has geometric dimension $\leqq k$ if there is a k-plane bundle β over X such that $\alpha + k = \beta$. For a smooth manifold M we denote by $\tau(M)$ the tangent bundle and $\tau_0(M)(=\tau(M)-\dim M)$ the stable tangent bundle. The following theorem of M.W. Hirsch [3] reduces the problem to that in homotopy theory.

(2.1) Theorem (M.W. Hirsch). *Let N and M be smooth manifolds with* $\dim N < \dim M$, *and $f\colon N \to M$ be a continuous map. Then f is homotopic to an immersion if and only if the stable bundle $f^!\tau_0(M) - \tau_0(N)$ has geometric dimension* $\leqq \dim M - \dim N$.

As for the geometric dimension of the stable bundle over the lens space, D. Sjerve [8] obtains the following result.

(2.2) Theorem (D. Sjerve). *Let q be an odd integer* >1, *and $\pi\colon S^{2n+1} \to L^n(q)$ be the natural projection. If $\xi \in \widetilde{KO}(L^n(q)) \cap \operatorname{Ker} \pi^!$, then* $\mathrm{g}\cdot\dim \xi \leqq 2[n/2]+1$.

The next result is a consequence of (2.1) and (2.2).

(2.3) Proposition. *Suppose q is odd. If* $m \geqq n+[n/2]+1$, *any map f: $L^n(q) \to L^m(q)$ is homotopic to an immersion.*

Proof. By (2.1) it is sufficient to show that the stable bundle $f^!\tau_0(L^m(q)) - \tau_0(L^n(q))$ has geometric dimension $\leqq 2m-2n$.

There is a map $\bar{f}$ which satisfies the following commutative diagram:

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{\bar{f}} & S^{2m+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ L^n(q) & \xrightarrow{f} & L^m(q) \end{array}$$

where π and π' are the natural projections. Thus we have

$$\begin{aligned} \pi^!(f^!\tau_0(L^m(q)) - \tau_0(L^n(q))) &= \bar{f}^!\pi'^!\tau_0(L^m(q)) - \pi^!\tau_0(L^n(q)) \\ &= \bar{f}^!\tau_0(S^{2m+1}) - \tau_0(S^{2n+1}) = 0, \end{aligned}$$

since the sphere is stably parallelizable. Hence, by (2.2),

$$\mathrm{g}\cdot\dim(f^!\tau_0(L^m(q)) - \tau_0(L^n(q))) \leqq 2[n/2]+1 < 2m-2n. \qquad \text{q.e.d.}$$

Let $\triangle : H^1(L^n(q); Z_q) \to H^2(L^n(q); Z_q)$ be the Bockstein homomorphism associated with the coefficient sequence: $0 \to Z_q \to Z_{q^2} \to Z_q \to 0$. For the distinguished generator $y_n \in H^1(L^n(q); Z_q)$, put $x_n = \triangle y_n \in H^2(L^n(q); Z_q)$, which is also called the distinguished generator. Since the Bockstein homomorphism is natural for maps and the above $\triangle$ is isomorphic, we have

(2.4) LEMMA. *A map $f: L^n(q) \to L^m(q)$ has degree $d \in Z_q$ if and only if $f^* x_m = d x_n$.*

Hereafter, we shall omit the subscripts and write x and y instead of x_n and y_n, respectively.

As is well-known, the cohomology algebra $H^*(L^n(q); Z_q)$ is the tensor product of the exterior algebra on y and the truncated polynomial algebra on x with relations: $\triangle y = x$ and $x^{n+1} = 0$.

The mod q Pontrjagin class p_i and mod q dual Pontrjagin class $\bar{p}_i$ of $L^n(q)$ are given by the following equations [12, Cor. 3.2]:

$$p = \sum_{i=0}^{[n/2]} p_i = (1+x^2)^{n+1}, \tag{2.5}$$

$$\bar{p} = \sum_{i=0}^{[n/2]} \bar{p}_i = (1+x^2)^{-n-1} = \sum_{i=0}^{[n/2]} (-1)^i \binom{n+i}{i} x^{2i}. \tag{2.6}$$

§ 3. Immersions of $L^n(q)$ in $L^m(q)$

(3.1) LEMMA. *Let $g: L^n(q) \to L^{n+k}(q)$, $k>0$, be an immersion with degree $d \in Z_q$. Denote by $\nu(g)$ the normal bundle of g. Then the Pontrjagin class $p(\nu(g))$ is given by the following:*

$$\begin{aligned} p(\nu(g)) &= \sum_{t=0}^{[n/2]} \left\{ \sum_{i+j=t} (-1)^i \binom{n+i}{i} \binom{n+k+1}{j} d^{2j} \right\} x^{2t} \\ &= \sum_{i=0}^{[n/2]} \binom{n+k+1}{i} (d^2-1)^i x^{2i} (1+x^2)^{k-i}. \end{aligned}$$

PROOF. By the assumption we have the Whitney sum decomposition:

$$\nu(g) \oplus \tau(L^n(q)) = g^! \tau(L^{n+k}(q)).$$

Since $H^*(L^n(q); Z)$ has no 2-torsion for odd q, it holds that

$$p(\nu(g)) p(\tau(L^n(q))) = g^* p(\tau(L^{n+k}(q))).$$

By (2.5) and (2.4), we obtain

$$p(\nu(g)) = (g^*(1+x^2)^{n+k+1})(1+x^2)^{-n-1} = (1+d^2x^2)^{n+k+1}(1+x^2)^{-n-1}$$

$$=\sum_{j=0}^{[n/2]}\binom{n+k+1}{j}d^{2j}x^{2j}\sum_{i=0}^{[n/2]}\binom{-n-1}{i}x^{2i}$$

$$=\sum_{t=0}^{[n/2]}\left\{\sum_{i+j=t}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}\right\}x^{2t}.$$

Also we have

$$p(\nu(g))=(1+x^2+(d^2-1)x^2)^{n+k+1}(1+x^2)^{-n-1}$$

$$=\sum_{i=0}^{[n/2]}\binom{n+k+1}{i}(d^2-1)^i x^{2i}(1+x^2)^{k-i} \qquad \text{q.e.d.}$$

Proof of Theorem A. Suppose that the map f is homotopic to an immersion $g: L^n(q)\to L^{n+k}(q)$. Then $\deg(g)=\deg(f)=d$. Let $\nu(g)$ be the normal bundle of g. By (3.1) we have

$$p_k(\nu(g))=\sum_{i+j=k}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}x^{2k}.$$

On the other hand, $p_k(\nu(g))=\chi(\nu(g))^2$, where $\chi(\nu(g))$ is the Euler class of $\nu(g)$. Thus we obtain the desired result. q.e.d.

Let $S^{2n+1}=\{(z_0,\cdots,z_n\in C^{n+1}\mid \sum_{i=0}^{n}|z_i|^2=1\}$ be the unit sphere in complex $(n+1)$-space C^{n+1}. The image of $(z_0,\cdots,z_n)$ by the natural projection $S^{2n+1}\to L^n(q)$ is denoted by $[z_0,\cdots,z_n]$.

Proof of Theorem B. Suppose that the map f is homotopic to an immersion $g: L^n(p)\to L^{n+k}(p)$. Then $\deg(g)=\deg(f)=d$. Let $i: L^{n+k}(p)\to L^{n+l-1}(p)$ be the standard inclusion defined by $i[z_0,\cdots,z_{n+k}]=[z_0,\cdots,z_{n+k},\overbrace{0,\cdots,0}^{l-k-1}]$. Then, clearly, the composite map $ig: L^n(p)\to L^{n+l-1}(p)$ is an immersion with degree d. Since the normal bundle $\nu(ig)$ is $(2l-2)$-dimensional and p is odd, we must have $p_j(\nu(ig))=0$ for $j\geqq l$. Thus it follows from the second equality of (3.1) that

$$p_j(\nu(ig))=\sum_{i=l}^{j}\binom{n+l}{i}\binom{l-1-i}{j-i}(d^2-1)^i x^{2j}=0, \qquad j\geqq l.$$

A simple calculation shows that, for any j with $l\leqq j\leqq[n/2]$,

$$\binom{n+l}{j}(d^2-1)^j\equiv 0 \qquad \mathrm{mod}\ p.$$

By the assumption, we see that $d=\pm1$, because p is a prime.

Conversely, if $\deg(f)=1$, f is homotopic to the standard inclusion which is obviously an immersion with degree 1, and if $\deg(f)=-1$, f is homotopic to the map $t: L^n(p)\to L^{n+k}(p)$ defined by $t[z_0,\cdots,z_n]=[\bar z_0,\cdots,\bar z_n,\overbrace{0,\cdots,0}^{k}]$ which is seen to be an immersion with degree -1. q.e.d.

§4. Embeddings of $L^n(q)$ in $L^m(q)$

(4.1) LEMMA. *If* $g: L^n(q)\to L^{n+k}(q)$, $k>0$, *is an embedding with degree* d, *then the* mod q *Euler class* $\chi(\nu(g))$ *of the normal bundle* $\nu(g)$ *of* g *is equal to* $d^{n+k+1}x^k$.

PROOF. Let $g_*: H^0(L^n(q); Z_q)\to H^{2k}(L^{n+k}(q); Z_q)$ be the Gysin homomorphism defined by the following commutative diagram:

$$
\begin{array}{ccc}
H^0(L^n(q); Z_q) & \xrightarrow{g_*} & H^{2k}((L^{n+k}(q); Z_q) \\
\downarrow & & \downarrow \\
H_{2n+1}(L^n(q); Z_q) & \xrightarrow{g_*} & H_{2n+1}(L^{n+k}(q); Z_q)
\end{array}
\tag{4.2}
$$

where the vertical maps are the Poincaré duality isomorphisms, and the lower horizontal map is the homomorphism induced by g. Then, according to S. Feder [2, Th. 1.3], we have

$$\chi(\nu(g))=g^*g_*(1)=g^*(d^{n+1}x^k)=d^{n+k+1}x^k,$$

by (4.2), (2.4) and the definition of the degree. q.e.d.

PROOF OF THEOREM C. Suppose that the map f is homotopic to an embedding $g: L^n(q)\to L^{n+k}(q)$. Then $\deg(g)=\deg(f)=d$. By the first equality of (3.1) we have

$$p_k(\nu(g))=\sum_{i+j=k}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}x^{2k}.$$

On the other hand, (4.1) shows that

$$p_k(\nu(g))=\chi(\nu(g))^2=d^{2n+2k+2}x^{2k}.$$

Thus we get the desired formula. q.e.d.

§5. Embeddability of $L^n(5)$ in R^m

First, we recall the properties of W.S. Massey's subalgebra. Suppose that a compact, connected, smooth manifold M of dimension m is embedded differentiably in $(m+l)$-sphere S^{m+l}. Let E be the total space of the normal S^{l-1}-bundle associated with the embedding, and let $p: E\to M$ be the projection. Then there is a subalgebra $A^*=\Sigma A^i$ of $H^*(E; Z_q)$ satisfying the following properties (cf. [4] and [5]):

(5.1) A^* is closed under cohomology operations.

(5.2) The following direct sum decomposition holds:

$$H^i(E; Z_q)=p^*H^i(M; Z_q)+A^i, \qquad 0<i<m+l-1.$$

(5.3) $A^i=0$ for $i\geqq m+l-1$.

PROOF OF THEOREM D. Suppose that $L^n(5)$ is embedded differentiably in R^{3n+2}. We may assume that $L^n(5)$ is embedded differentiably in S^{3n+2}. Let E be the total space of the normal S^n-bundle associated with the embedding, and let $p: E\to L^n(5)$ be the projection. Then there exists a subalgebra $A^*=\Sigma A^i$ of $H^*(E; Z_5)$ which satisfies the properties (5.1-3) for $M=L^n(5)$ and $l=n+1$.

Since the Euler class of the normal bundle of the embedding vanishes, the Gysin sequence breaks up into pieces of length 3 as follows:

$$0\longrightarrow H^i(L^n(5); Z_5)\xrightarrow{p^*} H^i(E; Z_5)\xrightarrow{\psi} H^{i-n}(L^n(5); Z_5)\longrightarrow 0.$$

Let $a\in H^n(E; Z_5)$ be an element such that $\psi(a)=1\in H^0(L^n(5); Z_5)$. Then we have the following direct sum decomposition (cf. [4, §8]):

$$H^i(E; Z_5)=p^*H^i(L^n(5); Z_5)+a\cdot p^*H^{i-n}(L^n(5); Z_5). \tag{5.4}$$

We may assume that $a\in A^n$. Then, by (5.1), $a^2\in A^{2n}$. By (5.4), there exist unique elements $\alpha\in H^{2n}(L^n(5); Z_5)$ and $\beta\in H^n(L^n(5); Z_5)$ such that

$$a^2=p^*\alpha+a\cdot p^*\beta. \tag{5.5}$$

According to [4, Th. IV], it holds that $4\alpha+\beta^2=\bar{p}_{n/2}$, and hence

$$\beta^2=x^n+\alpha, \tag{5.6}$$

because $\bar{p}_{n/2}=x^n$. It follows from (5.2) and (5.4) that

$$A^i\cong H^{i-n}(L^n(5); Z_5)\cong Z_5 \qquad \text{for } n\leqq i\leqq 3n. \tag{5.7}$$

Note that $\alpha=\pm x^n$, $\pm 2x^n$ or 0. If $\alpha=x^n$ or $2x^n$, then $\beta^2=2x^n$ or $-2x^n$, respectively, by (5.6). But this is impossible. If $\alpha=-x^n$, then $\beta=0$, by (5.6), and so, $a^2=-p^*x^n\neq 0$, by (5.5). This is inconsistent with the fact that $a^2\in A^{2n}$ and the direct sum decomposition (5.2).

Assume that $\alpha=-2x^n$. Then, by (5.6), $\beta=\pm 2x^{n/2}$. Thus, by (5.5),

$$a^2=p^*(-2x^n)+a\cdot p^*(\pm 2x^{n/2}).$$

Let $p^*u+a\cdot p^*x^{n/2}y$ be a non-zero element of A^{2n+1}, where $u\in H^{2n+1}(L^n(5); Z_5)$. Then $a(p^*u+a\cdot p^*x^{n/2}y)=a\cdot p^*(u\pm 2x^ny)\in A^{3n+1}$. Since $A^n\cdot A^{2n+1}\subset A^{3n+1}=0$ by (5.3), we have $u=\mp 2x^ny$. Therefore, we see, by (5.7), that A^{2n+1} is generated by $p^*(\mp 2x^ny)+a\cdot p^*x^{n/2}y$.

Put $s=n/8=2\cdot 5^t$, and let $\mathscr{P}^s: A^n\to A^{2n}$ be the reduced power operation mod 5. According to [6, Th. 1], we have

$$\mathscr{P}^s a = p^* \alpha_s + a \cdot p^* Q_s,$$

where α_s is an element of $H^{2n}(L^n(5); Z_5)$, and where $Q_s \in H^n(L^n(5); Z_5)$ is the characteristic class defined by $Q_s = \phi^{-1}\mathscr{P}^s U$ (ϕ is the Thom isomorphism of the normal bundle and U is its Thom class). By an easy calculation, we can see that $Q_s = -2x^{n/2}$. Finally, consider the operation $\mathscr{P}^s$: $A^{2n+1} \to A^{3n+1}$. Then, $\mathscr{P}^s(p^*(\mp 2x^n y) + a \cdot p^* x^{n/2} y) = a \cdot p^* x^n y \neq 0$. While, $A^{3n+1} = 0$, by (5.3). This is a contradiction.

In the case $\alpha = 0$, we get a contradiction in the same way. q.e.d.

References

[1] M. Adachi, *A remark on submersions and immersions with codimension one or two,* J. Math. Kyoto Univ. **9** (1969), 393–404.

[2] S. Feder, *Immersions and embeddings in complex projective spaces,* Topology **4** (1965), 143–158.

[3] M. W. Hirsch, *Immersions of manifolds,* Trans. Amer. Math. Soc. **93** (1959), 242–276.

[4] W. S. Massey, *On the cohomology ring of a sphere bundle,* J. Math. Mech. **7** (1958), 265–289.

[5] W. S. Massey, *On the imbeddability of the real projective spaces in Euclidean space,* Pacific J. Math. **9** (1959), 783–789.

[6] R. Nakagawa and T. Kobayashi, *Non-embeddability of lens spaces* mod 3, J. Math. Kyoto Univ. **5** (1966), 313–324.

[7] E. Rees, *Embedding odd torsion manifolds,* Bull. London Math. Soc. **3** (1971), 356–362.

[8] D. Sjerve, *Geometric dimension of vector bundles over lens spaces,* Trans. Amer. Math. Soc. **134** (1968), 545–557.

[9] D. Sjerve, *Vector bundles over orbit manifolds,* Trans. Amer. Math. Soc. **138** (1969), 97–106.

[10] H. Suzuki, *Higher order non-singular immersions in projective spaces,* Quart. J. Math. Oxford (2), **20** (1969), 33–44.

[11] H. Suzuki, *On higher order non-singular immersions of RP^n in CP^m,* J. Fac. Sci. Hokkaido Univ. **22** (1972), 161–170.

[12] R. H. Szczarba, *On tangent bundles of fibre spaces and quotient spaces,* Amer. J. Math. **86** (1964), 685–697.

[13] E. Thomas, *On the existence of immersions and submersions,* Trans. Amer. Math. Soc. **132** (1968), 387–394.

[14] E. Thomas, *Submersions and immersions with codimension one or two,* Proc. Amer. Math. Soc. **19** (1968), 859–863.

Department of Mathematics,
Faculty of Science,
Hiroshima University