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Remarks on the m-Accretiveness of Nonlinear Operators

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Introduction

Let X be a real Banach space and let A be a multivalued operator from X into X, that is, to each $x\in X$ a subset Ax of X be assigned. We define $D(A)=\{x\in X;\ Ax\neq\phi\}$, $R(A)=\bigcup_{x\in X} Ax$ and $G(A)=\{[x, x']\in X\times X;\ x'\in Ax\}$. We denote by F the *duality* mapping of X into the dual space X^*, i.e., it is defined by $Fx=\{x^*\in X^*;\ <x, x^*>=\|x\|^2=\|x^*\|^2\}$ for $x\in X$, where $<\ ,\ >$ denotes the natural pairing between X and X^* and $\|\cdot\|$ denotes the norms in X and X^*. An operator A is called *accretive* in X, if for any $[x_i, x_i']\in G(A)$, $i=1, 2$, there is an element $f\in F(x_1-x_2)$ such that $<x_1'-x_2', f>\geq 0$, or equivalently,

$$\lim_{h\downarrow 0}\frac{1}{h}[\|x_1-x_2+h(x_1'-x_2')\|-\|x_1-x_2\|]\geq 0 \tag{1}$$

(see R. H. Martin, Jr. [7]). An accretive operator A is called *m-accretive*, if $R(A+I)=X$.

It was shown in [6; THEOREM 1] that, under the uniform convexity of X^*, an accretive operator A is m-accretive if and only if it is demiclosed (i.e., for any sequence $\{[x_n, x_n']\}\subset G(A)$, $x_n\to x$ strongly and $x_n'\to x'$ weakly in X imply that $[x, x']\in G(A)$) and for each $z\in X$ and each $x\in D(A)$, the initial value problem: $u'(t)+Au(t)+z\ni 0$, $u(0)=x$ has a strong solution on $[0, \infty)$. In this note we do not require the uniform convexity of X^* and shall show an analogue of the above result in more general spaces, namely, in reflexive Banach spaces, by making use of the inequality (1) for accretiveness.

1. Main results

Let A be an operator from X into X and $\Omega=[0, r)$ or $[0, r]$ where $0<r\leq\infty$. Then an X-valued function u on Ω is called a *strong solution* of the initial value problem

$$u'(t)+Au(t)\ni 0, \qquad u(0)=a,$$

if $u(t)$ is strongly absolutely continuous on any bounded closed interval contained in Ω, $u(0)=a$ and the strong derivative $u'(t)$ exists, $u(t)\in D(A)$ and $u'(t)+Au(t)\ni 0$ for a.e. $t\in\Omega$. We denote by $\hat{D}(A)$ the set

$\{x \in X$; there is a sequence $\{[x_n, x'_n]\} \subset G(A)$ such that

$x_n \xrightarrow{s} x$ in X as $n \to \infty$ and $\{\|x'_n\|\}$ is bounded$\}$,

where " $\xrightarrow{s}$ " means convergence in the strong topology. We say that A is *almost demiclosed*, if $\hat{D}(A) = D(A)$. It is obvious that if A is demiclosed, then it is almost demiclosed, provided that X is reflexive.

THEOREM 1. *Suppose that X is reflexive. Let A be an accretive operator from X into X. Then the following statements are equivalent to each other*:
(a_1) *A is m-accretive.*
(a_2) *A is almost demiclosed, and for each $x \in D(A)$ and each $z \in X$ the initial value problem*

$$u'(t) + Au(t) + z \ni 0, \qquad u(0) = x$$

has a strong solution on $[0, \infty)$.
(a_3) *For each $x \in \hat{D}(A)$ and each $z \in X$, the initial value problem*

$$u'(t) + Au(t) + z \ni 0, \qquad u(0) = x \tag{2}$$

has a strong solution on $[0, \infty)$.

Let X_0 be a subset of X and let $T = \{T(t); t \geq 0\}$ be a family of singlevalued operators from X_0 into X_0. We say that T is a *contraction semigroup* on X_0, if
(i) $T(t+t')x = T(t)T(t')x$ for $t, t' \geq 0$ and $x \in X_0$,
(ii) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for $t \geq 0$ and $x, y \in X_0$,
(iii) $T(0)x = x$ for $x \in X_0$,
(iv) the function $t \to T(t)x$ is strongly continuous on $[0, \infty)$ for each $x \in X_0$.
We define the *strong* (resp. *weak*) *infinitesimal generator* G_s (resp. G_w) of T by

$$G_s x = \text{s-}\lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \left(\text{resp. } G_w x = \text{w-}\lim_{t \downarrow 0} \frac{T(t)x - x}{t}\right)$$

whenever the limit exists. Here, the symbol "s-lim" (resp. "w-lim") means convergence in the strong (resp. weak) topology.

THEOREM 2. *Suppose that X is reflexive. Let A be an accretive operator from X into X. Then the following statements are equivalent to each other*:
(b_1) *A is m-accretive.*
(b_2) *For each $z \in X$, there is a contraction semigroup $T^{(z)} = \{T^{(z)}(t); t \geq 0\}$ on $\overline{D(A)}$ such that $G(-G_s^{(z)}) \subset G(A+z)$ and*

$$D(A) \subset \left\{x \in \overline{D(A)};\ \liminf_{t \downarrow 0} \frac{\|T^{(z)}(t)x - x\|}{t} < \infty\right\}, \tag{3}$$

where $G_s^{(z)}$ is the strong infinitesimal generator of $T^{(z)}$.

(b_3) *For each* $z \in X$, *there is a contraction semigroup* $T^{(z)} = \{T^{(z)}(t);\ t \geq 0\}$ *on* $\overline{D(A)}$ *with the property* (3) *such that* $G(-G_w^{(z)}) \subset G(A+z)$, *where* $G_w^{(z)}$ *is the weak infinitesimal generator of* $T^{(z)}$.

The following two corollaries are obtained from Theorem 2 by the same method as in the proofs of Corollaries 1 and 2 in [6].

COROLLARY 1. (*F. E. Browder* [2]) *Suppose that* X *is reflexive. Let* A *be a singlevalued accretive operator from* X *into* X. *Then* A *is m-accretive if and only if for each* $z \in X$ *there is a contraction semigroup on* $\overline{D(A)}$ *whose weak infinitesimal generator is* $-(A+z)$.

For an operator B from X into X we define B^0 by $B^0 x = \{x' \in Bx;\ \|x'\| = |||Bx|||\}$, where $|||E||| = \inf_{y \in E} \|y\|$ for a subset E of X.

COROLLARY 2. *Suppose that* X *is reflexive and* X *and* X^* *are strictly convex. Let* A *be an accretive operator from* X *into* X. *Then* A *is m-accretive if and only if for each* $z \in X$ *the operator* $(A+z)^0$ *is singlevalued,* $D((A+z)^0) = D(A)$ *and there is a contraction semigroup on* $\overline{D(A)}$ *whose weak infinitesimal generator is* $-(A+z)^0$.

2. **Proof of Theorem** 1.

Hereafter we assume that X is reflexive. For the proof of the assertion $(a_1) \to (a_2)$ of Theorem 1 we first show the following lemma.

LEMMA 1. *If* A *is m-accretive, then it is almost demiclosed.*

PROOF. First we recall the generation theorem by M. G. Crandall and T. M. Liggett [3; THEOREM I]. The theorem says that if A is m-accretive, then there is a contraction semigroup $T = \{T(t);\ t \geq 0\}$ on $\overline{D(A)}$ such that

$$T(t)x = \operatorname*{s-lim}_{n \to \infty} \left(I + \frac{t}{n} A\right)^{-n} x \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in \overline{D(A)}$$

and this contraction semigroup has the following property:

$$\|T(t)x - T(t')x\| \leq |||Ax|||\ |t - t'| \quad \text{for} \quad t, t' \geq 0 \quad \text{and} \quad x \in D(A).$$

Now, assume that $[x_n, x'_n] \in G(A)$, $x_n \xrightarrow{s} x$ as $n \to \infty$ and $\|x'_n\| \leq M$ for all n. Then, from the above property of T it follows that $\|T(t)x_n - x_n\| \leq \|x'_n\| t$ for $t \geq 0$. Hence, letting $n \to \infty$, we have $\|T(t)x - x\| \leq Mt$ for $t \geq 0$. By Corollary 1 in I. Miyadera [8] we have $x \in D(A)$. Thus A is almost demiclosed. q.e.d.

The assertion $(a_1) \to (a_2)$ of Theorem 1 easily follows from the above lemma and Theorems I and II in [3], and the assertion $(a_2) \to (a_3)$ of Theorem 1 is trivial.

Next we shall prove the assertion $(a_3)\to(a_1)$ of Theorem 1 by means of a sequence of lemmas which are valid under the assumption (a_3). Thus, hereafter, assume (a_3).

LEMMA 2. (α) *A is closed (i.e., $[x_n, x'_n]\in G(A)$, $x_n \xrightarrow{s} x$ and $x'_n \xrightarrow{s} x'$ in X imply that $[x, x']\in G(A)$).*
(β) *Let $\tilde{A}$ be any accretive operator such that $G(\tilde{A})\supset G(A)$. Then $\hat{D}(A)\cap D(\tilde{A}) = D(A)$ and $\tilde{A}x = Ax$ for every $x\in D(A)$.*

PROOF. Assume that $[x_n, x'_n]\in G(A)$, $x_n \xrightarrow{s} x$ and $x'_n \xrightarrow{s} x'$ in X as $n\to\infty$. Then $x\in\hat{D}(A)$. By (a_3), the initial value problem: $u'(t)+Au(t)-x'\ni 0$, $u(0)=x$ has a strong solution $u(t)$ on $[0, \infty)$. Let B be the operator given by $G(B)=G(A)\cup\{[x, x']\}$. Then $u(t)$ is also a strong solution of the initial value problem: $u'(t)+Bu(t)-x'\ni 0$, $u(0)=x$. Therefore, since B is also accretive, the uniqueness of a strong solution (cf., T. Kato [5; LEMMA 6.2] or H. Brezis and A. Pazy [1; LEMMA 2.2]) implies that $u(t)=x$ for all $t\geq 0$, and hence $[x, x']\in G(A)$. Thus (α) is proved, and (β) is also proved just as (α).

q.e.d.

Now we consider the initial value problem

$$u'(t)+Au(t)+u(t)\ni 0, \qquad u(0)=a \tag{4}$$

and shall show that (4) has a strong solution on $[0, \infty)$ for each $a\in D(A)$.

Let $a\in D(A)$. For a positive integer n we define an X-valued function u_n as follows. Let $v(t)$ be a strong solution of (2) with $x=z=a$ and choose a positive number δ_n^1 such that $\frac{1}{n}-\frac{1}{n^2}\leq\delta_n^1\leq\frac{1}{n}$, $-v'(\delta_n^1)\in Av(\delta_n^1)+a$ and $\|v'(\delta_n^1)\|=|||Av(\delta_n^1)+a|||\leq|||Aa+a|||$. In fact, in view of Lemma 2.2 in [1], such δ_n^1 exists. Let us define $u_n(t)=v(t)$ if $t\in[0, t_n^1]$, $t_n^1=\delta_n^1$. Next we assume that u_n is already defined on $[0, t_n^k]$, $1\leq k<n$. Let $w(t)$ be a strong solution of (2) with $x=z=u_n(t_n^k)$, and choose a positive number δ_n^{k+1} such that $\frac{1}{n}-\frac{1}{n^2}\leq\delta_n^{k+1}\leq\frac{1}{n}$, $-w'(\delta_n^{k+1})\in Aw(\delta_n^{k+1})+u_n(t_n^k)$ and $\|w'(\delta_n^{k+1})\|=|||Aw(\delta_n^{k+1})+u_n(t_n^k)|||\leq|||Au_n(t_k^n)+u_n(t_n^k)|||$. Let us define $u_n(t)=w(t-t_n^k)$ if $t\in[t_n^k, t_n^{k+1}]$, $t_n^{k+1}=\delta_n^{k+1}+t_n^k$. Thus by induction u_n is defined on $[0, t_n^n]$. Clearly $1-\frac{1}{n}\leq t_n^n\leq 1$. We see that u_n is strongly absolutely continuous on $[0, t_n^n]$ and satisfies

$$u'_n(t)+Au_n(t)+u_n(t_n^k)\ni 0 \qquad \textit{a.e. on} \quad [t_n^k, t_n^{k+1}]$$

for $k=0, 1, \ldots, n-1$.

LEMMA 3. *Set $K=|||Aa+a|||$. Then for each n*

(5) $$\|u_n'(t)\| \leq eK \qquad a.e. \ on \quad [0, t_n^n].$$

This lemma is obtained by a simple modification of the proof of Lemma 6 in [6].

LEMMA 4. *The sequence* $\{u_n\}_{n=1}^{\infty}$ *is strongly uniformly convergent on* $[0, 1)$, *and the limit* $u(t)$ *satisfies*

(6) $$\|u(t)-u(t')\| \leq eK|t-t'| \qquad for\ t,\ t' \geq 0,$$

(7) $$u(0)=a \qquad and \quad u(t) \in \hat{D}(A) \qquad for\ all\ t \geq 0.$$

PROOF. Set $P_{n,m}(t)=\|u_n(t)-u_m(t)\|$ on $\left[0,\ 1-\frac{1}{n}-\frac{1}{m}\right]$. If $s \in (t_n^i,\ t_n^{i+1}]$, $s \in (t_m^j,\ t_m^{j+1}]$, $u_n'(s)+U_n(s)+u_n(t_n^i)=0$ and $u_m'(s)+U_m(s)+u_m(t_m^j)=0$, where $U_n(s) \in Au_n(s)$ and $U_m(s) \in Au_m(s)$, then

$$\begin{aligned} P_{n,m}'(s) &= \lim_{h \downarrow 0} -\frac{1}{h}[\|u_n(s)-u_m(s)+h(U_n(s)+u_n(t_n^i)-U_m(s)-u_m(t_m^j))\| \\ &\quad -\|u_n(s)-u_m(s)\|] \\ &\leq \lim_{h \downarrow 0} -\frac{1}{h}[\|u_n(s)-u_m(s)+h(U_n(s)+u_n(s)-U_m(s)-u_m(s))\| \\ &\quad -\|u_n(s)-u_m(s)\|]+\|u_n(s)-u_n(t_n^i)\|+\|u_m(s)-u_m(t_m^j)\|. \end{aligned}$$

Now, $U_n(s)+u_n(s) \in (A+I)u_n(s)$ and $U_m(s)+u_m(s) \in (A+I)u_m(s)$. Since $A+I$ is also accretive, it follows from (1) in the introduction that

$$\begin{aligned} &\lim_{h \downarrow 0} \frac{1}{h}[\|u_n(s)-u_m(s)+h(U_n(s)+u_n(s)-U_m(s)-u_m(s))\| \\ &\qquad -\|u_n(s)-u_m(s)\|] \geq 0. \end{aligned}$$

Hence, by (5),

$$P_{n,m}'(s) \leq \|u_n(s)-u_n(t_n^i)\|+\|u_m(s)-u_m(t_m^j)\| \leq eK\left(\frac{1}{n}+\frac{1}{m}\right).$$

Thus,

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| \leq eK\left(\frac{1}{n}+\frac{1}{m}\right) \quad for \quad a.e. \quad t \in \left[0,\ 1-\frac{1}{n}-\frac{1}{m}\right].$$

Hence $\|u_n(t)-u_m(t)\| \leq eK\left(\frac{1}{n}+\frac{1}{m}\right)$ for all $t \in \left[0,\ 1-\frac{1}{n}-\frac{1}{m}\right]$ and hence $\|u_n(t)-u_m(t)\| \to 0$ uniformly on $[0, 1)$ as $n,\ m \to \infty$. Let $u(t)$ be the limit. Since

$\|u_n(t)-u_n(t')\|\leq eK|t-t'|$ for any $t,\ t'\in\left[0,\ 1-\frac{1}{n}\right]$ by (5), by letting $n\to\infty$ we have (6). Clearly $u(0)=a$. The fact that $u(t)\in\hat{D}(A)$ for all $t\geq 0$ follows from (5). Thus we have (7). q.e.d.

We define $<x,\ y>_s=\sup_{y^*\in Fy}<x,\ y^*>$ for $x,\ y\in X$. Then $<,\ >_s\colon X\times X\to(-\infty,\ \infty)$ is upper semicontinuous in the strong topology of $X\times X$(see [3; LEMMA 2.16]). Then the limit function u of $\{u_n\}$ has the following property:

LEMMA 5. *For any* $[x,\ x']\in G(A)$ *and any* $t,\ s\in[0,\ 1)$ *with* $t\geq s$,

$$\|u(t)-x\|^2-\|u(s)-x\|^2\leq 2\int_s^t<-x'-u(\tau),\ u(\tau)-x>_s d\tau. \tag{8}$$

PROOF. By the definition of u_n, $u_n'(t)+U_n(t)+u_n(t_n^k)=0$ a.e. on $[t_n^k,\ t_n^{k+1}]$, $k=0,1,\dots,n-1$, where $U_n(t)\in Au_n(t)$ a.e. on $[0,\ t_n^n)$. For each t, by the accretiveness of A, there is $S_n(t)\in F(u_n(t)-x)$ such that $<U_n(t)-x',\ S_n(t)>\geq 0$. Hence, by using Lemma 1.3 of T. Kato [4] and Lemma 3, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_n(t)-x\|^2 &= <u_n'(t),\ S_n(t)> \\
&= <-U_n(t)-u_n(t_n^k),\ S_n(t)> \\
&\leq <-x'-u_n(t),\ S_n(t)>+<u_n(t)-u_n(t_n^k),\ S_n(t)> \\
&\leq <-x'-u_n(t),\ u_n(t)-x>_s+\frac{eK}{n}\|u_n(t)-x\| \\
&\leq <-x'-u_n(t),\ u_n(t)-x>_s+\frac{eK}{n}(\|x\|+\|a\|+eK).
\end{aligned}$$

Integrating the first and the last members of the above inequalities on $[s,\ t]$, we have

$$\begin{aligned}
&\|u_n(t)-x\|^2-\|u_n(s)-x\|^2 \\
&\leq 2\int_s^t<-x'-u_n(\tau),\ u_n(\tau)-x>_s d\tau+\frac{2}{n}eK|t-s|(\|x\|+\|a\|+eK).
\end{aligned} \tag{9}$$

On the other hand, since $u_n\xrightarrow{s}u$ and $\{u_n\}$ is uniformly bounded on $[0,\ 1)$, it follows from Fatou's lemma and the upper semicontinuity of $<,\ >_s\colon X\times X\to R$ that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_s^t<-x'-u_n(\tau),\ u_n(\tau)-x>_s d\tau \\
&\leq\int_s^t\limsup_{n\to\infty}<-x'-u_n(\tau),\ u_n(\tau)-x>_s d\tau \\
&\leq\int_s^t<-x'-u(\tau),\ u(\tau)-x>_s d\tau.
\end{aligned}$$

Therefore, letting $n\to\infty$ in (9), we obtain (8). q.e.d.

LEMMA 6. *$u(t)$ is a strong solution on $[0, 1)$ of $u'(t)+Au(t)+u(t)\ni 0$, $u(0)=a$.*

PROOF. We shall prove that

$$<-u'(t)-u(t)-x', u(t)-x>_s\geq 0 \qquad \textit{for a.e.} \quad t\in[0, 1) \tag{10}$$

for any $[x, x']\in G(A)$. In fact, let $[x, x']\in G(A)$ be an arbitrary element. Then we first observe that for $s, t\geq 0$ with $s>t$

$$\begin{aligned}
&<u(s)-u(t), u(t)-x>_s \\
&\leq <u(s)-x, u(t)-x>_s - \|u(t)-x\|^2 \\
&\leq \|u(s)-x\| \; \|u(t)-x\| - \|u(t)-x\|^2 \\
&\leq \frac{1}{2}\|u(s)-x\|^2 - \frac{1}{2}\|u(t)-x\|^2.
\end{aligned}$$

Hence from (8) we obtain

$$<\frac{u(s)-u(t)}{s-t}, u(t)-x>_s \leq \frac{1}{s-t}\int_t^s <-x'-u(\tau), u(\tau)-x>_s d\tau.$$

Here, if u is strongly differentiable at t, then we infer from the above inequality and the upper semicontinuity of $<, >_s$ that

$$<u'(t), u(t)-x>_s \leq <-x'-u(t), u(t)-x>_s.$$

Thus (10) holds. Next, fix any t at which u is strongly differentiable and define an operator $\tilde{A}$ by $G(\tilde{A})=G(A)\cup\{[u(t), -u'(t)-u(t)]\}$. Then (10) implies that $\tilde{A}$ is accretive. Applying (β) of Lemma 2 for this $\tilde{A}$, we have $u(t)\in D(A)$ and $\tilde{A}u(t)=Au(t)$, since $u(t)\in\hat{D}(A)$ by (7). Thus

$$-u'(t)-u(t)\in Au(t) \qquad \textit{a.e.} \quad \textit{on} \quad [0, 1).$$

PROOF of the assertion $(a_3)\to(a_1)$ of Theorem 1: We have seen that for each $a\in D(A)$ the initial value problem (4) has a local strong solution $u(t)$. By using a standard argument we deduce that $u(t)$ can be extended to a strong solution of (4) on $[0, \infty)$. Therefore, by Lemma 9 in [6] and (α) of Lemma 2, $0\in R(A+I)$. For an arbitrary point $z\in X$, replacing A by $A-z$ in the above argument, we conclude that $z\in R(A+I)$. Thus $R(A+I)=X$. q.e.d.

REMARK. The assertion of Theorem 1 is false without the reflexivity of the space X; in fact there are a non-reflexive Banach space X and an m-accretive operator A in X such that the Cauchy problem: $u'(t)+Au(t)\ni 0$, $u(0)=a$ does not have a strong solution, even if $a \in D(A)$. For an example, see G. F. Webb [9].

3. Proof of Theorem 2.

We can prove Theorem 2 just as Theorem 2 in [6], using Theorem 1.

References

[1] H. Brezis and A. Pazy, Accretive sets and differential equations in Banach spaces, Israel J. Math., **8** (1970), 367–383.

[2] F. E. Browder, Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces, Bull. Amer. Math. Soc., **73** (1967), 867–874.

[3] M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, Amer. J. Math., **93** (1971), 265–298.

[4] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, **19** (1967), 508–520.

[5] T. Kato, Accretive operators and nonlinear evolution equations in Banach spaces, Proc. Symp. Nonlinear Functional Analysis, A. M. S., **Part 1** (1970), 138–161.

[6] N. Kenmochi, On the m-accretiveness of nonlinear operators in Banach spaces, Hiroshima Math. J., **2** (1972), 299–311.

[7] R. H. Martin, Jr., A global existence theorem for autonomous differential equations in a Banach space, Proc. Amer. Math. Soc., **26** (1970), 307–314.

[8] I. Miyadera, Some remarks on semigroups of nonlinear operators, Tôhoku Math. J., **23** (1971), 245–258.

[9] G. F. Webb, Continuous nonlinear perturbations of linear accretive operators in Banach spaces, J. Functional Analysis, **10** (1972), 191–203.

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