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A Finite-Difference Method on a Riemann Surface

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Introduction.

In the present paper we aim to discuss a method of finite-differences from the point of view of applications to the function theory. Since we speak of harmonic and analytic differentials and functions on a Riemann surface, we need to construct a theory of finite-differences on a *Polyhedron*.

Let u be a function defined at the points of the complex plane whose coordinates are integers. As a definition of a discrete harmonic function u on a plane, the so-called five-point formula

$$u(z+1)+u(z+i)+u(z-1)+u(z-i)-4u(z)=0$$

is generally used. How we define the conjugate discrete harmonic function and a discrete analytic function so that their definition match with the above definition of a discrete harmonic function, is an important problem. It is desirable, based on the definitions, to construct a theory of rich contents of discrete harmonic and analytic functions. As the works with this intention, we can mention Blanc [3], Lelong-Ferrand [10], [11], Isaacs [7], [8], Duffin [5], Hundhausen [6], etc.

Blanc [3] introduced the concepts of a *rèseau Riemannien* and a *rèseau conjuguè* on a plane, and introduced very general definitions of a discrete harmonic function and its conjugate function. He developed an interesting analogy with the type problem of a Riemann surface, and also he [2] developed an analogy with Nevanlinna's first and second fundamental theorems. However, it seems that he did not intend to make an effective use of a conjugate harmonic function on a rèseau conjuguè. Our definitions of a harmonic function and its conjugate function are similar to Blanc's.

Let f be a complex-valued function defined at the points of the complex plane whose coordinates are integers. Then Lelong-Ferrand [10], [11] introduced the following definition of a discrete analytic function f:

$$\frac{f(z+1+i)-f(z)}{1+i}=\frac{f(z+i)-f(z+1)}{i-1}. \tag{1}$$

If we set $f=u+iu^*$ where u and u^* are real, then it is seen that the discrete analyticity of f implies that u and u^* are discrete harmonic and satisfy a pair of difference equations which are analogous to the Cauchy-Riemann equations. She developed several interesting analogies with ordinary analytic functions. With the last definition of analyticity, Duffin [5] brought new developments which included the Cauchy integral formula, Liouville's theorem, Harnack's inequality, polynomial expansions and Hilbert transforms.

Isaacs [7], [8] developed a theory of discrete analytic functions based on the following definitions of analyticity:

$$f(z+1)-f(z)=\frac{f(z+i)-f(z)}{i}$$

or

$$f(z+1)-f(z-1)=\frac{f(z+i)-f(z-i)}{i}.$$

Hundhausen [6] introduced a more general criterion of discrete analyticity based on a discrete formulation of Morera's theorem and showed a new approach.

In Chapter I of the present paper, we aim to construct a theory of discrete harmonic and analytic differences on a polyhedron, where our definition of a polyhedron differs from the ordinary one based on a triangulation and admits also a polygon and a lune as 2-simplices (cf. § 1. **1**). A function (a zero order difference), a first order difference and a second order difference on a polyhedron are defined as functions which take a complex value at each oriented 0-simplex, 1-simplex and 2-simplex respectively (cf. § 2. **1**). In order to set the definitions of a conjugate harmonic difference (function) and an analytic difference (function) which answer our purpose, we introduce concepts of a conjugate polyhedron and a complex polyhedron (cf. § 1. **3**). The method of differentials on a Riemann surface (cf. Chapter V of Ahlfors & Sario [1]) has been very valuable in the process of construction of the present theory. Many methods and results analogous to the theory of harmonic and analytic differentials (functions) of the continuous case are developed, and a model of the function theory is constructed on a complex polyhedron.

In Chapter II, we shall concern ourselves with the problem of approximating harmonic and analytic differentials on a Riemann surface by harmonic and analytic differences respectively. We define a Riemann surface based on a normal quadrangulation (cf. § 1. **8**). More generally, we define a normal quadrangulation of a subregion of a generic Riemann surface W by trajectories and orthogonal trajectories of a quadratic differential Ψ, and further we define an exhaustion of W by a sequence of normal subdivisions by Ψ (cf. § 1. **9** $\sim$ § 1. **11**). Then we shall discuss the norm convergence of smooth extensions of harmonic and analytic differences on the sequence of normal subdivisions to harmonic and analytic differentials on W respectively (cf. Theorems 7. 1, 7. 2, 7. 3, 8. 1 and Corollary 8. 1, and cf. § 7. **1** for the definition of smooth extension).

Courant-Friedrichs-Lewy [4] showed that a solution of the Dirichlet problem on a plane region can be uniformly approximated by a corresponding solution of the discrete case. Their method is essentially to show the equi-continuity of a sequence of discrete harmonic functions. Lelong-Ferrand [11] discussed the problem of approximating uniformly an analytic function on a plane by discrete analytic functions in the sense of the definition (1). For this problem she made use of the Cauchy integral theorem for a discrete analytic function. In our present method, the harmonicity of the limit differentials of smooth extensions of discrete harmonic differences and its conjugate differences is simultaneously shown and thus the analyticity of the limit differentials of smooth extensions of discrete analytic differences is also shown. Our method is based on the facts that the smooth extensions of a discrete harmonic difference and its conjugate difference are *closed* differentials in the sense of Ahlfors & Sario (cf. Chapter V of [1]), their limit differentials in the sense of the Dirichlet norm are a pair of closed and conjugate closed ones, and thus a pair of harmonic and conjugate harmonic ones. The method of orthogonal projection of differences and differentials is also effectively used (cf. Theorem 7. 3 and Corollary 8. 1).

Finally, as an application of our results to numerical calculation, we shall discuss in § 9 the problem of determining Riemann's period matrix of a closed Riemann surface. With respect to the problems of this type, Opfer [14], [15] dealt with the problem of determining the modulus of a doubly connected domain by means of finite-difference method, and Mizumoto [12], [13] dealt with the corresponding problem for a general multiply connected domain.

Chapter I Theory of differences on a polyhedron.

§ 1. Foundation of topology.

1. Polyangulation. Let E^2 be the euclidean plane. By a euclidean 0-simplex we mean a point on E^2. By a euclidean 1-simplex we mean a closed line segment or a closed circular arc. By a euclidean 2-simplex we mean a closed polygon surrounded by a finite number ($\geqq 2$) of segments and circular arcs. A lune (biangle) and a triangle are also admitted as a euclidean 2-simplex.

Let F be a 2-dimensional orientable manifold. By 0-simplex q, 1-simplex a and 2-simplex M on F we mean a pair of euclidean 0-simplex q^e, 1-simplex a^e and 2-simplex M^e respectively, and one-to-one bicontinuous mappings ϕ of q^e, a^e and M^e respectively into F. We shall write $q=[q^e, \phi]$, $a=[a^e, \phi]$ and $M=[M^e, \phi]$. The images of q^e, a^e and M^e under ϕ are called the *carriers* of q, a and M respectively, and are denoted by $|q|$, $|a|$ and $|M|$ respectively; that is, $\phi(q^e)=|q|$, $\phi(a^e)=|a|$ and $\phi(M^e)=|M|$. M is called a *polygon* on F, and the images of the edges and vertices of M^e are called *edges* and *vertices* of M. Each edge of M is a 1-simplex and each vertex of M is a 0-simplex. We say that a point p

belongs to q, a and M when $p \in |q|$, $p \in |a|$ and $p \in |M|$ respectively.

Let us suppose that a collection K of polygons (2-simplices) is defined on F such that each point p on F belongs to at least one polygon in K and such that the following conditions (i), (ii), (iii) are satisfied:

(i) if p belongs to a polygon M of K but is not on an edge of M, then M is the only polygon containing p and $|M|$ is a neighborhood of p;

(ii) if p belongs to an edge a of a polygon M_1 in K but is not a vertex of M_1, then there is exactly one other polygon M_2 in K such that $|a| \subset |M_1| \cap |M_2|$, M_1 and M_2 are the only polygons containing p, and $|M_1| \cup |M_2|$ is a neighborhood of p;

(iii) if p is a vertex of M_1, there is a finite number of polygons $M_1, \ldots, M_\kappa$ ($\kappa \geqq 2$), each having p as a vertex, such that each successive pair of polygons M_j, M_{j+1} ($j=1, \ldots, \kappa$; $M_{\kappa+1}=M_1$) have at least one edge in common, $M_1, \ldots, M_\kappa$ are the only polygons containing p, and $|M_1| \cup \ldots \cup |M_\kappa|$ forms a neighborhood of p, where it is permitted that some pair of polygons have two or more edges in common. Then, K is called a *polyangulation* of F or a *polyhedron*,[1] and F on which a polyangulation is defined, is called a *polyangulated manifold*. If each polygon M of a polyhedron K is a quadrangle, then K is called a *quadrangulation* of F or *quadratic polyhedron*, and F is called a *quadrangulated manifold*.

Let Ω be a compact bordered subregion of F whose boundary consists of edges of K. Then the collection of polygons of K having their carriers in Ω is called a *compact bordered polyhedron*. If F is closed (open resp.), then K is said to be *closed* (*open resp.*).

Let K and L be two polyhedra. If every polygon of L is a polygon of K, then L is called a *subpolyhedron* of K, K is said to *contain* L, and it is denoted by $L \subset K$. Furthermore, if $|L|$ is a regular (canonical resp.) subregion of $|K|$ (see p. 26, p. 61 and p. 80 of [1] for the definition), then L is said to be *regular* (*canonical* resp.).

Let a sequence of polyhedra $\{K_n\}_{n=0}^{\infty}$ be an exhaustion of an open polyhedron K. If each K_n is regular (canonical resp.), then $\{K_n\}_{n=0}^{\infty}$ is said to be *regular* (*canonical* resp.).

2. Homology. On a polyhedron we can define a homology in the same manner as the case of a triangulated polyhedron. An *ordered* n-simplex ($n=0$, 1, 2) is defined in the similar way. An ordered n-simplex ($n=0$, 1, 2) is denoted by the same notations q, a, M, etc. as an n-simplex. The orientation of simplices induces an orientation of the manifold F.

For a fixed dimension n ($n=0$, 1, 2) a free Abelian group $C_n(K)$ is defined by the following conditions (i), (ii):

1) Throughout the present paper, the terminology "polyhedron" will be taken for this sense.

(i) all ordered n-simplices are generators of $C_n(K)$;

(ii) each element of $C_n(K)$ can be represented by the form of finite sum

$$\sum_j x_j q_j, \qquad \sum_j x_j a_j, \qquad \sum_j x_j M_j$$

for $n=0, 1, 2$ respectively, where x_j are integers. Each element of $C_n(K)$ is called an *n-dimensional chain* or an *n-chain.*

The boundary ∂ of an n-simplex for $n=0, 1$ is also similarly defined. For $n=2$, the boundary of a 2-simplex M is defined by

$$\partial M = a_1 + \dots + a_\kappa \qquad (\kappa \geqq 2),$$

where $a_1,\dots, a_\kappa$ are edges of M with the orientation induced by the orientation of M. The boundary of a chain is defined by

$$\partial(\sum_j x_j q_j) = \sum_j x_j \partial q_j, \qquad \partial(\sum_j x_j a_j) = \sum_j x_j \partial a_j,$$

$$\partial(\sum_j x_j M_j) = \sum_j x_j \partial M_j$$

for $n=0, 1, 2$, respectively. A 1-chain whose boundary is zero, is called a *cycle.*

Provided any confusion does not occur, for the present case of polyhedron we shall use the same usual terminologies of homology (see Ch. I, §4 of Ahlfors & Sario [1] for the definition of terminologies).

3. Complex polyhedron. If two open or closed polyangulations K and K^* of a common manifold F satisfy the following conditions (i), (ii), then K^* (K resp.) is called the *conjugate polyhedron* of K (K^* resp.):

(i) To each 0-simplex q (q^* resp.) of K (K^* resp.), there is exactly one 2-simplex M^* (M resp.) of K^* (K resp.) such that $|q| \in |M^*|$ ($|q^*| \in |M|$ resp.). M^*, M, q^* and q are said to be *conjugate* to q, q^*, M and M^* respectively.

(ii) To each 1-simplex a (a^* resp.) of K (K^* resp.), there is exactly one 1-simplex a^* (a resp.) of K^* (K resp.) such that $|a|$ intersects $|a^*|$ at only one point. If the oriented 1-simplex a^* runs through the oriented 1-simplex a from the right to the left, then a^* ($-a$ resp.) is said to be *conjugate* to a (a^* resp.). Throughout the present paper, the notation a^* will always express the conjugate of a 1-simplex a. Thus $a^{**} = (a^*)^* = -a$.

The pair of K and K^* is called a *complex polyangulation* of F or *complex polyhedron*, and is denoted by $\boldsymbol{K} = <K, K^*>$. A manifold F on which a complex polyangulation is defined, is called a *complex polyangulated manifold.* If F is open or closed, then $\boldsymbol{K} = <K, K^*>$ is said to be *open* or *closed* respectively. Let L be a compact bordered subpolyhedron of K and L^* be the sum of polygons of K^* having their carriers in $|L|$. Let us suppose that L^* is connected. Then L^* is the maximal compact bordered subpolyhedron of K^* under the condition

$|L^*| \subset |L|$. The pair $\boldsymbol{L} = \langle L, L^* \rangle$ is called a *compact bordered complex polyhedron.*

If each polygon of a complex polyhedron $\boldsymbol{K}$ is a $4n$-angle (n: a positive integer), then $\boldsymbol{K}$ is said to be *latticed.*

Let $\boldsymbol{K} = \langle K, K^* \rangle$ and $\boldsymbol{L} = \langle L, L^* \rangle$ be two complex polyhedra. If L and L^* are subpolyhedra of K and K^* respectively, then $\boldsymbol{L}$ is called a *complex subpolyhedron* of $\boldsymbol{K}$. Furthermore, if L is regular (canonical resp.), then $\boldsymbol{L}$ is said to be *regular* (*canonical* resp.).

If $\{\boldsymbol{K}_n = \langle K_n, K_n^* \rangle\}_{n=1}^{\infty}$ is a sequence of complex subpolyhedra of an open complex polyhedron $\boldsymbol{K} = \langle K, K^* \rangle$ such that $\{K_n\}_{n=1}^{\infty}$ defines an exhaustion of K, then $\{\boldsymbol{K}_n\}_{n=1}^{\infty}$ is called an *exhaustion of* $\boldsymbol{K}$. An exhaustion $\{\boldsymbol{K}_n\}_{n=1}^{\infty}$ such that each $\boldsymbol{K}_n$ is regular (canonical resp.), is said to be *regular* (*canonical* resp.).

An n-simplex or an n-chain ($n=0, 1, 2$) is said to be *in the interior of* $\boldsymbol{K} = \langle K, K^* \rangle$, if its carrier is in the interior of $|K|$.

4. Homology on a complex polyhedron. By an *n-chain X* ($n=0, 1, 2$) *of* $\boldsymbol{K}$, we mean a formal sum $X = X_1 + X_2$ of an n-chain X_1 of K and an n-chain X_2 of K^*. Here we agree that if $\boldsymbol{K}$ is compact bordered then the conjugate 1-simplex a^* of each $a \in \partial K$ and its boundary ∂a^* are admitted as a generator of $C_1(K^*)$ and that of $C_0(K^*)$ respectively, and thus X_2 is precisely an n-chain of $K^* + \{a^* | a \in \partial K\}$. The *boundary* ∂X is defined by $\partial X = \partial X_1 + \partial X_2$. X is said to be *homologous to zero*, denoted by $X \sim 0$, if and only if $X_1 \sim 0$ and $X_2 \sim 0$.

Let γ_1, γ_2 be 1-chains of a complex polyhedron $\boldsymbol{K} = \langle K, K^* \rangle$. We shall define the *intersection number* $\gamma_1 \times \gamma_2$ of γ_1 and γ_2 as follows:

(i) if γ_1 and γ_2 are both in K or both in K^*, then $\gamma_1 \times \gamma_2 = 0$;

(ii) if γ_1 is contained in K (K^* resp.), and if γ_2 is contained in K^* (K resp.), then $\gamma_1 \times \gamma_2$ is defined by the ordinary method;

(iii) for generic γ_1 and γ_2, $\gamma_1 \times \gamma_2 = \gamma_1' \times \gamma_2'' + \gamma_1'' \times \gamma_2'$ ($\gamma_1 = \gamma_1' + \gamma_1''$, $\gamma_2 = \gamma_2' + \gamma_2''$; $\gamma_1', \gamma_2' \subset K$, $\gamma_1'', \gamma_2'' \subset K^*$).

A system of cycles $\{A_n, B_n, A_n^*, B_n^*\}_{n=1}^{\mu}$ ($\mu \leqq \infty$) on $\boldsymbol{K}$ satisfying the following conditions (i), (ii) is called a *canonical homology basis of* $\boldsymbol{K}$ if $\boldsymbol{K}$ is closed and a *canonical homology basis of* $\boldsymbol{K}$ *modulo the border* $\partial\boldsymbol{K} = \langle \partial K, \partial K^* \rangle$ *or the ideal boundary* if $\boldsymbol{K}$ is compact bordered or open respectively:

(i) $\{A_n, B_n\}$ and $\{A_n^*, B_n^*\}$ are bases of cycles on K and K^* respectively if $\boldsymbol{K}$ is closed, and bases of cycles on K and K^* respectively modulo the border or the ideal boundary if $\boldsymbol{K}$ is compact bordered or open respectively;

(ii) $A_m \times A_n^* = B_m \times B_n^* = 0$, $A_m \times B_n^* = A_m^* \times B_n = \delta_{mn}$ where δ_{mn} is Kronecker's symbol.

Let us suppose that $\boldsymbol{K}$ is compact bordered or open. Let γ be a finite or infinite 1-chain on $\boldsymbol{K}$ (cf. p. 72 of [1]). If $\partial\gamma = 0$, or $\partial\gamma$ is at most a collection of 0-simplices of ∂K and 0-simplices whose carriers are outside of $|K|$, then γ is called a *relative cycle.*

We can find a basis of dividing cycles $\{c_n, c_n^*\}_{n=1}^{\nu}(\nu \leqq \infty)$ on $\boldsymbol{K}$ with a system of relative cycles $\{d_n, d_n^*\}_{n=1}^{\nu}$ as follows (cf. pp. 65–75 of Ahlfors & Sario [1]):

(i) $\{c_n\}$ and $\{c_n^*\}$ are bases of dividing cycles on K and K^* respectively;

(ii) $c_m \times d_n^* = c_m^* \times d_n = \delta_{mn}$.

The basis $\{c_n, c_n^*\}_{n=1}^{\nu}$ is called a *canonical homology basis of dividing cycles of* $\boldsymbol{K}$, and d_n^* (d_n resp.) is called a *conjugate relative cycle* of c_n (c_n^* resp.).

5. Complex cycles. Let a be an oriented 1-simplex of a complex polyhedron $\boldsymbol{K} = <K, K^*>$. $|a^*|$ is divided into two portions by the point $p = |a| \cap |a^*|$. We divide a^* into two 1-simplicies b and b' whose carriers are the portions of $|a^*|$ lying on the right side and the left side of a respectively. b (b' resp.) is called the *conjugate right* (*left* resp.) *half* 1-*simplex of a.*

Let q be a 0-simplex of $\boldsymbol{K}$, and let a' and a'' be two successive 1-simplices such that q is the terminal and initial vertex of a' and a'' respectively. Let $M_1,\dots, M_\kappa$ $(\kappa \geqq 1)$ be the collection of 2-simplices of $\boldsymbol{K}$ having q as their common vertex and lying on the left side of the chain $a' + a''$ such that a' and a'' are the edges of M_1 and M_κ respectively, and such that each successive pair M_j, M_{j+1} of 2-simplices has a common edge a_j with the terminal vertex q, where if $\kappa = 1$ then a' and a'' are the edges of the common 2-simplex $M_1 = M_\kappa$, and $\{a_j\}_{j=1}^{\kappa-1} = \emptyset$.

Let b' and b'' be the conjugate left half 1-simplices of a' and a'' respectively, and let q' and q'' be the initial vertices of b' and b'' lying on a' and a'' respectively. We define a new 1-simplex b with $\partial b = q'' - q'$ whose carrier is the union of the portion of $|a'|$ between $|q'|$ and $|q|$, and the portion of $|a''|$ between $|q|$ and $|q''|$. The 1-simplex b and the 0-simplex q are said to be *dual* each other with respect to the 1-chain $a' + a''$, and also the 0-simplex q' and the 1-simplex a' are said to be *dual* each other. Furthermore we define a new 2-simplex M'^* such that

$$\partial M'^* = b + b'' - \sum_{j=1}^{\kappa-1} a_j^* - b'.$$

The 2-simplex M'^* is called the *conjugate left half* 2-*simplex of q with respect to the* 1-*chain* $a' + a''$.

Let $\gamma = \sum_{j=1}^{\nu} a_j$ be a cycle on $\boldsymbol{K}$ such that a_j and a_{j+1} have a common vertex q_j as their terminal and initial vertices respectively where $a_{\nu+1} = a_1$. A generic cycle can be expressed as a sum of such cycles γ. Let b_j be the dual 1-simplex of q_j w.r.t. $a_j + a_{j+1}$. Then the cycle

$$\gamma^* \equiv \sum_{j=1}^{\nu} b_j$$

is called the *dual cycle* of γ. A pair of γ and γ^* is called a *complex cycle* over γ and is denoted by $\boldsymbol{\gamma} = <\gamma, \gamma^*>$.

Now we restrict $\boldsymbol{K}=<K, K^*>$ to be compact bordered. The (*simple*) *boundary* $\partial\boldsymbol{K}=<\partial K, \partial K^*>$ of $\boldsymbol{K}$ is defined by the sum of the 1-chains ∂K and ∂K^*. Next, by K^{*+} we denote the sum of all 2-simplices of K^* and the conjugate left half 2-simplices of all $q\in\partial K$ with respect to ∂K. Then $|K^{*+}|=|K|$.

By the dual cycle of the boundary ∂K we define the *boundary* ∂K^{*+} of K^{*+}. The sum of ∂K and ∂K^{*+} is called the *complex boundary* of $\boldsymbol{K}$ and denoted by $\boldsymbol{\partial K}=<\partial K, \partial K^{*+}>$. Throughout the present paper we shall preserve these notations.

6. Subdivision of a polyhedron. Let M be an arbitrary 2-simplex of a polyhedron K and let $a_1,\ldots, a_\kappa$ be the edges of M denoted cyclically such that $\partial a_j=q_{j+1}-q_j$ $(q_{\kappa+1}=q_1)$. Then the sequence $a_1,\ldots, a_\kappa$ is called a *cyclic sequence of edges* of M. Let p and p_j $(j=1,\ldots, \kappa)$ be fixed interior points of $|M|$ and $|a_j|$ respectively. We define subdivision of M and new simplices as follows. First, we define new 0-simplices q and q_j^0 $(j=1,\ldots, \kappa)$ whose carriers are p and p_j $(j=1, \ldots, \kappa)$. Each a_j $(j=1,\ldots, \kappa)$ is subdivided into two 1-simplices a_{j1}, a_{j2} so that $\partial a_{j1}=q_j^0-q_j$, $\partial a_{j2}=q_{j+1}-q_j^0$. New 1-simplices a_{j3} $(j=1, \ldots, \kappa)$ are defined as those with $\partial a_{j3}=q-q_j^0$ whose carriers are arcs between p_j and p, and disjoint to each other except for the common point p respectively. And new 2-simplices M_j $(j=1,\ldots, \kappa)$ are defined as ones satisfying $\partial M_j=a_{j2}+a_{j+1,1}+a_{j+1,3}-a_{j3}$ $(a_{\kappa+1,k}=a_{1k})$. We carry out this procedure for all 2-simplices $M\in K$ so that if a 1-simplex a is a common edge of two 2-simplices M_1, M_2, then by subdivision of M_1 and M_2 a common subdivision of a is induced. Then we obtain a new polyhedron K_1 which is called the *subdivision* of K. A subdivision K_1 of an arbitrary polyhedron K is always quadratic, but the conjugate polyhedron K_1^* of K_1 is not quadratic provided either K or K^* is not so.

Let $\boldsymbol{K}=<K, K^*>$ be a complex polyhedron, K_1 be a subdivision of K and K_1^* be the conjugate polyhedron of K_1. Then the complex polyhedron $\boldsymbol{K}_1=<K_1, K_1^*>$ is called the *subdivision of* $\boldsymbol{K}$, where we should note that K_1^* is not a subdivision of K^*.

7. Normal coordinates. Let K be a quadratic polyhedron and $M=[M^e, \phi]$ be an arbitrary 2-simplex of K. We can choose the mapping ϕ so that M^e is a square. Let P_1, P_2, P_3, P_4 be the vertices of M^e successively denoted anti-clockwise and let (x_j, y_j) $(j=1,\ldots, 4)$ be the cartesian coordinates of P_j.

Let $P=(x, y)$ be a point of M^e. The point $P=(x, y)$ is said to have the *normal coordinates* $(\mu_1, \mu_2, \mu_3, \mu_4)$ if $\mu_j\geqq 0$ $(j=1,\ldots, 4)$, $\mu_1+\mu_2=1$, $\mu_3+\mu_4=1$ and

$$\begin{cases} x=\mu_1\mu_3x_1+\mu_2\mu_3x_2+\mu_2\mu_4x_3+\mu_1\mu_4x_4, \\ y=\mu_1\mu_3y_1+\mu_2\mu_3y_2+\mu_2\mu_4y_3+\mu_1\mu_4y_4. \end{cases}$$

Obviously, the points P_1, P_2, P_3, P_4 have the normal coordinates $(1, 0, 1, 0)$, (0, 1, 1, 0), (0, 1, 0, 1), (1, 0, 0, 1) respectively. And a point P on the edges P_1P_2, P_2P_3, P_3P_4, P_4P_1 has the normal coordinates $(\mu_1, \mu_2, 1, 0)$, $(0, 1, \mu_3, \mu_4)$, $(\mu_1, \mu_2, 0, 1)$, $(1, 0, \mu_3, \mu_4)$ respectively. Then the pairs (μ_1, μ_2), (μ_3, μ_4), (μ_1, μ_2), (μ_3, μ_4) respectively are called *normal coordinates of the point P on the edges* P_1P_2, P_2P_3, P_3P_4, P_4P_1 respectively (*induced by the normal coordinates of* M^e). The normal coordinates of a point of M^e are invariant by an affine transformation.

Let $M=[M^e, \phi]$ be a 2-simplex of K with normal coordinates assigned on M^e. Then we can assign the normal coordinates to each point p of M by giving the normal coordinates of $\phi^{-1}(p)\in M^e$ to the point p.

A point of M having the normal coordinates (1/2, 1/2, 1/2, 1/2) is called a *middle point* of M. The set of points having normal coordinates $(1/2, 1/2, \mu_3, \mu_4)$ and the set of points having normal coordinates $(\mu_1, \mu_2, 1/2, 1/2)$ respectively are called *median lines* of M. A point of an edge a of M having the normal coordinates (1/2, 1/2) is called a *middle point* of a.

By the method analogous to the case of normal coordinates of triangulation, we can prove that for the collection $\{M=[M^e, \phi]\}$ of 2-simplices of a quadrangulation K, a set of mappings ϕ can be so chosen that for each 1-simplex a which is a common edge of two 2-simplices M_1, M_2, the normal coordinates of each point on a induced by M_1 and M_2 respectively are the same (for the case of triangulation, e.g. cf. Theorem 5–7 of Springer [16]). A set of normal coordinates chosen in this way is called *normal coordinates of K*. A quadratic polyhedron K to which such normal coordinates are assigned, is said to be *normal*.

Let $\boldsymbol{K}=<K, K^*>$ be a complex polyhedron such that K is a normal quadratic polyhedron. If for the conjugate q^* of each 2-simplex $M\in K$, $|q^*|$ is the middle point of M, and if for each 1-simplex $a\in K$ which is a common edge of two 2-simplices M_1, M_2, $|a^*|$ lies on the median lines of M_1 and M_2, then K^* and $\boldsymbol{K}$ are called a *normal conjugate polyhedron of* K and a *normal complex polyhedron* respectively.

Let K be a normal quadratic polyhedron. If a subdivision K_1 of K satisfies the following conditions (i), (ii), then K_1 is called a *normal subdivision* of K:

(i) for each 0-simplex $q\in K_1$, q is a 0-simplex of K, or $|q|$ is a middle point of a 2-simplex $M\in K$ or a 1-simplex $a\in K$;

(ii) for each 1-simplex $a\in K_1$, a is a subdivision of a 1-simplex of K, or $|a|$ lies on a median line of a 2-simplex $M\in K$.

8. A Riemann surface based on a normal quadrangulation. Let F be a quadrangulated manifold on which a normal quadrangulation K is defined. Then we can define an analytic structure of F and make F into a Riemann surface by the following procedure (i), (ii), (iii) (in the case of triangulation, cf. pp. 113–114 of Springer [16]).

(i) We fix a square M^e on E^2 and map each 2-simplex $M\in K$ onto the square M^e preserving the normal coordinates of M. By these mappings, a local uniformizing parameter in a neighborhood of each point in the interior of each 2-simplex of K is defined.

(ii) If a point p lies on an edge of a 2-simplex $M_1 \in K$ and p is not any vertex of M_1, then there exists another 2-simplex $M_2 \in K$ which also has p on its edge. Then we map $|M_1 \cup |M_2|$ onto a union of two adjacent squares M_1^e, M_2^e preserving the normal coordinates of M_1 and M_2. The point p is mapped into a point $P \in M_1^e \cap M_2^e$. By this mapping, a local uniformizing parameter in a neighborhood of p is defined. This local uniformizing parameter is an analytic function of the local uniformizing parameter of M_1 or M_2 defined in (i).

(iii) Let a point p be a vertex at which ν 2-simplices $M_1, .., M_\nu$ of K meet. For each j $(j=1,..., \nu)$, we map M_j, preserving the normal coordinates of M_j, onto a square M_j^e in E^2(z-plane) whose vertex corresponding to p is at the origin and whose edge corresponding to $|M_{j-1}| \cap |M_j|$ is in common with M_{j-1}^e, where for $j=1$ the latter condition for the edge is omitted. Then the image of the chain of M_1^e, .., M_ν^e by the mapping $\zeta = z^{4/\nu}$ forms a neighborhood of the origin in the ζ-plane. A local uniformizing parameter about p is defined by the coordinates of corresponding points in the ζ-plane. This local uniformizing parameter is an analytic function of the local uniformizing parameter defined in (i) or (ii).

The Riemann surface W constructed by the above procedure (i), (ii), (iii) is called a *Riemann surface based on a normal quadrangulation K.*

9. Normal quadrangulation by a quadratic differential. Let W be an open or closed Riemann surface. By a *quadratic differential* on W we mean a differential Ψ satisfying the following conditions (i), (ii):

(i) To each local uniformizing parameter z of W, a meromorphic function $Q(z)$ is assigned;

(ii) By a transformation $\zeta = \zeta(z)$ of local uniformizing parameters, the meromorphic function $Q(z)$ assigned to z is transformed to a meromorphic function $Q_1(\zeta)$ assigned to ζ under the condition:

$$Q(z) = Q_1(\zeta)\left(\frac{d\zeta}{dz}\right)^2.$$

Then the quadratic differential is expressed by the symbol $\Psi = Q(z)dz^2$.

If for a local uniformizing parameter z about a point p $Q(z)$ has a zero or a pole of order k at the point corresponding to p, then the point p is called a *zero* or a *pole of order k* of the quadratic differential $\Psi = Q(z)dz^2$ respectively. Zeros and poles of Ψ are called the *critical points* of Ψ, zeros and simple poles of Ψ are called *finite critical points*, and poles of order at least two are called *infinite critical points*. For the properties of quadratic differentials and especially of their critical points, we can refer to the chapter three of Jenkins [9].

Let $\Psi=Q(z)dz^2$ be a quadratic differential on a Riemann surface W and p_0 be a fixed point of W being a regular or finite critical point of Ψ. We suppose that there exist a subregion Ω of W and a quadrangulation K of Ω satisfying the following conditions (i)$\sim$(v):

(i) the point p_0 is the carrier of some 0-simplex $q_0 \in K$;

(ii) the carrier $|q|$ of each 0-simplex $q \in K$ is a regular or finite critical point of Ψ;

(iii) the carrier $|a|$ of each 1-simplex $a \in K$ lies on a trajectory or an orthogonal trajectory of Ψ, no interior point of $|a|$ is a critical point of Ψ, and on taking a as an oriented curve

$$\left|\int_a \Psi^{1/2}\right| = \left|\int_a Q(z)^{1/2}dz\right| = 1;$$

(iv) no interior point of the carrier $|M|$ of each 2-simplex $M \in K$ is a critical point of Ψ;

(v) each component of the relative boundary $\partial\Omega$ of Ω is either a carrier of a finite or infinite 1-chain of K if it is a continuum, or an infinite critical point of Ψ if it is a point. As the mapping function ϕ^{-1} of each 2-simplex $M=[M^e, \phi] \in K$, we can adopt the function

$$\phi^{-1}(p(z)) = \int_{z_0}^{z} Q(z)^{1/2}dz,$$

where z is a local uniformizing parameter of $p \in |M|$ and z_0 is a fixed point of the local uniformizing parameter. Then, by the conditions (iii) and (iv), M^e is a unit square, and thus by the set of the functions ϕ we can introduce normal coordinates to K. The normal quadrangulation K of the subregion Ω is called a *normal quadrangulation* (*with the original vertex* p_0) *of a subregion* Ω *by a quadratic differential* Ψ, and denoted by $K=K(\Omega, \Psi, p_0)=K(\Omega, \Psi)$. The subregion Ω is said to be *normally quadrangulable by* Ψ *for the original vertex* p_0. The normal quadrangulation $K(\Omega, \Psi, p_0)$ is uniquely determined by Ω, Ψ and p_0 provided it exists.

Let W be the Riemann surface based on a normal quadrangulation K such that each polygon of its conjugate polyhedron K^* is $2n$-angle (n: a positive integer). Then we can easily find a quadratic differential Ψ on W such that K is the normal quadrangulation of W by Ψ, i.e. $K=K(W, \Psi)$.

10. Exhaustion of a Riemann surface by a sequence of normal subdivisions. Let W be an open or closed Riemann surface. We fix an original vertex p_0 in the following. If there exists the maximal normal quadrangulation $K(\Omega_0, \Psi)$ among all normal quadrangulations by a quadratic differential Ψ which contain a fixed $K(\Omega, \Psi)$, then $K(\Omega_0, \Psi)$ is said to be *relatively maximal* (with respect to $K(\Omega, \Psi)$).

If $\{\Omega_n\}_{n=0}^{\infty}$ is an increasing sequence of subregions of W, and each Ω_n ($n=0, 1,\dots$) is normally quadrangulable by the quadratic differential $4^n\Psi$ respectively, then the sequence of normal quadrangulations $\{K_n=K(\Omega_n, 4^n\Psi)\}_{n=0}^{\infty}$ is called an *increasing sequence of normal subdivisions by* Ψ. Here we should note that $K(\Omega_{n-1}, 4^n\Psi)$ is a normal subdivision of $K(\Omega_{n-1}, 4^{n-1}\Psi)$. If each K_n is relatively maximal, then $\{K_n\}_{n=0}^{\infty}$ is said to be *relatively maximal*. Now let us suppose that W is open. If the sequence $\{\Omega_n\}_{n=0}^{\infty}$ is an exhaustion of W, then the sequence of normal subdivisions $\{K_n\}_{n=0}^{\infty}$ is said to define an *exhaustion of* W. Furthermore if each K_n is regular (canonical resp.), then the exhaustion $\{K_n\}_{n=0}^{\infty}$ is said to be *regular* (*canonical* resp.). If $\{K_n\}_{n=0}^{\infty}$ defines an exhaustion of W, then we can always find a regular or canonical exhaustion $\{K_n'\}_{n=0}^{\infty}$ of W such that $K_n'\subset K_n$ for every n.

11. Normal quadrangulation by the differential of a meromorphic function

Let f be a meromorphic function on an open or closed Riemann surface W. As a quadratic differential Ψ in **9** and **10**, we can adopt the *quadratic differential of* f

$$df^2\equiv\left(\frac{df}{dz}\right)^2dz^2$$

and the *logarithmic quadratic differential of* f

$$dLf^2\equiv\left(\frac{N}{2\pi}d\lg f\right)^2\equiv\left(\frac{N}{2\pi}\frac{1}{f}\frac{df}{dz}\right)^2dz^2,$$

where z is a local uniformizing parameter and N is a positive integer.

Let $K=K(\Omega, df^2, p_0)$ and $K'=K(\Omega', dLf^2, p_0)$ be the normal quadrangulations by the quadratic differential of f and the logarithmic quadratic differential of f respectively. Then the image of each 0-simplex $q\in K$ by f is a lattice point of the type $w=w_0+m+in$ (m, n: integers; $w_0=f(p_0)$) and the image of each 1-simplex $a\in K$ by f lies on a straight line $\operatorname{Re} w=\operatorname{Re} w_0+m$ or $\operatorname{Im} w=\operatorname{Im} w_0+n$. The image of each 0-simplex $q\in K'$ by f is a lattice point of the type

$$|w|=|w_0|\exp\left(\frac{2\pi m}{N}\right),\quad \arg w=\arg w_0+\frac{2\pi n}{N}$$
$$(m, n:\ \text{integers};\ w_0=f(p_0))$$

and the image of each 1-simplex $a\in K'$ by f lies on a circle $|w|=|w_0|\exp(2\pi m/N)$ or a ray $\arg w=\arg w_0+2\pi n/N$.

Let K^* and K'^* be the conjugate polyhedra of the K and the K' respectively. Then we know that the complex polyhedra $\boldsymbol{K}=\langle K, K^*\rangle$ and $\boldsymbol{K}'=\langle K', K'^*\rangle$

are latticed.

We can easily see that there always exists the relatively maximal with respect to each normal quadrangulation by the differential df^2 or the logarithmic differential dLf^2.

LEMMA 1.1 *Let $\{K_n\}_{n=0}^{\infty}$ be a relatively maximal sequence of normal subdivisions by the quadratic differential df^2 or the logarithmic quadratic differential dLf^2 of a meromorphic function f, let e be the set of critical points of df^2 or dLf^2 respectively outside $|K_n|$ for all n, and let $W'=W-e$. Then there exists a canonical exhaustion $\{K_n'\}_{n=0}^{\infty}$ of W' such that $K_n' \subset K_n$ for every n.*

PROOF. We shall prove the lemma in the case of the differential df^2. The proof in the case of dLf^2 is quite similar.

We can put $K_n = K(\Omega_n, 4^n df^2)$ $(n=0, 1,\ldots)$. Let p' be an arbitrary regular point of df^2 in W' and let p_1 be a fixed regular point of df^2 in Ω_0. We can connect p' to p_1 by a smooth curve γ in W' so that each point of γ is a regular point of df^2. Then there exists a positive number $r>0$ such that for the r-neighborhood G_r of γ measured by f (i.e. G_r is the connected component containing γ of $\{p \mid |f(p)-f(p'')|<r,\ p\in W,\ p''\in\gamma\}$), each point of G_r is a regular point of df^2. Then for all sufficiently large n, there exists a 2-chain X_n of the normal quadrangulation by $4^n df^2$ such that $\gamma\subset|X_n|\subset G_r$, thus $X_n\subset K_n$ and thus $p'\in\Omega_n=|K_n|$, for each K_n is relatively maximal. Hence we see that $|K_{n-1}|\subset|K_n|$ for every n and $\cup_{n=0}^{\infty}|K_n|=W'$. Then we can easily find a canonical exhaustion $\{K_n'\}_{n=0}^{\infty}$ of W' such that $K_n'\subset K_n$ for every n.

§ 2. Differences on a polyhedron.

1. Difference calculus. Let $\boldsymbol{K} = <K, K^*>$ be an arbitrary complex polyhedron.

By a *function* on $\boldsymbol{K}$ we mean the complex valued function f on the set of oriented 0-simplices of $\boldsymbol{K}$ such that f has a value $f(q)=f_q$ for each oriented 0-simplex q and $f(-q)=-f(q)$.

By a *first order difference* or 1-*difference* ω on $\boldsymbol{K}$ we mean the complex valued function ω on the set of oriented 1-simplices of $\boldsymbol{K}$ such that ω has a value $\omega(a)=\omega_a$ for each oriented 1-simplex a and $\omega(-a)=-\omega(a)$.

By a *second order difference* or 2-*difference* Ω on $\boldsymbol{K}$ we mean the complex valued function Ω on the set of oriented 2-simplices of $\boldsymbol{K}$ such that Ω has a value $\Omega(M)=\Omega_M$ for each oriented 2-simplex M and $\Omega(-M)=-\Omega(M)$.

For the conformity, a function on $\boldsymbol{K}$ is called a *zero order difference* or 0-difference.

We assume that differences of arbitrary order satisfy the linearity condition, e.g. for two first order differences ω_1, ω_2

$$(c_1\omega_1+c_2\omega_2)(a)=c_1\cdot\omega_1(a)+c_2\cdot\omega_2(a)$$

$$(c_1,\ c_2\text{: complex constants}).$$

A multiplication of a 1-difference ω with a function f is defined as a 1-difference satisfying the condition

$$f\omega(a)=\omega f(a)=\frac{1}{2}(f_1+f_2)\omega_a \qquad \text{for each 1-simplex } a\in \boldsymbol{K},$$

where $f_j=f(q_j)$ $(j=1,\ 2)$ and $\partial a^*=q_2-q_1$. A multiplication of a 2-difference Ω with a function f is defined as a 2-difference satisfying the condition

$$f\Omega(M)=f_q\Omega_M \qquad \text{for each 2-simplex } M\in \boldsymbol{K},$$

where q is the conjugate of M.

The complex conjugate $\bar{\omega}$ of a 1-difference ω is defined by $\bar{\omega}(a)=\overline{\omega(a)}$.

The *difference of a function* f is defined as a 1-difference $\omega=\Delta f$ satisfying the condition

$$\Delta f(a)=f_2-f_1 \qquad \text{for each 1-simplex } a\in \boldsymbol{K},$$

where $f_j=f(q_j)$ $(j=1,\ 2)$ and $\partial a=q_2-q_1$. If for a 1-difference ω there exists a function f such that $\omega=\Delta f$, then ω is said to be *exact*. The *difference of a* 1-*difference* ω is defined as a 2-difference $\Omega=\Delta\omega$ satisfying the condition

$$\Delta\omega(M)=\sum_{j=1}^{\kappa}\omega_j \qquad \text{for each 2-simplex } M\in \boldsymbol{K},$$

where $\omega_j=\omega(a_j)$ $(j=1,\ldots,\ \kappa)$ and $\partial M=\sum_{j=1}^{\kappa}a_j$. If $\Delta\omega=0$, then ω is said to be *closed*. Obviously, if ω is exact, then ω is closed.

The *exterior product* of two 1-differences ω_1, ω_2 is defined as a 2-difference $\Omega=\omega_1\omega_2$ satisfying the condition

$$\omega_1\omega_2(M)=-\frac{1}{2}\sum_{j=1}^{\kappa}\omega_1(a_j^*)\omega_2(a_j) \qquad \text{for each 2-simplex } M\in \boldsymbol{K},$$

where $\partial M=\sum_{j=1}^{\kappa}a_j$. We can easily verify that the partial difference formula

$$\Delta(f\omega)=(\Delta f)\omega+f\Delta\omega \tag{2.1}$$

holds.

2. Summation of differences. We can define the *sum* of an n-difference $(n=0,\ 1,\ 2)$ over an n-chain. Let $\lambda=\sum x_jq_j$, $\gamma=\sum x_ja_j$ and $X=\sum x_jM_j$ be a 0-chain, a 1-chain and a 2-chain respectively of a complex polyhedron $\boldsymbol{K}$. The *sum of* 0-*difference* f, 1-*difference* ω *and* 2-*difference* Ω *over* λ, γ *and* X respectively are defined by

$$\mathfrak{S}_{\lambda} f=\sum x_j f(q_j)$$

$$\mathfrak{S}_{\gamma} \omega=\sum x_j \omega(a_j)$$

and

$$\mathfrak{S}_{X} \Omega=\sum x_j \Omega(M_j)$$

respectively. The basic duality between a chain and a difference

$$\mathfrak{S}_{X} \Delta\omega=\mathfrak{S}_{\partial X} \omega \tag{2.2}$$

is obvious, where X is a 2-chain and ω is a 1-difference. The formula for partial summation

$$\mathfrak{S}_{X} (\Delta f)\omega=\mathfrak{S}_{\partial X} f\omega-\mathfrak{S}_{X} f\Delta\omega \tag{2.3}$$

follows from (2.1) and (2.2).

The following two criteria are also obvious:

A 1-difference ω is exact if and only if $\mathfrak{S}_{\gamma} \omega=0$ for every cycle γ;

A 1-difference ω is closed if and only if $\mathfrak{S}_{\gamma} \omega=0$ for every cycle γ that is homologous to 0.

If ω is closed, then the *period of ω along a cycle γ* is defined by $\mathfrak{S}_{\gamma} \omega$, which depends only on the homology class of γ.

Now we shall define the sum of 2-difference over a complex polyhedron $\boldsymbol{K}=\langle K, K^*\rangle$. If $\boldsymbol{K}$ is compact bordered or closed, then the sum of a 2-difference Ω over $\boldsymbol{K}$

$$\mathfrak{S}_{\boldsymbol{K}} \Omega$$

is defined as the sum of Ω over the 2-chain $\boldsymbol{K}$ because $\boldsymbol{K}$ is itself a 2-chain. If $\boldsymbol{K}$ is open, then we can set

$$\mathfrak{S}_{\boldsymbol{K}} \Omega=\lim_{X\to\boldsymbol{K}} \mathfrak{S}_{X} \Omega \tag{2.4}$$

provided that the limit exists, where X is a 2-chain of $\boldsymbol{K}$ such that $X\subset\boldsymbol{K}$. Let us define $|\Omega|$ by $|\Omega|(M)=|\Omega(M)|$ for each $M\in\boldsymbol{K}$. Then (2.4) exists if and only if the limit

$$\mathfrak{S}_{\boldsymbol{K}} |\Omega|=\lim_{X\to\boldsymbol{K}} \mathfrak{S}_{X} |\Omega| \tag{2.5}$$

is finite.

3. Conjugate differences. Let ω be a 1-difference on a complex polyhedron $\boldsymbol{K}$. Then the *conjugate difference* ω^* of ω is defined as a 1-difference satisfying the condition

$$\omega^*(a^*)=\omega(a) \qquad \text{for each 1-simplex } a\in\boldsymbol{K}.$$

Then we can easily see that

$$\omega^{**}=-\omega, \tag{2.6}$$

$$\omega_1^*\omega_2^*=-\omega_2\omega_1. \tag{2.7}$$

A 1-difference ω is said to be *harmonic* if ω and ω^* are both closed. By (2.6) and the definition, ω and ω^* are simultaneously harmonic. Let u be a function on $\boldsymbol{K}$. u is called a *harmonic function on* $\boldsymbol{K}$ if the difference Δu is harmonic. u is harmonic on $\boldsymbol{K}$ if and only if

$$\sum_{j=1}^{\kappa}u_j-\kappa u_0=0$$

for each 0-simplex q_0 in the interior of $\boldsymbol{K}$, where $u_j=u(q_j)$ $(j=0,\dots,\kappa)$, $\partial a_j=q_j-q_0$ $(j=1,\dots,\kappa)$ and a_j $(j=1,\dots,\kappa)$ are all 1-simplices having q_0 as a vertex.

A 1-difference ϕ is said to be *pure* if $\phi^*=-i\phi$. Thus a difference ϕ is pure if and only if $\phi_{a^*}=i\phi_a$ for every $a\in\boldsymbol{K}$. A difference ϕ is said to be *analytic* if it is closed and pure. A function f on $\boldsymbol{K}$ is said to be *analytic* if the difference Δf is analytic. If a difference ω is harmonic, then $\omega+i\omega^*$ is analytic. The complex conjugate $\bar{\phi}$ of an analytic difference ϕ is said to be *antianalytic*. Every harmonic difference can be uniquely written as the sum of an analytic difference and an antianalytic difference.

§ 3. The Hilbert space of differences.

1. The inner product. Let ω_1, ω_2 be two 1-differences on a complex polyhedron $\boldsymbol{K}=\langle K, K^*\rangle$. We shall define the *inner product* $(\omega_1,\omega_2)=(\omega_1,\omega_2)_{\boldsymbol{K}}$ of ω_1 and ω_2. If $\boldsymbol{K}$ is closed, then it is defined by

$$(\omega_1,\omega_2)_{\boldsymbol{K}}=\sum_{a\in\boldsymbol{K}}\omega_1(a)\overline{\omega_2(a)}. \tag{3.1}$$

If $\boldsymbol{K}$ is compact bordered, then it is defined by

$$(\omega_1,\omega_2)_{\boldsymbol{K}}=\sum_{a\in K-\partial K}(\omega_a^1\bar{\omega}_a^2+\omega_{a^*}^1\bar{\omega}_{a^*}^2)+\frac{1}{2}\sum_{a\in\partial K}(\omega_a^1\bar{\omega}_a^2+\omega_{a^*}^1\bar{\omega}_{a^*}^2), \tag{3.2}$$

where $\omega_a^1=\omega_1(a)$, etc. If $\boldsymbol{K}$ is open, then it is defined by the limit process

$$(\omega_1,\omega_2)_{\boldsymbol{K}}=\lim_{\boldsymbol{L}\to\boldsymbol{K}}(\omega_1,\omega_2)_{\boldsymbol{L}}, \tag{3.3}$$

provided that the limit exists, where $\boldsymbol{L}=<L, L^*>$ is a compact bordered complex polyhedron such that $\boldsymbol{L}\subset\boldsymbol{K}$. In the definition of inner product (ω_1, ω_2) we agree that for the sense-reversed $-\boldsymbol{K}$ of $\boldsymbol{K}$

$$(\omega_1, \omega_2)_{-\boldsymbol{K}}=-(\omega_1, \omega_2)_{\boldsymbol{K}}.$$

If $\boldsymbol{K}$ is closed or open, then we can see that

$$(\omega_1, \omega_2)_{\boldsymbol{K}}=\mathfrak{S}_{\boldsymbol{K}}\omega_1\bar{\omega}_2^*. \tag{3.4}$$

If $\boldsymbol{K}$ is compact bordered, then we can similarly see that

$$(\omega_1, \omega_2)_{\boldsymbol{K}}=\mathfrak{S}_{\boldsymbol{K}}\omega_1\bar{\omega}_2^*+\frac{1}{2}\sum_{a\in\partial K}\omega_a^1\bar{\omega}_a^2+\frac{1}{2}\sum_{a\in\partial K^*}\omega_{a^*}^1\bar{\omega}_{a^*}^2. \tag{3.5}$$

By the definitions (3.1), (3.2) and (3.3), for every case of $\boldsymbol{K}$ we have that

$$(\omega_2, \omega_1)=(\bar{\omega}_1, \bar{\omega}_2), \tag{3.6}$$

$$(\omega_1^*, \omega_2^*)=(\omega_1, \omega_2). \tag{3.7}$$

Let ω be a 1-difference on a complex polyhedron $\boldsymbol{K}$. Then the *norm* $\|\omega\|=\|\omega\|_{\boldsymbol{K}}$ of ω is defined by

$$\|\omega\|_{\boldsymbol{K}}=(\omega, \omega)_{\boldsymbol{K}}^{1/2}. \tag{3.8}$$

Let us denote the *Hilbert space of all* 1-*differences* ω on $\boldsymbol{K}$ with $\|\omega\|<\infty$ by $\Gamma=\Gamma(\boldsymbol{K})$. Furthermore, we define the closed subspaces of Γ as follows:

$$\Gamma_c=\{\omega|\ \omega \text{ is closed}, \omega\in\Gamma\},$$

$$\Gamma_e=\{\omega|\ \omega \text{ is exact}, \omega\in\Gamma\},$$

$$\Gamma_h=\{\omega|\ \omega \text{ is harmonic}, \omega\in\Gamma\},$$

$$\Gamma_a=\{\phi|\ \phi \text{ is analytic}, \phi\in\Gamma\},$$

$$\Gamma_c^*=\{\omega^*|\ \omega\in\Gamma_c\},$$

$$\Gamma_e^*=\{\omega^*|\ \omega\in\Gamma_e\},$$

$$\Gamma_h^*=\{\omega^*|\ \omega\in\Gamma_h\}.$$

Then it is obvious that $\Gamma_h^*=\Gamma_h$, $\Gamma_e\subset\Gamma_c$, $\Gamma_h=\Gamma_c\cap\Gamma_c^*$ and $\Gamma_a\subset\Gamma_h$.

Because of Schwarz's inequality

$$|(\omega_1, \omega_2)|\leqq\|\omega_1\|\cdot\|\omega_2\|,$$

the inner product (ω_1, ω_2) for any pair $\omega_1, \omega_2\in\Gamma$ always exists.

2. Green's formula. Let $\boldsymbol{\gamma}=<\gamma, \gamma^*>$ be a complex cycle on a complex polyhedron $\boldsymbol{K}=<K, K^*>$. Let f and ω be a function and a 1-difference on $\boldsymbol{K}$ respectively. We shall define f and ω on γ^*.

Let q be an arbitrary 0-simplex of γ^*, let a be the dual 1-simplex of q and let $\partial a^*=q_2-q_1$. Then we define the *value of f at $q\in\gamma^*$* by

$$f(q)=\frac{1}{2}(f(q_1)+f(q_2)). \tag{3.9}$$

Let b be an arbitrary 1-simplex of γ^*, let q be the dual 0-simplex of b, let M'^* be the conjugate left half 2-simplex of q w.r.t. γ and let

$$\partial M'^*=b+b''-\sum_{j=1}^{\kappa-1} a_j^*-b'$$

with the notations defined in § 1. 5. Then we define the *value of ω along $b\in\gamma^*$* by

$$\omega(b)=\frac{1}{2}\,\omega(a'^*)+\sum_{j=1}^{\kappa-1}\omega(a_j^*)-\frac{1}{2}\,\omega(a''^*), \tag{3.10}$$

with the notations a', a'' defined in § 1. 5.

The multiplication of ω with f on $\boldsymbol{\gamma}$ is defined by

$$f\omega(a)=f_q\omega_a \qquad \text{for each 1-simplex } a\in\boldsymbol{\gamma}, \tag{3.11}$$

where q is the dual 0-simplex of a.

LEMMA 3.1. (Green's formula.) *If $\boldsymbol{K}$ is compact bordered or closed, then we have*

$$(\Delta f,\,\omega)_{\boldsymbol{K}}=\mathfrak{S}_{\partial\boldsymbol{K}} f\bar{\omega}^*-\mathfrak{S}_{\boldsymbol{K}} f\Delta\bar{\omega}^*, \tag{3.12}$$

where $\partial\boldsymbol{K}$ is the complex boundary of $\boldsymbol{K}$, and if $\boldsymbol{K}$ is closed then the first term of the right side vanishes.

PROOF. By (3.5), (2.3) we have

$$\begin{aligned}(\Delta f,\,\omega)_{\boldsymbol{K}}&=\Big(\mathfrak{S}_{\partial\boldsymbol{K}} f\bar{\omega}^*+\frac{1}{2}\sum_{a\in\partial K}\Delta f_a\bar{\omega}_a+\frac{1}{2}\sum_{a\in\partial K^*}\Delta f_{a^*}\bar{\omega}_{a^*}\Big)-\mathfrak{S}_{\boldsymbol{K}} f\Delta\bar{\omega}^*\\ &=\Big(\mathfrak{S}_{\partial\boldsymbol{K}} f\bar{\omega}^*+\frac{1}{2}\sum_{a\in\partial K^*}(f_1+f_2)\bar{\omega}_a^*+\frac{1}{2}\sum_{a\in\partial K}\Delta f_a\bar{\omega}_{a^*}^*\\ &\quad-\frac{1}{2}\sum_{a\in\partial K^*}\Delta f_{a^*}\bar{\omega}_a^*\Big)-\mathfrak{S}_{\boldsymbol{K}} f\Delta\bar{\omega}^*\\ &=\Big(\mathfrak{S}_{\partial\boldsymbol{K}} f\bar{\omega}^*+\sum_{a\in\partial K^*} f_1\bar{\omega}_a^*+\sum_{a\in\partial K}\Delta f_a\cdot\frac{1}{2}\,\bar{\omega}_{a^*}^*\Big)-\mathfrak{S}_{\boldsymbol{K}} f\Delta\bar{\omega}^*,\end{aligned}$$

where $f_j=f(q_j)$ $(j=1, 2)$ and $\partial a^*=q_2-q_1$. By the definitions (3.10), (3.11) and simple calculation we find that

$$\int_{\partial K^{*+}} f\bar{\omega}^* = \sum_{a\in\partial K^*} f_1\bar{\omega}^*_a + \sum_{a\in\partial K} \Delta f_a\cdot\frac{1}{2}\bar{\omega}^*_{a^*}.$$

Hence we obtain the present lemma.

By (3.12) the equation

$$\int_{\partial K}(f\Delta\bar{g}^* - \bar{g}\Delta f^*) = \int_K (f\Delta(\Delta\bar{g}^*) - \bar{g}\Delta(\Delta f^*)) \tag{3.13}$$

holds for a pair of functions f, g on $\boldsymbol{K}$.

In Courant-Friedrichs-Lewy [4], Blanc [3] and Mizumoto [12], we can find some different types of Green's formulas for discrete functions. In the present paper we shall find that the present Green formula constructed on a complex polyhedron is very convenient.

3. **Orthogonal projection on a compact polyhedron.** In **3**$\sim$**7**, we shall briefly state the method of orthogonal projection of the Hilbert space of differences which is analogous to the case of differentials (cf. Ch. V of Ahlfors & Sario [1]).

Let $\boldsymbol{K}$ be a closed complex polyhedron. By (3.6), (3.7), (3.12), we have

$$(\omega, \Delta f^*) = -\int_K \bar{f}\Delta\omega$$

for a function f and a 1-difference ω on $\boldsymbol{K}$. Hence $\Delta\omega=0$ implies $(\omega, \Delta f^*)=0$. Conversely, if

$$\int_K \bar{f}\Delta\omega = 0$$

holds for all functions f on $\boldsymbol{K}$, then we have $\Delta\omega=0$ on $\boldsymbol{K}$. Hence on a closed complex polyhedron $\boldsymbol{K}$, Γ_c (and Γ_c^*) is the orthogonal complement of Γ_e^* (and Γ_e resp.). Then by the general theory, we have the orthogonal decompositions

$$\Gamma = \Gamma_c \dot{+} \Gamma_e^* = \Gamma_c^* \dot{+} \Gamma_e, \tag{3.14}$$

and hence we have immediately the orthogonal decomposition

$$\Gamma = \Gamma_h \dot{+} \Gamma_e \dot{+} \Gamma_e^*.$$

A 1-difference ω on a complex polyhedron $\boldsymbol{K}$ is said to vanish *along* a 1-chain γ if $\omega(a)=0$ for every 1-simplex $a\in\gamma$.

Let $\boldsymbol{K}$ be a compact bordered complex polyhedron. A closed 1-difference ω on $\boldsymbol{K}$ is said to belong to the subspace $\Gamma_{c0}(\boldsymbol{K})$ if $\omega=0$ along the complex boundary $\boldsymbol{\partial K}$. Similarly $\omega=\Delta f$ is said to belong to the subspace $\Gamma_{e0}(\boldsymbol{K})$ if $f=0$ on the complex boundary $\partial\boldsymbol{K}$. By making use of (3.12) and the similar argument to the case of (3.14), on a compact bordered complex polyhedron $\boldsymbol{K}$ we have the orthogonal decompositions

$$\Gamma=\Gamma_{c0}\dotplus\Gamma_e^*=\Gamma_{c0}^*\dotplus\Gamma_e, \tag{3.15}$$

$$\Gamma=\Gamma_c\dotplus\Gamma_{e0}^*=\Gamma_c^*\dotplus\Gamma_{e0}, \tag{3.16}$$

and hence we have immediately the orthogonal decomposition

$$\Gamma=\Gamma_h\dotplus\Gamma_{e0}\dotplus\Gamma_{e0}^*.$$

A 1-difference ω on a compact bordered complex polyhedron $\boldsymbol{K}$ is said to be *semiexact* if ω is closed and its period along each contour of K and K^* vanishes. The subspaces Γ_{se} and Γ_{se}^* of Γ are defined by $\Gamma_{se}=\{\omega|\ \omega \text{ is semiexact}, \omega\in\Gamma\}$ and $\Gamma_{se}^*=\{\omega^*|\ \omega\in\Gamma_{se}\}$. Then we have the orthogonal decompositions

$$\Gamma=\Gamma_{c0}\cap\Gamma_e\dotplus\Gamma_{se}^*=(\Gamma_{c0}\cap\Gamma_e)^*\dotplus\Gamma_{se}. \tag{3.17}$$

4. Orthogonal projection on a generic polyhedron. Let us suppose that $\boldsymbol{K}$ is an open or closed complex polyhedron. A 1-difference ω (a function f resp.) on $\boldsymbol{K}$ is said to have *compact support* if $\omega(a)=0$ for all 1-simplices $a\in\boldsymbol{K}$ ($f(q)=0$ for all 0-simplices $q\in\boldsymbol{K}$ resp.) except for a finite number of 1-simplices (0-simplices resp.) of $\boldsymbol{K}$.

Let Γ_{e0}' be the subclass of Γ_e consisting of the 1-differences ω such that $\omega=\Delta f$ for a function f with compact support. We define the subspace Γ_{e0} of Γ as the closure in Γ of Γ_{e0}'. The subspace Γ_{c0} is defined as the orthogonal complement of Γ_e^*. From the definitions it follows that $\Gamma_{e0}=\Gamma_e$ and $\Gamma_{c0}=\Gamma_c$ for a closed complex polyhedron $\boldsymbol{K}$.

On an arbitrary complex polyhedron $\boldsymbol{K}$ we have the following orthogonal decompositions:

$$\Gamma=\Gamma_e\dotplus\Gamma_{c0}^*=\Gamma_e^*\dotplus\Gamma_{c0}, \tag{3.18}$$

$$\Gamma=\Gamma_{e0}\dotplus\Gamma_c^*=\Gamma_{e0}^*\dotplus\Gamma_c, \tag{3.19}$$

$$\Gamma=\Gamma_h\dotplus\Gamma_{e0}\dotplus\Gamma_{e0}^*, \tag{3.20}$$

$$\Gamma_c=\Gamma_h\dotplus\Gamma_{e0}, \tag{3.21}$$

$$\Gamma_e=\Gamma_{he}\dotplus\Gamma_{e0}, \tag{3.22}$$

$$\Gamma_{c0}=\Gamma_{h0}\dotplus\Gamma_{e0}, \tag{3.23}$$

$$\Gamma_h=\Gamma_{he}\dotplus\Gamma_{h0}^*=\Gamma_{h0}\dotplus{}^*_{he}, \tag{3.24}$$

where $\Gamma_{he}=\Gamma_h\cap\Gamma_e$ and $\Gamma_{h0}=\Gamma_h\cap\Gamma_{c0}$.

If ω_1 and ω_2 are two 1-differences such that $\omega_1 - \omega_2 \in \Gamma_{e0}$, then ω_1 is said to have the *same boundary behavior and the same periodicity as* ω_2. The decompositions (3.21), (3.22) and (3.23) assure that for any difference τ of Γ_c, Γ_e or Γ_{c0} there exists the difference ω of Γ_h, Γ_{he} or Γ_{h0} respectively with the same boundary behavior and the same periodicity as τ.

5. The double of a polyhedron. Let $\boldsymbol{K} = <K, K^*>$ be a compact bordered complex polyhedron. Let $\hat{F}$ be the double of the manifold $F = |K|$ and let j be the involutory mapping of $\hat{F}$ onto itself (cf. pp. 26–27 & p. 290 of Ahlfors & Sario [1] for the definition). We can always define a polyangulation $\hat{K}$ of $\hat{F}$ and its conjugate polyhedron $\hat{K}^*$ satisfying the following conditions (i), (ii), (iii):

(i) Each n-simplex ($n = 0, 1, 2$) of K (K^* resp.) and its image by the mapping j are n-simplices of $\hat{K}$ ($\hat{K}^*$ resp.);

(ii) For each 0-simplex q and each 1-simplex a of ∂K, there exist a 2-simplex M^* and a 1-simplex a^* of $\hat{K}^*$ which are the conjugates of q and a respectively and which are mapped onto themselves but sense-reversed by the involutory mapping j;

(iii) The collection of n-simplices of $\hat{K}$ defined in (i) ($\hat{K}^*$ defined in (i), (ii) resp.) forms the whole class of n-simplices of $\hat{K}$ ($\hat{K}^*$ resp.).

$\hat{K}$ and $\hat{K}^*$ are called the *doubles* of K and K^* respectively, and $\hat{\boldsymbol{K}} = <\hat{K}, \hat{K}^*>$ is called the *double* of $\boldsymbol{K} = <K, K^*>$. $\hat{K}$, $\hat{K}^*$ and $\hat{\boldsymbol{K}}$ are closed.

6. Schottky differences. With each 1-difference ω on a double $\hat{\boldsymbol{K}}$ we can associate a new difference $\omega^\sim$ by

$$\omega^\sim(j(a)) = \omega(a) \qquad \text{for each 1-simplex } a \in \hat{\boldsymbol{K}}.$$

We can easily verify the invariance of exterior product:

$$\omega_1^\sim \omega_2^\sim(-j(M)) = \omega_1 \omega_2(M) \qquad \text{for each 2-simplex } M \in \hat{\boldsymbol{K}},$$

and

$$\omega^{*\sim} = -\omega^{\sim *}.$$

We shall say that ω is *even* if $\omega^\sim = \omega$, and *odd* if $\omega^\sim = -\omega$. If ω is even, then ω^* is odd. If ω is closed and odd then ω vanishes along the complex boundary $\boldsymbol{\partial K} = <\partial K, \partial K^{*+}>$, for it follows immediately for ∂K and from the definition (3.10) for ∂K^{*+}.

Any 1-difference ω on $\hat{\boldsymbol{K}}$ can be uniquely decomposed into $\omega = \omega_1 + \omega_2^*$ where ω_1 and ω_2 are even. If $\omega \in \Gamma_h(\hat{\boldsymbol{K}})$, then we can immediately see that in the decomposition $\omega = \omega_1 + \omega_2^*$, $\omega_2^* \in \Gamma_{h0}(\boldsymbol{K})$ and $\omega_1 \in \Gamma_{h0}^*(\boldsymbol{K})$. Furthermore we can immediately see that the whole class of differences on $\boldsymbol{K}$ which have a harmonic extension to $\hat{\boldsymbol{K}}$ is given by the direct sum $\Gamma_{h0}(\boldsymbol{K}) \dotplus \Gamma_{h0}^*(\boldsymbol{K})$. Such differences will be called *Schottky differences.*

Let $\boldsymbol{K}$ be an open complex polyhedron. The following lemma is proved by the orthogonal projection method.

LEMMA 3.2. *$\omega\in\Gamma_{h0}(\boldsymbol{K})$ if and only if for every $\varepsilon>0$ and every finite 2-chain $X\subset\boldsymbol{K}$ there exist a regular complex subpolyhedron $\boldsymbol{K}_0$ of $\boldsymbol{K}$ and $\omega_0\in\Gamma_{h0}(\boldsymbol{K})$ such that $X\subset\boldsymbol{K}_0$ and $\|\omega-\omega_0\|_{\boldsymbol{K}_0}<\varepsilon$.*

A 1-difference ω on an open complex polyhedron $\boldsymbol{K}$ is called a *Schottky difference* on $\boldsymbol{K}$ if for every $\varepsilon>0$ and every finite 2-chain $X\subset\boldsymbol{K}$ there exist a regular complex subpolyhedron $\boldsymbol{K}_0$ of $\boldsymbol{K}$ and $\omega_0\in\Gamma_{h0}(\boldsymbol{K}_0)\dotplus\Gamma_{h0}^*(\boldsymbol{K}_0)$ such that $X\subset\boldsymbol{K}_0$ and $\|\omega-\omega_0\|_{\boldsymbol{K}_0}<\varepsilon$. The class of all Schottky differences on $\boldsymbol{K}$ is denoted by $\Gamma_S=\Gamma_S(\boldsymbol{K})$. The following representation is obtained:

$$\Gamma_S=\mathrm{Cl}(\Gamma_{h0}+\Gamma_{h0}^*), \tag{3.25}$$

where by Cl we denote the closure.

7. Harmonic measures. Let $\boldsymbol{K}$ be a compact bordered complex polyhedron. Then the space of harmonic measures Γ_{hm} is defined by $\Gamma_{hm}=\Gamma_{h0}\cap\Gamma_{he}$. By (3.17), $\Gamma_h=\Gamma_{hse}\dotplus\Gamma_{hm}^*$, where $\Gamma_{hse}=\Gamma_h\cap\Gamma_{se}$.

Next, let $\boldsymbol{K}$ be an open complex polyhedron. A 1-difference $\omega\in\Gamma(\boldsymbol{K})$ is said to be *semiexact* if ω is closed and $\oint_\gamma\omega=0$ for every dividing cycle γ on $\boldsymbol{K}$. The space of all semiexact harmonic differences on $\boldsymbol{K}$ is denoted by $\Gamma_{hse}=\Gamma_{hse}(\boldsymbol{K})$. The space of harmonic measures $\Gamma_{hm}=\Gamma_{hm}(\boldsymbol{K})$ is defined as follows: $\omega\in\Gamma_{hm}$ if and only if for every $\varepsilon>0$ and every finite 2-chain $X\subset\boldsymbol{K}$ there exist a *canonical* complex subpolyhedron $\boldsymbol{K}_0$ of $\boldsymbol{K}$ and harmonic measure $\omega_0\in\Gamma_{hm}(\boldsymbol{K}_0)$ such that $X\subset\boldsymbol{K}_0$ and $\|\omega-\omega_0\|_{\boldsymbol{K}_0}<\varepsilon$. By the orthogonal projection method, the following orthogonal decomposition is proved:

$$\Gamma_h=\Gamma_{hse}\dotplus\Gamma_{hm}^*=\Gamma_{hse}^*\dotplus\Gamma_{hm}. \tag{3.26}$$

§ 4. Singularities and periods.

1. Singularities. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be a complex polyhedron. Let $\{q_n\}_{n=1}^{\nu}$ ($\nu\leqq\infty$) be an arbitrary collection of 0-simplices in the interior of $\boldsymbol{K}$ and M_n^* ($n=1,\dots,\nu$) be the conjugate 2-simplex of q_n respectively. Let Θ be a 1-difference on $\boldsymbol{K}$. If Θ is closed on $\boldsymbol{K}-\sum_{n=1}^{\nu}M_n^*$, then Θ is called the *closed difference with the closed singularities* q_n ($n=1,\dots,\nu$) and

$$P(\Theta, q_n)\equiv\oint_{\partial M_n^*}\Theta \qquad (n=1,\dots,\nu)$$

is called the *singular part of the closed difference Θ at q_n* respectively. Two closed differences Θ_1 and Θ_2 with singularities at a 0-simplex q are said to define the *same singularity at* q if $P(\Theta_1, q)=P(\Theta_2, q)$.

If both differences Θ and Θ^* are closed on $\boldsymbol{K}-\sum_{n=1}^{\nu} M_n^*$, then Θ is called the *harmonic difference with the harmonic singularities* q_n $(n=1,\ldots,\nu)$, and

$$P(\Theta, q_n)\equiv \oint_{\partial M_n^*}\Theta \quad\text{and}\quad P(\Theta^*, q_n)\equiv \oint_{\partial M_n^*}\Theta^* \qquad (n=1,\ldots,\nu)$$

are called the *singular* and the *conjugate singular parts of the harmonic difference* Θ *at* q_n respectively. Two harmonic differences Θ_1 and Θ_2 with singularities at a 0-simplex q are said to define the *same singularity at* q if $P(\Theta_1, q)=P(\Theta_2, q)$ and $P(\Theta_1^*, q)=P(\Theta_2^*, q)$. We agree that a *harmonic* difference Θ_1 and a *closed* difference Θ_2 with a singularity at q are said to define the *same singularity at* q if $P(\Theta_1, q)=P(\Theta_2, q)$ and $P(\Theta_1^*, q)=0$.

If Θ is closed on $\boldsymbol{K}-\sum_{n=1}^{\nu} M_n^*$ and pure on $\boldsymbol{K}$, then Θ is called the *analytic difference with the analytic singularities* q_n $(n=1,\ldots,\nu)$ and

$$\operatorname{Res}(\Theta, q_n)\equiv\frac{1}{2\pi i}\oint_{\partial M_n^*}\Theta \qquad (n=1,\ldots,\nu)$$

is called the *residue of* Θ *at* q_n respectively. Two analytic differences Θ_1 and Θ_2 with singularities at a 0-simplex q are said to define the *same singularity at* q if $\operatorname{Res}(\Theta_1, q)=\operatorname{Res}(\Theta_2, q)$. A harmonic difference Θ with a harmonic singularity q is said to be with an *analytic* singularity q if

$$P(\Theta^*, q)=-iP(\Theta, q)$$

and in this case the residue $\operatorname{Res}(\Theta, q)$ can be also defined.

2. The existence of singular differences.

THEOREM 4.1. *Let* Θ *be a closed difference of* $\Gamma(\boldsymbol{K})$ *with a finite number of singularities* $\{q_n\}_{n=1}^{\nu}$ $(\nu<\infty)$. *Then there exists a unique harmonic difference* τ *with the same singularities as* Θ *at* q_n $(n=1,\ldots,\nu)$, *with the same boundary behavior and the same periodicity as* Θ.

PROOF. The uniqueness is clear by (3.21).

By the orthogonal decomposition (3.20) we have the expression

$$\Theta=\omega_h+\omega_{e0}+\omega_{e0}^*$$

or

$$\Theta-\omega_{e0}=\omega_h+\omega_{e0}^*, \tag{4.1}$$

where $\omega_h\in\Gamma_h$, $\omega_{e0}\in\Gamma_{e0}$ and $\omega_{e0}^*\in\Gamma_{e0}^*$, and ω_{e0}^* need not be the conjugate of

ω_{e0}. The left-hand side of (4.1) is a closed difference with the same singularities as Θ at every q_n $(n=1,\dots,\nu)$ and the conjugate of the right-hand side is closed everywhere on $\boldsymbol{K}$. Hence $\tau=\Theta-\omega_{e0}$ is the desired difference.

COROLLARY 4.1. (Cf. Blanc [2], [3].) *Under the same assumption as Theorem* 4.1 *there exists a unique harmonic difference* σ *with the harmonic singularities* q_n $(n=1,\dots,\nu)$ *with the conjugate singular parts*

$$P(\sigma^*, q_n)=P(\Theta, q_n) \qquad (n=1,\dots,\nu)$$

such that $\sigma\in\Gamma_{e0}(\boldsymbol{K})$.

PROOF. The uniqueness is clear.

For the difference τ of Theorem 4.1 we note that $\tau_1=-\tau^*\in\Gamma_c$ and

$$P(\tau_1^*, q_n)=P(\tau, q_n)=P(\Theta, q_n) \qquad (n=1,\dots,\nu).$$

By the orthogonal decomposition (3.21), we have

$$\tau_1=\omega_h+\omega_{e0} \qquad (\omega_h\in\Gamma_h,\ \omega_{e0}\in\Gamma_{e0}).$$

$\sigma=\omega_{e0}$ satisfies the condition of the corollary.

COROLLARY 4.2. *Under the same assumption as Theorem* 4.1 *there exists a unique harmonic difference* χ *with the analytic singularities* q_n $(n=1,\dots,\nu)$ *with the residues*

$$\operatorname{Res}(\chi, q_n)=\frac{1}{2\pi i}P(\Theta, q_n) \qquad (n=1,\dots,\nu)$$

such that $\chi-\Theta\in\Gamma_{e0}(\boldsymbol{K})$.

PROOF. $\chi=-i\sigma+\tau$ is the desired one, where τ, σ are the differences of Theorem 4.1 and Corollary 4.1 respectively.

Let a be an oriented 1-simplex of $\boldsymbol{K}$, and q_1 and q_2 be the 0-simplices with $\partial a=q_2-q_1$. We define the *singular difference* Θ^a *associated to* 1*-simplex* a by the condition

$$\Theta^a(a^*)=-1,$$

$$\Theta^a(a')=0 \qquad \text{for every 1-simplex } a'\neq\pm a^*.$$

Θ^a is a closed difference of $\Gamma(\boldsymbol{K})$ with the singularities q_1 and q_2 with the singular parts $P(\Theta^a, q_1)=-1$ and $P(\Theta^a, q_2)=1$ provided q_1 and q_2 are in the interior of $\boldsymbol{K}$.

Let γ be an arbitrary 1-chain in the interior of $\boldsymbol{K}$. We can write as $\gamma=\sum_{j=1}^{\kappa}x_j a_j$ (a_j: a 1-simplex, x_j: an integer) and then define the *singular difference* Θ^γ *associated to* γ by

$$\Theta^{\gamma}=\sum_{j=1}^{\kappa}x_j\Theta^{a_j}.$$

Let q_1 and q_2 be an arbitrary pair of 0-simplices in the interior of K (and K^*). Then there exists a 1-chain γ such that $\partial\gamma=q_2-q_1$ and the singular difference Θ^{γ} associated to γ gives the closed difference of $\Gamma(\boldsymbol{K})$ with the singularities q_1 and q_2 with $P(\Theta^{\gamma}, q_1)=-1$ and $P(\Theta^{\gamma}, q_2)=1$.

The following lemma is easily verified.

LEMMA 4.1. *Let $\{q_n\}_{n=1}^{\mu}$ and $\{q_n\}_{n=\mu+1}^{\nu}$ be arbitrary finite collections of* 0-*simplices of K and K^* respectively being in the interior of $\boldsymbol{K}$, and α_n $(n=1,\dots,\nu)$ be real or complex numbers. A necessary and sufficient condition in order that there exists a closed difference Θ of $\Gamma(\boldsymbol{K})$ with singularities q_n $(n=1,\dots,\nu)$ with $P(\Theta, q_n)=\alpha_n$ which vanishes along $\partial\boldsymbol{K}$ if $\boldsymbol{K}$ is compact bordered and which is identically zero outside of a finite chain if $\boldsymbol{K}$ is open or closed, is*

$$\sum_{n=1}^{\mu}\alpha_n=0 \qquad \text{and} \qquad \sum_{n=\mu+1}^{\nu}\alpha_n=0. \tag{4.2}$$

The difference Θ of Lemma 4.1 is said to have a *vanishing singular part sum* if it satisfies (4.2).

COROLLARY 4.3. *Let Θ be the difference with a vanishing singular part sum which exists by Lemma* 4.1. *Then, we obtain the following* (i), (ii), (iii):

(i) *there exists a unique harmonic difference σ with the singularities q_n $(n=1,\dots,\nu)$ with the conjugate singular parts*

$$P(\sigma^*, q_n)=\alpha_n \qquad (n=1,\dots,\nu)$$

such that $\sigma\in\Gamma_{e0}(\boldsymbol{K})$;

(ii) *there exists a unique harmonic difference τ with the same singularities at q_n $(n=1,\dots,\nu)$ as Θ such that $\tau-\Theta\in\Gamma_{e0}$;*

(iii) *there exists a unique harmonic difference χ with the analytic singularities at q_n $(n=1,\dots,\nu)$ with the residues* $\mathrm{Res}(\chi, q_n)=\alpha_n/(2\pi i)$ *such that $\chi-\Theta\in\Gamma_{e0}$.*

3. Chains and differences. Let q_1, q_2 be two 0-simplices of K (and K^*) in the interior of $\boldsymbol{K}$, γ be a 1-chain like $\partial\gamma=q_2-q_1$ and Θ^{γ} be the singular difference associated to γ. By (iii) of Corollary 4.3, there exists a unique harmonic difference χ^{γ} with the analytic singularities at q_1, q_2 with $\mathrm{Res}(\chi^{\gamma}, q_n)=P(\Theta^{\gamma}, q_n)/(2\pi i)$ $(n=1, 2)$ such that $\chi^{\gamma}-\Theta^{\gamma}\in\Gamma_{e0}$. Noting the definition in § 1. **4**, we can immediately verify that the period of Θ^{γ} (or χ^{γ}) along any cycle γ' on $\boldsymbol{K}-\{q_1, q_2\}$ is equal to $\gamma\times\gamma'$;

$$\mathfrak{S}_{\gamma'}\Theta^{\gamma}=\mathfrak{S}_{\gamma'}\chi^{\gamma}=\gamma\times\gamma'. \tag{4.3}$$

The periods of χ^{γ} depend only on the homology class of γ (for fixed q_1, q_2) and if $\gamma' \sim \gamma$ then $\chi^{\gamma'} = \chi^{\gamma}$.

Let γ be an arbitrary finite chain in the interior of $\boldsymbol{K}$ and Θ^{γ} be the singular difference associated to γ. Let χ^{γ} be the unique harmonic difference, constructed in Corollary 4.2, for the present Θ^{γ}. We shall use the unique representation

$$\chi^{\gamma} = \phi^{\gamma} + \overline{\psi^{\gamma}},$$

where ϕ^{γ} and ψ^{γ} are analytic except at most the singularities of χ^{γ}. We find that ϕ^{γ} has the same analytic singularities as χ^{γ} while ψ^{γ} is everywhere analytic on $\boldsymbol{K}$. The singularities of χ^{γ} and ϕ^{γ} depend only on the boundary $\partial\gamma$. χ^{γ} and ϕ^{γ} have analytic singularities with the residues equal to the coefficients, divided by $2\pi i$, in $\partial\gamma$. χ^{γ} and ϕ^{γ} are harmonic and analytic respectively on $\boldsymbol{K}$ if and only if γ is a cycle. The mappings $\gamma \to \chi^{\gamma}$, $\gamma \to \phi^{\gamma}$ and $\gamma \to \psi^{\gamma}$ are linear.

4. Reproducing property. Let Θ^{a}, χ^{a}, ϕ^{a} and ψ^{a} be the differences defined in **2** and **3** for a 1-simplex $\gamma = a$ in the interior of $\boldsymbol{K}$. Then for an arbitrary difference ϕ of $\Gamma_a(\boldsymbol{K})$ we have

$$(\phi, \psi^{a}) = (\phi, \overline{\chi^{a} - \phi^{a}}) = (\phi, \overline{\chi^{a}}),$$

for ϕ is analytic on $\boldsymbol{K}$ and $\overline{\phi^{a}}$ is antianalytic on $\boldsymbol{K} - q_1 - q_2$ where $\partial a = q_2 - q_1$. Further, by the orthogonal decomposition (3.21) and the definition of Θ^{a} we have

$$(\phi, \overline{\chi^{a}}) = (\phi, \overline{\Theta^{a}}) = -\phi(a^*) = -i\phi(a),$$

since $\phi \in \Gamma_a$ and $\overline{\chi^{a} - \Theta^{a}} \in \Gamma_{e0}$. Hence we have

$$(\phi, \psi^{a}) = -i\phi(a).$$

Let χ^{γ}, ϕ^{γ} and ψ^{γ} be the differences defined in **3** for a 1-chain γ in the interior of $\boldsymbol{K}$. Then, because of the linearity of ψ^{γ} with respect to γ we have

$$(\phi, \psi^{\gamma}) = -i \int_{\gamma} \phi. \tag{4.4}$$

By a similar method we have

$$(\phi, \phi^{\gamma}) = -i \int_{\gamma} \phi. \tag{4.5}$$

Let ω be an arbitrary difference of Γ_h. By making use of the unique representation $\omega = \phi + \overline{\psi}$ where ϕ and ψ are analytic, we have

$$(\omega, (\chi^{\gamma})^*) = (\phi + \overline{\psi}, (\phi^{\gamma})^* + (\overline{\psi^{\gamma}})^*) = i(\phi, \phi^{\gamma}) - i(\overline{\psi}, \overline{\psi^{\gamma}}).$$

Hence by (4.4), (4.5) we find

(4.6) $$(\omega, (\chi^{\gamma})^*) = \mathfrak{S}_{\gamma} \omega.$$

Now we suppose that γ is a cycle in the interior of $\boldsymbol{K}$. For this case, χ^{γ} is harmonic, and ϕ^{γ} and ψ^{γ} are analytic on $\boldsymbol{K}$. For this case, we can omit the assumption that γ is in the interior of $\boldsymbol{K}$, for χ^{γ}, ϕ^{γ}, ψ^{γ} depend only on the homology class of γ. We find that $\chi^{\gamma} \in \Gamma_{h0}$ and $\phi^{\gamma}, \psi^{\gamma} \in \Gamma_{aS} = \Gamma_a \cap \Gamma_S$ and under these conditions, χ^{γ}, ϕ^{γ}, ψ^{γ} are the unique reproducing differences satisfying (4.6), (4.5), (4.4) respectively. Since χ^{γ} is real and $\phi^{\gamma} - \psi^{\gamma}$ is analytic, we have $\psi^{\gamma} = \phi^{\gamma}$ and $\chi^{\gamma} = 2\,\mathrm{Re}\,\phi^{\gamma} = 2\,\mathrm{Re}\,\psi^{\gamma}$.

By (4.3), (4.6) we find that

(4.7) $$(\chi^{\gamma}, (\chi^{\gamma'})^*) = \mathfrak{S}_{\gamma'} \chi^{\gamma} = \gamma \times \gamma'$$

for cycles γ, γ'.

5. The classical theorems. Let $\boldsymbol{K}$ be a closed complex polyhedron. An analytic difference ϕ with analytic singularities is said to be *of rational type* if every residue of ϕ is an integer and every period of ϕ is an integral multiple of $2\pi i$. We denote the singularities of the difference ϕ of rational type by $q_1, \ldots, q_\nu$ and the residue of ϕ at q_n $(n = 1, \ldots, \nu)$ by m_n respectively. Then the 0-chain $\sum_{n=1}^{\nu} m_n q_n$ is called the *divisor of* ϕ.

Abel's theorem. A necessary and sufficient condition in order that a given 0-*chain A is the divisor of a difference ϕ of rational type, is that there exists a* 1-*chain γ such that $\partial\gamma = A$ and*

$$\mathfrak{S}_{\gamma} \psi = 0$$

for all analytic differences ψ on $\boldsymbol{K}$.

The proof is analogous to the continuous case when we make use the consequence in **4**. We can construct various types of the extensions of Abel's theorem to an open complex polyhedron. It seems that the present theorem has a sense when we consider it as an approximation of Abel's theorem of the continuous case.

Now we shall mention a type of the bilinear relation.

The bilinear relation. Let $\boldsymbol{K}$ be an open complex polyhedron. Let $\omega \in \Gamma_{c0}$ and $\sigma \in \Gamma_{se}$. We suppose that σ has only a finite number of nonzero periods with respect to a given canonical homology basis A_n, B_n, A_n^*, B_n^* of $\boldsymbol{K}$ modulo the ideal boundary. Then we have that

$$(\omega, \sigma^*) = \sum_n \left(\mathfrak{S}_{A_n} \omega \mathfrak{S}_{B_n^*} \bar{\sigma} + \mathfrak{S}_{A_n^*} \omega \mathfrak{S}_{B_n} \bar{\sigma} - \mathfrak{S}_{B_n} \omega \mathfrak{S}_{A_n^*} \bar{\sigma} - \mathfrak{S}_{B_n^*} \omega \mathfrak{S}_{A_n} \bar{\sigma} \right).$$

This is easily proved by making use of the reproducing differences χ^{A_n}, χ^{B_n},

$\chi^{A_n^*}$, $\chi^{B_n^*}$ defined in **4**.

§ 5. The theory of harmonic functions and analytic functions.

1. Harmonic functions. In the present section, as an application of the method of orthogonal projection of § 3 and Green's formula (3.12), several analogies with the classical theory of harmonic and analytic functions are developed (cf. Courant-Friedrichs-Lewy [4], Blanc [2], [3], Lelong-Ferrand [10], [11], Isaacs [8], Duffin [5], Hundhausen [6], etc.).

Let $\boldsymbol{K}=<K, K^*>$ be a complex polyhedron. We can state the *maximum and minimum principle of a harmonic function on* $\boldsymbol{K}$ as follows: The restriction to K (K^* resp.) of a real harmonic function u on $\boldsymbol{K}$ does not take the maximum or minimum value in the interior of K (K^* resp.) provided the restriction is non-constant in the interior, and if $\boldsymbol{K}$ is compact bordered then the maximum and minimum values are taken on ∂K (∂K^* resp.). This follows immediately from the definition of a harmonic function.

The following statement is also obvious: *if a sequence* $\{u_n\}_{n=1}^{\infty}$ *of harmonic functions on* $\boldsymbol{K}$ *pointwise converges, then the limit function* u *is harmonic.*

Let us suppose that $\boldsymbol{K}$ is compact bordered. Given a function f on the complex boundary $\boldsymbol{\partial K}$, the problem to find the unique harmonic function which takes the boundary value f on $\boldsymbol{\partial K}$, is called the *Dirichlet problem.* Concerning the *Neumann problem* we interpret it as the problem to find the harmonic function u uniquely determined except for an additive constant satisfying $\Delta u^*=\Theta$ along $\boldsymbol{\partial K}$ for an arbitrarily given 1-difference Θ under the condition

$$\text{(5.1)} \qquad \mathfrak{S}_{\partial K}\Theta=0 \quad \text{and} \quad \mathfrak{S}_{\partial K^{*+}}\Theta=0.$$

The existence of the solution of the Dirichlet problem is assured by the orthogonal decomposition (3.22), and further by (3.22) we know that the so-called *Dirichlet principle* holds.

Now we shall verify the existence of the solution of the Neumann problem. If, for a 1-difference Θ given under (5.1), we can construct a closed difference σ with the same boundary behavior as Θ: $\sigma=\Theta$ along $\boldsymbol{\partial K}$, then by the orthogonal decompositions (3.21), (3.24) we have the representation

$$-\sigma^*=\omega_{he}+\omega_{h0}^*+\omega_{e0}^*,$$

where $\omega_{he}\in\Gamma_{he}$, $\omega_{h0}^*\in\Gamma_{h0}^*$ and $\omega_{e0}^*\in\Gamma_{e0}^*$, and then the component ω_{he} gives the solution Δu.

We shall construct the closed difference σ. Let $\{c_j, c_j^*\}_{j=1}^{\nu}$ ($\nu<\infty$) be a canonical homology basis of dividing cycles of $\boldsymbol{K}$, and d_j^* and d_j ($j=1,\dots,\nu$) be conjugate relative cycles of c_j and c_j^* respectively. We set

(5.2) $$\mathfrak{S}_{c_j}\Theta=\alpha_j \quad \text{and} \quad \mathfrak{S}_{c_j^*}\Theta=\alpha_j^* \quad (j=1,\dots, \nu).$$

If all α_j and α_j^* vanish, then σ is easily constructed. Otherwise,

$$\tau=\sum_{j=1}^{\nu}(\alpha_j\Theta^{d_j^*}+\alpha_j^*\Theta^{d_j})$$

is a closed difference with the periods $-\alpha_j$ and $-\alpha_j^*$ along c_j and c_j^* respectively where $\Theta^{d_j^*}$ and Θ^{d_j} are the singular differences associated to d_j^* and d_j respectivey thus $\Theta+\tau$ has the vanishing period along every c_j and c_j^*, and thus the problem is reduced to the former case.

2. Green function. The *Green difference* $\Delta g=\Delta g_{q_0}$ of a compact bordered or open complex polyhedron $\boldsymbol{K}$ with a singularity q_0 is defined as a 1-difference on $\boldsymbol{K}$ satisfying the following conditions (i), (ii):

(i) Δg is a harmonic difference with an only singularity q_0 with the conjugate singular part $P(\Delta g^*, q_0)=-2\pi$;

(ii) $\Delta g\in\Gamma_{e0}$.

The *Green function* $g=g_{q_0}=g(q, q_0)$ with the singularity q_0 is the function whose difference is the Green difference with the singularity q_0 which has the vanishing boundary value if $\boldsymbol{K}$ is compact bordered and which is the pointwise limit of functions f_n with compact support such that $\|\Delta g-\Delta f_n\|\to 0$ $(n\to\infty)$ if $\boldsymbol{K}$ is open.

Obviously, if for a given $\boldsymbol{K}$ there exists the Green function g on $\boldsymbol{K}$, then it is unique. Further we find that if $q_0\in K$ ($q_0\in K^*$ resp.) then the support of g is K (K^* resp.).

If $\boldsymbol{K}$ is compact bordered, then the Green function of $\boldsymbol{K}$ always exists and can be constructed as follows. Let γ be a 1-chain on $\boldsymbol{K}$ such that $\partial\gamma=q_0-q_1$ where q_1 is a 0-simplex of ∂K or a 0-simplex whose carrier is outside of $|K|$. Then the singular difference Θ^γ associated to γ is a closed difference with the only closed singularity q_0 with the singular part $P(\Theta^\gamma, q_0)=1$. The difference σ which is constructed in Corollary 4.1 for $\Theta=-2\pi\Theta^\gamma$, gives Δg. We can also verify the existence of the Green function by making use of (i) of Corollary 4.3 for the double $\hat{\boldsymbol{K}}$ of $\boldsymbol{K}$. The Green function g is positive in the interior of $\boldsymbol{K}$ by the maximum principle.

Let $\boldsymbol{K}$ be open, $\{\boldsymbol{K}_n\}_{n=0}^\infty$ be a regular exhaustion of $\boldsymbol{K}$ and $g_{\boldsymbol{K}_n}=g_{\boldsymbol{K}_n}(q, q_0)$ $(n=0, 1,\dots;\ q_0\in\boldsymbol{K}_0)$ be the Green function of $\boldsymbol{K}_n$. Then $g_{\boldsymbol{K}_n}$ is monotone increasing with n. Hence there exists the pointwise limit $\lim_{n\to\infty}g_{\boldsymbol{K}_n}$ which is everywhere finite or identically infinite. There exists the Green function g of $\boldsymbol{K}$ with the singularity q_0 and $g\equiv\lim_{n\to\infty}g_{\boldsymbol{K}_n}$ if and only if $\lim_{n\to\infty}g_{\boldsymbol{K}_n}<\infty$. By (i) of Corollary 4.3, the existence of the Green function does not depend on a particular choice of the singularity q_0 in K (or K^*).

3. Neumann function. The *Neumann difference* $\Delta n = \Delta n_{q_0}$ of a compact bordered complex polyhedron $\boldsymbol{K}$ with a singularity q_0 is defined as a 1-difference satisfying the following conditions (i), (ii):

(i) Δn is an exact harmonic difference with the only singularity q_0 with the conjugate singular part $P(\Delta n^*, q_0) = -2\pi$;

(ii) $\Delta n^* \equiv c_1$ along ∂K and $\Delta n^* \equiv c_2$ along ∂K^{*+}, where if $q_0 \in K$ then $c_1 = 0$ and $c_2 = -2\pi/\nu$, if $q_0 \in K^*$ then $c_1 = -2\pi/\nu$ and $c_2 = 0$, and ν is the number of 1-simplices of ∂K. The *Neumann function* $n = n_{q_0} = n(q, q_0)$ is the function determined except for an additive constant whose difference is Δn. We shall normalize the Neumann function by the condition

$$\mathfrak{S}_{\lambda} n_{q_0} = 0 \qquad \text{and} \qquad \mathfrak{S}_{\lambda^*} n_{q_0} = 0, \tag{5.3}$$

where λ and λ^* denote the 0-chains being the sum of all 0-simplices of ∂K and ∂K^{*+}, respectively. The uniqueness of the Neumann difference follows from $\Delta n - \Delta n' \in \Gamma_{he} \cap \Gamma^*_{h0}$ for another one $\Delta n'$. If $q_0 \in K$ ($q_0 \in K^*$ resp.) then the support of n is K (K^* resp.).

The existence of the Neumann difference is verified as follows. Let Δg be the Green difference with the singularity q_0. We construct the solution u of the Neumann problem satisfying the boundary condition: $\Delta u^* = c_1 - \Delta g^*$ along ∂K and $\Delta u^* = c_2 - \Delta g^*$ along ∂K^{*+}. Then $n = g + u$ gives the Neumann function.

4. The properties of Green function. *Let $\boldsymbol{K}$ be a compact bordered complex polyhedron and Δg_q be the Green difference of $\boldsymbol{K}$ with the singularity q. Then the solution u of the Dirichlet problem of* **1** *is given by*

$$u(q) = -\frac{1}{2\pi} \mathfrak{S}_{\partial K} f \Delta g_q^* \tag{5.4}$$

for each 0*-simplex q in the interior of $\boldsymbol{K}$.*

Let M^* be the conjugate 2-simplex of q. By Green's formula (3.13) we have

$$\mathfrak{S}_{\partial K} (u \Delta g_q^* - g_q \Delta u^*) = \mathfrak{S}_{K} (u \Delta(\Delta g_q^*) - g_q \Delta(\Delta u^*)). \tag{5.5}$$

The left-hand side of (5.5) is equal to

$$\mathfrak{S}_{\partial K} u \Delta g_q^*,$$

since $g_q = 0$ on $\partial \boldsymbol{K}$. The right-hand side of (5.5) is equal to

$$u\Delta(\Delta g_q^*)(M^*) = u(q) P(\Delta g_q^*, q) = -2\pi u(q),$$

since Δg_q^* and Δu^* are closed on $\boldsymbol{K} - M^*$ and $\boldsymbol{K}$ respectively.

The symmetric property of the Green function:

$$g(q, r)=g(r, q) \tag{5.6}$$

holds.

If $\boldsymbol{K}$ is compact bordered, then (5.6) follows from Green's formula

$$\int_{\partial \boldsymbol{K}}(g_q\Delta g_r^* - g_r\Delta g_q^*)=\int_{\boldsymbol{K}}(g_q\Delta(\Delta g_r^*)-g_r\Delta(\Delta g_q^*)).$$

If $\boldsymbol{K}$ is open, then (5.6) is obtained by a limit process of the compact bordered case.

5. Harnack's inequality. Let u be a non-negative harmonic function on a compact bordered complex polyhedron $\boldsymbol{K}$. Let q_0 and q be a fixed and a generic 0-simplices respectively in the interior of the common K or K^*. We set

$$M\equiv M(q, q_0, \boldsymbol{K})=\max_{a\in\gamma}\frac{\Delta g_q^*(a)}{\Delta g_{q_0}^*(a)},$$

$$\mu\equiv\mu(q, q_0, \boldsymbol{K})=\min_{a\in\gamma}\frac{\Delta g_q^*(a)}{\Delta g_{q_0}^*(a)},$$

where $\gamma=\{a|\ a\in\partial\boldsymbol{K},\ \Delta g_{q_0}^*(a)\neq 0\}$. Here M and μ are positive numbers depending on only $\boldsymbol{K}$, q_0 and q. If we note that $\Delta g_{q_0}^*$ and Δg_q^* simultaneously vanish on $\partial\boldsymbol{K}$, then by (5.4) we have

$$u(q)=-\frac{1}{2\pi}\int_{\partial\boldsymbol{K}}u\Delta g_q^*\leqq -\frac{M}{2\pi}\int_{\partial\boldsymbol{K}}u\Delta g_{q_0}^*=Mu(q_0).$$

With an analogous inequality we obtain *Harnack's inequality* (cf. Duffin [5])

$$\mu u(q_0)\leqq u(q)\leqq Mu(q_0). \tag{5.7}$$

6. The properties of Neumann function. *Let $\boldsymbol{K}$ be a compact bordered complex polyhedron and n_q be the Neumann function of $\boldsymbol{K}$ with a singularity q. Then the solution u of the Neumann problem of* **1** *is given by*

$$u(q)=\frac{1}{2\pi}\int_{\partial\boldsymbol{K}}n_q\Theta+c, \tag{5.8}$$

where

$$c=\begin{cases}\dfrac{1}{\nu}\displaystyle\int_{\lambda}u & (q\in K),\\[2ex] \dfrac{1}{\nu}\displaystyle\int_{\lambda^*}u & (q\in K^*),\end{cases}$$

λ and λ^* are the 0-chains being the sum of all 0-simplices of ∂K and ∂K^{*+} respectively, and ν is the number of $q \in \lambda$.

By Green's formula (3.13) we have

$$\mathfrak{S}_{\boldsymbol{\partial K}}(u \Delta n_q^* - n_q \Delta u^*) = \mathfrak{S}_{\boldsymbol{K}}(u \Delta(\Delta n_q^*) - n_q \Delta(\Delta u^*)). \tag{5.9}$$

Here

$$\mathfrak{S}_{\boldsymbol{\partial K}} u \Delta n_q^* = c_1 \mathfrak{S}_{\lambda^*} u + c_2 \mathfrak{S}_{\lambda} u,$$

and the right-hand side of (5.9) is equal to $-2\pi u(q)$.

The symmetric property of the Neumann function:

$$n(q, r) = n(r, q) \tag{5.10}$$

holds.

It follows from Green's formula

$$\mathfrak{S}_{\boldsymbol{\partial K}}(n_q \Delta n_r^* - n_r \Delta n_q^*) = \mathfrak{S}_{\boldsymbol{K}}(n_q \Delta(\Delta n_r^*) - n_r \Delta(\Delta n_q^*)).$$

7. Cauchy's summation theorem. *Let $\boldsymbol{K}$ be a complex polyhedron, $\boldsymbol{\gamma}$ be a complex cycle on $\boldsymbol{K}$ such that $\boldsymbol{\gamma} \sim 0$, f be an analytic function on $\boldsymbol{K}$ and ϕ be an analytic difference on $\boldsymbol{K}$. Then we have*

$$\mathfrak{S}_{\boldsymbol{\gamma}} f\phi = 0. \tag{5.11}$$

We may suppose that $\boldsymbol{\gamma}$ is the complex boundary of a compact bordered polyhedron $\boldsymbol{K}_0$: $\boldsymbol{\partial K}_0 = \boldsymbol{\gamma}$, Then by Green's formula (3.12) we have

$$\mathfrak{S}_{\boldsymbol{\gamma}} f\phi = -(\Delta f, \bar{\phi}^*)_{\boldsymbol{K}_0} = 0,$$

since Δf is analytic and $\bar{\phi}^*$ is antianalytic.

Remark. We can also verify (5.11) by an immediate calculation of

$$\mathfrak{S}_{\boldsymbol{\partial M}} f\phi$$

for a 2-simplex M, where $\boldsymbol{\partial M}$ means the complex cycle being the sum of ∂M and its dual cycle.

Now let us suppose that $\boldsymbol{K}$ is latticed. Let a be a fixed 1-simplex of $\boldsymbol{K}$. Then we can uniquely define the 1-difference ε on $\boldsymbol{K}$ by the following conditions (i), (ii):

(i) $\varepsilon(a)=1$;

(ii) For any 2-simplex $M \in \boldsymbol{K}$, let $a_1, .., a_{4n}$ be a cyclic sequence of edges of M. If $\varepsilon(a_1)=1$, then

$$\begin{array}{llll} \varepsilon(a_{4j-3})=1, & \varepsilon(a_{4j-2})=0, & \varepsilon(a_{4j-1})=-1, & \varepsilon(a_{4j})=0, \\ \varepsilon(a^*_{4j-3})=0, & \varepsilon(a^*_{4j-2})=-1, & \varepsilon(a^*_{4j-1})=0, & \varepsilon(a^*_{4j})=1 \\ & & & (j=1,\dots, n). \end{array}$$

The difference ε is harmonic and thus $\delta=\varepsilon+i\varepsilon^*$ is an analytic difference. The differences ε and δ are called the *uniformizing harmonic* and *analytic differences* of $\boldsymbol{K}$ respectively.

Cauchy's summation theorem (cf. Lelong-Ferrand [11], Duffin [5], Hundhausen [6]): *Let $\boldsymbol{K}$ be a latticed complex polyhedron, $\boldsymbol{\gamma}$ be a complex cycle on $\boldsymbol{K}$ such that $\boldsymbol{\gamma}\sim 0$ and f be an analytic function on $\boldsymbol{K}$. Then we have*

$$\mathfrak{S}_{\gamma} f\delta=0. \tag{5.12}$$

8. Cauchy's summation formula, Residue theorem. *Cauchy's summation formula* (cf. Duffin [5]): *Let $\boldsymbol{K}$ be a compact bordered complex polyhedron, f be an analytic function on $\boldsymbol{K}$ and ϕ be an analytic difference with the singularity q of the residue* 1. *Then we have*

$$f(q)=\frac{1}{2\pi i}\mathfrak{S}_{\partial \boldsymbol{K}} f\phi. \tag{5.13}$$

By Green's formula (3.12) we have

$$\begin{aligned}\mathfrak{S}_{\partial \boldsymbol{K}} f\phi &= -(\Delta f, \bar{\phi}^*)+\mathfrak{S}_{\boldsymbol{K}} f\Delta\phi \\ &= f\Delta\phi(M^*)=f(q)\cdot P(\phi, q)=2\pi i f(q),\end{aligned}$$

where M^* is the conjugate 2-simplex of q.

In (5.13), we can choose the analytic Green difference $\psi_q=\Delta g_q+i(\Delta g_q)^*$ as $-\phi$. Then we have

$$f(q)=-\frac{1}{2\pi i}\mathfrak{S}_{\partial \boldsymbol{K}} f\psi_q. \tag{5.14}$$

Residue theorem: *Let $\boldsymbol{K}$ be a compact bordered complex polyhedron and ϕ be an analytic difference on $\boldsymbol{K}$ with the singularities q_n $(n=1,\dots, \nu)$ with the residues b_n. Then we have*

$$\mathfrak{S}_{\partial \boldsymbol{K}} \phi=2\pi i\sum_{n=1}^{\nu} b_n. \tag{5.15}$$

The proof is similar to that of (5.13).

Chapter II A finite-difference method on a Riemann surface.

§ 6. The convergence of differences with respect to subdivisions.

1. Natural extension of a difference. Let $\boldsymbol{K}=<K, K^{*}>$ be a complex polyhedron such that K is quadratic and $\boldsymbol{K}_1=<K_1, K_1^{*}>$ be a subdivision of $\boldsymbol{K}$. Let σ be a difference of $\Gamma_c(\boldsymbol{K})$ with support K. We shall define the *natural extension* $\sigma^{\natural}$ *of* σ *to the subdivision* $\boldsymbol{K}_1$ as follows: with the notations in § 1.**6**, for the subdivision of each $M \in K$ we set

$$\text{(i)} \qquad \sigma^{\natural}_{j1}=\sigma^{\natural}_{j2}=\frac{1}{2}\sigma_j \qquad (j=1,\dots, 4);$$

$$\text{(ii)} \qquad \sigma^{\natural}_{j3}=\frac{1}{4}(\sigma_{j+1}-\sigma_{j-1}) \qquad (j=1,\dots, 4 \ (\mathrm{mod}\, 4)),$$

where $\sigma_j=\sigma(a_j)$, $\sigma^{\natural}_{jk}=\sigma^{\natural}(a_{jk})$, and we set $\sigma^{\natural}=0$ on K_1^{*}. Then we see that $\sigma^{\natural} \in \Gamma_c(\boldsymbol{K}_1)$ (cf. Lemma 6.1).

Let $\{\boldsymbol{K}_n=<K_n, K_n^{*}>\}_{n=0}^{\infty}$ be a sequence of complex polyhedra such that K_0 is quadratic and $\boldsymbol{K}_n$ is a subdivision of $\boldsymbol{K}_{n-1}$ respectively. Let σ be a difference of $\Gamma_c(\boldsymbol{K}_0)$ with support K_0. Let $\sigma^{\natural(n)}$ $(n=1, 2,\dots)$ be the natural extension of $\sigma^{\natural(n-1)}$ to $\boldsymbol{K}_n$ where $\sigma^{\natural(0)}=\sigma$. Then we shall define the *natural extension* $\sigma^{\natural}$ *of* σ *to* $\{\boldsymbol{K}_n\}_{n=0}^{\infty}$ as an operator which maps σ to the 1-difference $\sigma^{\natural(n)}$ on $\boldsymbol{K}_n$ for each n, that is, as an operator such that $\sigma^{\natural}=\sigma^{\natural(n)}$ on $\boldsymbol{K}_n$ for each n. We shall use the common notation $\sigma^{\natural}$ for the both definitions of the natural extension.

Lemma 6.1.

$$\|\sigma\|^2_{\boldsymbol{K}_0}-\|\sigma^{\natural}\|^2_{\boldsymbol{K}_n}=\frac{1}{3}\left(1-\frac{1}{4^n}\right)\sum_{M\in K_0}|\sigma_1+\sigma_3|^2,$$

where $\sigma_j=\sigma(a_j)$ *and* a_1, a_2, a_3, a_4 *is a cyclic sequence of edges of* M.

Proof. When we note that $-\sigma^{*}\bar{\sigma}(M)=(1/2)\sum_{j=1}^{4}|\sigma_j|^2$, we have

$$\begin{aligned}\mathfrak{S}_{M^1}(-\sigma^{\natural *}\overline{\sigma^{\natural}}) &= -\sum_{j=1}^{4}\sigma^{\natural *}\overline{\sigma^{\natural}}(M_j)\\ &=\frac{1}{2}\sum_{j=1}^{4}(|\sigma^{\natural}_{j1}|^2+|\sigma^{\natural}_{j2}|^2+2|\sigma^{\natural}_{j3}|^2)\\ &=\frac{1}{2}\left(\frac{1}{2}\sum_{j=1}^{4}|\sigma_j|^2+\frac{1}{8}\sum_{j=1}^{4}|\sigma_{j+1}-\sigma_{j-1}|^2\right),\end{aligned}$$

where $M^1=\sum_{j=1}^{4}M_j$ and $M_1,\dots, M_4$ are the 2-simplices defined in § 1. **6**. Because of $\sigma_1+\sigma_2+\sigma_3+\sigma_4=0$, we have

$$-\sigma^*\bar{\sigma}(M)-\iint_{M^1}(-\sigma^{\natural *}\overline{\sigma^{\natural}})$$
$$=\frac{1}{16}\Big(2\sum_{j=1}^{4}|\sigma_j|^2+\sum_{j=1}^{4}(\sigma_{j-1}\bar{\sigma}_{j+1}+\bar{\sigma}_{j-1}\sigma_{j+1})\Big)$$
$$=\frac{1}{16}\sum_{j=1}^{4}|\sigma_{j-1}+\sigma_{j+1}|^2=\frac{1}{4}|\sigma_1+\sigma_3|^2\geqq 0.$$

Hence

$$\|\sigma\|_{K_0}^2-\|\sigma^{\natural}\|_{K_1}^2=\iint_{K_0}(-\sigma^*\bar{\sigma})-\iint_{K_1}(-\sigma^{\natural *}\overline{\sigma^{\natural}})$$
$$=\frac{1}{4}\sum_{M\in K_0}|\sigma_1+\sigma_3|^2\geqq 0.$$

By a further elementary calculation we have

$$\|\sigma^{\natural}\|_{K_n}^2-\|\sigma^{\natural}\|_{K_{n+1}}^2=\frac{1}{4^{n+1}}\sum_{M\in K_0}|\sigma_1+\sigma_3|^2.$$

2. Norm convergence with respect to subdivision. With the notation in **1**, let σ^n $(n=0, 1, .\ )$ be a difference of $\Gamma_c(\boldsymbol{K}_n)$ with support K_n. By $\sigma^{n\natural}$ we denote the natural extension of σ^n to $\{\boldsymbol{K}_m\}_{m=n}^{\infty}$.

Lemma 6.2. *If the orthogonality*

$$(\sigma^n-\sigma^{m\natural},\ \sigma^n)_{\boldsymbol{K}_n}=0 \tag{6.1}$$

holds for every m, n $(n>m)$, then the following hold;

(i) $\|\sigma^n\|_{\boldsymbol{K}_n}$ *is monotone decreasing with n;*

(ii) $\lim\limits_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_n}=\lim\limits_{m,n\to\infty}\|\sigma^{m\natural}\|_{\boldsymbol{K}_n}$;

(iii) $\lim\limits_{m,n\to\infty}\|\sigma^n-\sigma^{m\natural}\|_{\boldsymbol{K}_n}=0$;

(iv) $\lim\limits_{n\to\infty}\sum\limits_{M\in K_n}|\sigma_1^n+\sigma_3^n|^2=0 \qquad (\sigma_j^n=\sigma^n(a_j))$.

Proof. By Lemma 6.1 we have

$$\|\sigma^m\|_{\boldsymbol{K}_m}^2-\|\sigma^{m\natural}\|_{\boldsymbol{K}_n}^2=\frac{1}{3}\Big(1-\frac{1}{4^{n-m}}\Big)\sum_{M\in K_m}|\sigma_1^m+\sigma_3^m|^2 \qquad (n>m). \tag{6.2}$$

(6.1) and (6.2) imply that

$$\|\sigma^n-\sigma^{m\natural}\|_{\boldsymbol{K}_n}^2=\|\sigma^{m\natural}\|_{\boldsymbol{K}_n}^2-\|\sigma^n\|_{\boldsymbol{K}_n}^2\leqq\|\sigma^m\|_{\boldsymbol{K}_m}^2-\|\sigma^n\|_{\boldsymbol{K}_n}^2.$$

Hence we have (i), (ii) and (iii). (ii) and (6.2) imply (iv).

Corollary 6.1. *If*

(i) $\sigma^n \in \Gamma_h(\boldsymbol{K}_n)$ *and* $\sigma^n - \sigma^{0\natural} \in \Gamma_{e0}(\boldsymbol{K}_n)$ $(n=0, 1,\dots)$,

(ii) $\sigma^n \in \Gamma^*_{hm}(\boldsymbol{K}_n)$ *and* $\sigma^n - \sigma^{0\natural} \in \Gamma_{se}(\boldsymbol{K}_n)$ $(n=0, 1,\dots)$,

or

(iii) $\sigma^n \in \Gamma^*_{h0}(\boldsymbol{K}_n) \cap \Gamma_{hse}(\boldsymbol{K}_n)$ *and* $\sigma^n - \sigma^{0\natural} \in \Gamma_e(\boldsymbol{K}_n)$ $(n=0, 1,\dots)$,

then (i), (ii), (iii) *and* (iv) *of Lemma* 6.2 *hold.*

§ 7. The approximation of a differential on the Riemann surface based on a normal quadrangulation.

1. Smooth extension of a difference. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be a normal complex polyhedron and W be the Riemann surface based on the normal quadrangulation K. Let σ be a difference of $\Gamma_c(\boldsymbol{K})$. For each quadrangular 2-simplex $M = [M^e, \phi]$ of $\boldsymbol{K}$ we can choose the mapping ϕ so that the normal coordinates of M are preserved and M^e is the square on the z-plane with $q_1^e = 0$, $q_2^e = 1$, $q_3^e = 1+i$, $q_4^e = i$, where $\partial M = \sum_{j=1}^4 a_j$, $\partial a_j = q_{j+1} - q_j$, $q_j^e = \phi^{-1}(q_j)$ $(j=1,\dots,4;\ q_5 = q_1)$. Then we define the *smooth extension* $\sigma^\#$ *of* σ *to* $|M|$ by the differential $\sigma^\#$ on $|M|$ satisfying

$$\sigma^\# = ((1-y)\sigma_1 - y\sigma_3)dx + (x\sigma_2 - (1-x)\sigma_4)dy \qquad (z = x+iy) \tag{7.1}$$

for the local uniformizing parameter $z = \phi^{-1}(p)$ of M where $\sigma_j = \sigma(a_j)$. Then we see that $\int_{a_j} \sigma^\# = \sigma_j$ $(j=1,\dots,4)$ on taking a_j as an oriented curve.

Let σ be a difference of $\Gamma_c(\boldsymbol{K})$ with support K. Then we can define the *smooth extension* $\sigma^\#$ *of* σ *to the Riemann surface* W by the differential on W which is the smooth extension $\sigma^\#$ of σ on each $M \in K$. Here the coefficients α, β of $\sigma^\# = \alpha dx + \beta dy$ are generally discontinuous at each point of the carrier $|a|$ of a 1-simplex $a \in K$. Then we define the coefficients α and β on $|a|$ by

$$\alpha(p_0) = \frac{1}{2}\Big(\lim_{p\to p_0,\, p\in|M_1|^\circ} \alpha(p) + \lim_{p\to p_0,\, p\in|M_2|^\circ} \alpha(p)\Big),$$

$$\beta(p_0) = \frac{1}{2}\Big(\lim_{p\to p_0,\, p\in|M_1|^\circ} \beta(p) + \lim_{p\to p_0,\, p\in|M_2|^\circ} \beta(p)\Big)$$

for a fixed local uniformizing parameter about $p_0 \in |a|$, where M_1 and M_2 are the 2-simplices with the common edge a and the interior of $|M_j|$ is denoted by $|M_j|^\circ$. Clearly the smooth extension $\sigma^\#$ defines a *closed* differential on W.[1)]

Let K_Q^* be the 2-chain defined as the sum of quadrangular 2-simplices of

1) Cf. Ch. V of Ahlfors & Sario [1] for the theory of differentials on a Riemann surface.

K^* and W_Q be the subset of W which is the carrier of K_Q^*. Let σ be a difference of $\Gamma_c(\boldsymbol{K})$ with support K^*. Then we can define the *smooth extension* $\sigma^\#$ *of* σ *to the subset* W_Q by the differential on W_Q which is the smooth extension $\sigma^\#$ of σ on each $M \in K_Q^*$.

Let σ be a generic difference of $\Gamma_c(\boldsymbol{K})$. Let σ_K and σ_{K^*} be the restrictions of σ to K and K^* respectively, and let $\sigma_K^\#$ and $\sigma_{K^*}^\#$ be the smooth extensions of σ_K and σ_{K^*} to W and W_Q respectively. By the differential $\sigma^\# = \sigma_K^\# + \sigma_{K^*}^\#$ on W_Q, we define the *smooth extension of* σ *to the subset* W_Q. $\sigma^\#$ defines a closed differential on W_Q.

2. The relation between σ and $\sigma^\#$.

LEMMA 7.1. *Let* σ, σ^1 *and* σ^2 *be differences of* $\Gamma_c(\boldsymbol{K})$ *with support* K. *Then the relations*

$$\|\sigma\|_{\boldsymbol{K}}^2 - \|\sigma^\#\|_W^2 = \frac{1}{3}\sum_{M \in K} |\sigma_1 + \sigma_3|^2 \geqq 0, \tag{7.2}$$

$$\|\sigma^\#\|_W^2 - \frac{1}{3}\|\sigma\|_{\boldsymbol{K}}^2 = \frac{1}{6}\sum_{M \in K} (|\sigma_1 - \sigma_3|^2 + |\sigma_2 - \sigma_4|^2) \geqq 0, \tag{7.3}$$

$$|(\sigma^1, \sigma^2)_{\boldsymbol{K}} - (\sigma^{1\#}, \sigma^{2\#})_W| \leqq ((\|\sigma^1\|_{\boldsymbol{K}}^2 - \|\sigma^{1\#}\|_W^2)(\|\sigma^2\|_{\boldsymbol{K}}^2 - \|\sigma^{2\#}\|_W^2))^{1/2} \tag{7.4}$$

hold, where by $(\ , \ )_W$ *and* $\| \ \|_W$ *we denote the inner product and the norm respectively in the Hilbert space of differentials on* W,[1] *and* $\sigma_j = \sigma(a_j)$ *for a cyclic sequence* a_1, a_2, a_3, a_4 *of edges of* M.

PROOF. For each $M \in K$, since $\sigma^{1\#}$ is harmonic on $|M|$, there exists a harmonic function u^1 on $|M|$ such that $du^1 = \sigma^{1\#}$. Hence

$$\int_{|M|} \sigma^{1\#}\overline{\sigma^{2\#*}} = \int_{\partial|M|} u^1 \overline{\sigma^{2\#*}}, \tag{7.5}$$

where $\partial|M|$ means the boundary of the region $|M|$ and the conjugate differential of $\sigma^{2\#}$ is denoted by $\sigma^{2\#*}$. By making use of (7.5) and $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0$ we can carry out the calculation

$$\int_{|M|} \sigma^{1\#}\overline{\sigma^{2\#*}}$$

$$= -\int_0^1 \sigma_1^1 x(x\overline{\sigma_2^2} - (1-x)\overline{\sigma_4^2})dx + \int_0^1 (\sigma_1^1 + \sigma_2^1 y)((1-y)\overline{\sigma_1^2} - y\overline{\sigma_3^2})dy$$

1) We shall use the common notations $(\ , \ )$ and $\| \ \|$ for both inner products and both norms of differences and differentials. If any confusion may occur, then we shall add the suffices $\boldsymbol{K}$ and W etc. like $(\ , \ )_{\boldsymbol{K}}$, $(\ , \ )_W$, $\| \ \|_{\boldsymbol{K}}$, $\| \ \|_W$.

$$+\int_0^1 \sigma_4^1 y((1-y)\overline{\sigma_1^2}-y\overline{\sigma_3^2})dy-\int_0^1(\sigma_4^1+\sigma_3^1 x)(x\overline{\sigma_2^2}-(1-x)\overline{\sigma_4^2})dx$$

$$=\frac{1}{6}\Big(\sum_{j=1}^4 \sigma_j^1\overline{\sigma_j^2}+(\sigma_1^1-\sigma_3^1)(\overline{\sigma_1^2}-\overline{\sigma_3^2})+(\sigma_2^1-\sigma_4^1)(\overline{\sigma_2^2}-\overline{\sigma_4^2})\Big),$$

where $\sigma_j^k=\sigma^k(a_j)$. Hence, when we note that $-\sigma^1{}^*\overline{\sigma^2}(M)=(1/2)\sum_{j=1}^4\sigma_j^1\overline{\sigma_j^2}$, we have

$$-\sigma^1{}^*\overline{\sigma^2}(M)-\int_{|M|}\sigma^{1\#}\overline{\sigma^{2\#*}}$$

$$=\frac{1}{6}((\sigma_1^1+\sigma_3^1)(\overline{\sigma_1^2}+\overline{\sigma_3^2})+(\overline{\sigma_2^1}+\overline{\sigma_4^1})(\overline{\sigma_2^2}+\overline{\sigma_4^2}))$$

$$=\frac{1}{3}(\sigma_1^1+\sigma_3^1)(\overline{\sigma_1^2}+\overline{\sigma_3^2}).$$

Therefore,

$$(\sigma^1,\sigma^2)_{\boldsymbol{K}}-(\sigma^{1\#},\sigma^{2\#})_W=\mathfrak{S}_K(-\sigma^1{}^*\overline{\sigma^2})-\int_W\sigma^{1\#}\overline{\sigma^{2\#*}}$$

$$=\frac{1}{3}\sum_{M\in K}(\sigma_1^1+\sigma_3^1)(\overline{\sigma_1^2}+\overline{\sigma_3^2}).$$

Hence we obtain (7.2) and further by Schwarz's inequality (7.4). Similarly, (7.3) is obtained.

Corollary 7.1. *A closed difference σ on $\boldsymbol{K}$ with support K belongs to $\Gamma_c(\boldsymbol{K})$ if and only if the smooth extension $\sigma^\#$ of σ is a differential of the space $\Gamma_c(W)$ of closed differentials on W with finite norm.*[1]

For the differences of $\Gamma_c(\boldsymbol{K})$ with support K^* we can also obtain the same results as (7.2), (7.3), (7.4) replacing K, W, $\|\sigma\|_{\boldsymbol{K}}^2$ and $(\sigma^1,\sigma^2)_{\boldsymbol{K}}$ by K_Q^*, W_Q, $\mathfrak{S}_{K_Q^*}(-\sigma^*\bar{\sigma})$ and $\mathfrak{S}_{K_Q^*}(-\sigma^1{}^*\overline{\sigma^2})$ respectively.

3. Courant-Friedrichs-Lewy's Lemma. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be a complex polyhedron. A 2-chain $Q\subset K$ is called a *quadrate* of K if there exists a one-to-one bicontinuous mapping ϕ of a square Q^e on the z-plane ($z=x+iy$) onto Q such that each $M\in Q$ is the image of a square M^e by ϕ, *i.e.* $M=[M^e,\phi]$. Here we may assume that each side of the square Q^e is parallel to either x-axis or y-axis. Let $\{Q_j\}_{j=0}^{\nu+1}$ $(\nu\geqq 1)$ be an increasing sequence of concentric quadrates of K such that Q_j $(j=1,\dots,\nu+1)$ is the minimum quadrate under $|Q_{j-1}|\subset|Q_j|^\circ$.

1) We shall use the common notation Γ with some suffix for both spaces of differences and differentials with finite norm. If any confusion may occur, then we shall indicate the polyhedron $\boldsymbol{K}$ and the Riemann surface W like $\Gamma(\boldsymbol{K})$ and $\Gamma(W)$ respectively.

Then, $\{\boldsymbol{Q}_j = <Q_j, Q_j^*>\}_{j=0}^{\nu+1}$ forms a sequence of compact bordered complex polyhedrons, where Q_j^* is the conjugate polyhedron of Q_j for each j. By a_x (a_y resp.) we denote an oriented 1-simplex of $Q_{\nu+1}$ such that $a_x^e = \phi^{-1}(a_x)$ ($a_y^e = \phi^{-1}(a_y)$ resp.) is a 1-simplex whose direction is the positive x-axis (y-axis resp.).

Let σ be a harmonic difference on $\boldsymbol{Q}_{\nu+1}$ with support $Q_{\nu+1}$. We can define a function (0-difference) u_x (u_y resp.) with support Q_ν by setting

$$u_x(q) = \sigma(a_x) \qquad (u_y(q) = \sigma(a_y) \text{ resp.})$$

for each a_x (a_y resp.) of $Q_{\nu+1}$, where $\partial a_x = q' - q$ ($\partial a_y = q' - q$ resp.). Further we define 1-differences σ_x and σ_y with support Q_ν by $\sigma_x = \Delta u_x$ and $\sigma_y = \Delta u_y$. It is easy to verify that σ_x and σ_y are harmonic.

Lemma 7.2. (Cf. pp. 49–51 of [4].)

$$\nu^2(\|\sigma_x\|_{Q_0}^2 + \|\sigma_y\|_{Q_0}^2) \leqq \|\sigma\|_{Q_{\nu+1}}^2.$$

Proof. By the formula (2.3) of partial summation we have

$$\|\sigma_x\|_{Q_0}^2 \leqq \int_{Q_j^*} \sigma_x \overline{\sigma_x^*} = \int_{\partial Q_j^*} u_x \overline{\sigma_x^*} \leqq \frac{1}{2}\left(\int_{\Lambda_j} |u_x|^2 - \int_{\Lambda_{j-1}} |u_x|^2\right)$$
$$(j = 1, \dots, \nu),$$

where Λ_j ($j = 0, \dots, \nu$) is the sum of all 0-simplices on ∂Q_j. Similarly,

$$\|\sigma_y\|_{Q_0}^2 \leqq \frac{1}{2}\left(\int_{\Lambda_j} |u_y|^2 - \int_{\Lambda_{j-1}} |u_y|^2\right) \qquad (j = 1, \dots, \nu).$$

Adding the last inequalities for j, we have

$$n(\|\sigma_x\|_{Q_0}^2 + \|\sigma_y\|_{Q_0}^2) \leqq \frac{1}{2}\left(\int_{\Lambda_n} (|u_x|^2 + |u_y|^2) - \int_{\Lambda_0} (|u_x|^2 + |u_y|^2)\right)$$
$$\leqq \frac{1}{2} \int_{\Lambda_n} (|u_x|^2 + |u_y|^2) \qquad (n = 1, \dots, \nu).$$

Furthermore, adding the last inequalities for n, we have

$$\nu(\nu+1)(\|\sigma_x\|_{Q_0}^2 + \|\sigma_y\|_{Q_0}^2) \leqq \sum_{n=1}^{\nu} \int_{\Lambda_n} (|u_x|^2 + |u_y|^2) \leqq \|\sigma\|_{Q_{\nu+1}}^2.$$

4. The estimation of $\|\sigma^{**} - \sigma^{**}\|$. Let $\boldsymbol{K} = <K, K^*>$ be a normal complex polyhedron and W be the Riemann surface based on the normal quadrangulation K. Let σ be a difference of $\Gamma_h(\boldsymbol{K})$ with support K.

We shall preserve the notations in **2**. Let M and N be a pair of 2-simplices such that $M \in K$, $N \in K_Q^*$ and $|M| \cap |N| \neq \emptyset$. Let a_j and b_j ($j = 1, \dots, 4$) be the cyclic sequences of edges of M and N respectively. We assume that $b_2=a_1^*$ and $b_3=-a_4^*$. Further we set $a_5=b_4^*$ and $a_6=b_1^*$.

We fix the local uniformizing parameter $z=x+iy$ on M defined in the first paragraph of **1**. Then by (7.1), on $|M|$

$$\sigma^{\#*}=-(x\sigma_2-(1-x)\sigma_4)dx+((1-y)\sigma_1-y\sigma_3)dy,$$

and further, on $D=|M|\cap|N|$

$$\sigma^{*\#}=\left(-\left(\frac{1}{2}-y\right)\sigma_6+\left(y+\frac{1}{2}\right)\sigma_4\right)dx+\left(\left(x+\frac{1}{2}\right)\sigma_1+\left(\frac{1}{2}-x\right)\sigma_5\right)dy,$$

where $\sigma_j=\sigma(a_j)$ $(j=1,\dots,6)$. We set $\tau=\sigma^{*\#}-\sigma^{\#*}$. Then

$$\|\tau\|_D^2=\int_D\tau\overline{\tau^*}=\int_{\partial D}u\overline{\tau^*}, \tag{7.6}$$

where u is a harmonic function on D like $du=\tau$. Noting that $\sigma_1+\sigma_2+\sigma_3+\sigma_4=0$ and $\sigma_1-\sigma_4-\sigma_5-\sigma_6=0$, we can carry out a calculation similar to Lemma 7.1 for the right side of (7.6). Then we have

$$\int_{\partial D}u\overline{\tau^*}=\frac{1}{3\cdot 2^6}(2|\sigma_1+\sigma_3|^2+2|\sigma_1-\sigma_5|^2+3|\sigma_2-\sigma_6|^2 \\ +3|\sigma_3+\sigma_5|^2). \tag{7.7}$$

Here we note that when we set $\sigma_j^*=\sigma^*(b_j)$ $(j=1,\dots,4)$,

$$\begin{aligned}&|\sigma_1-\sigma_5|^2=|\sigma_1^*+\sigma_3^*|^2,\\ &|\sigma_2-\sigma_6|^2,\ |\sigma_3+\sigma_5|^2\\ &\qquad\leqq|\sigma_1+\sigma_3|^2+|\sigma_1^*+\sigma_3^*|^2+2|\sigma_1+\sigma_3||\sigma_1^*+\sigma_3^*|\end{aligned}$$

hold. Hence we have

$$\|\tau\|_D^2\leqq\frac{1}{3\cdot 2^4}(2|\sigma_1+\sigma_3|^2+2|\sigma_1^*+\sigma_3^*|^2+3|\sigma_1+\sigma_3|\ |\sigma_1^*+\sigma_3^*|). \tag{7.8}$$

Adding (7.8) for all pairs M, N with $|M|\cap|N|\neq\emptyset$ and by making use of Schwarz's inequality, we obtain

$$\|\sigma^{*\#}-\sigma^{\#*}\|_{W_Q}^2\leqq\frac{1}{6}\sum_{M\in K}|\sigma_1+\sigma_3|^2+\frac{1}{6}\sum_{N\in K_Q^*}|\sigma_1^*+\sigma_3^*|^2 \\ +\frac{1}{4}\left(\sum_{M\in K}|\sigma_1+\sigma_3|^2\sum_{N\in K_Q^*}|\sigma_1^*+\sigma_3^*|^2\right)^{1/2}. \tag{7.9}$$

Now we prepare a useful lemma. Let $\boldsymbol{K}_0=\langle K_0, K_0^*\rangle$ be a normal complex polyhedron and W be the Riemann surface based on the normal quadrangulation K_0. Let $\{\boldsymbol{K}_n = <K_n, K_n^*>\}_{n=0}^{\infty}$ be the sequence of normal complex polyhedra such that K_n is the normal subdivision of K_{n-1} for each n. Let K_{nQ}^* ($n=0, 1,\dots$) be the 2-chain of K_n^* defined as the sum of whole quadrangular 2-simplices of K_n^*, and let W_n ($n=0, 1,\dots$) be the subset of W which is the carrier of K_{nQ}^*. Further, let K_{nm} and K_{nm}^* ($n>m$) be the 2-chains of K_n and K_n^* respectively defined as the sums of 2-simplices of K_n and K_n^* respectively having their carriers on W_m. There exists a number n_0 such that W_n is a subregion of W for every $n \geqq n_0$, and $\{W_n\}_{n=0}^{\infty}$ is an increasing sequence exhausting W', where $W' = W - |\Lambda_B|$ and Λ_B is the sum of 0-simplices of K_0 whose conjugate 2-simplices are not quadrangles.

LEMMA 7.3. *Let σ^n ($n=0, 1,\dots$) be a difference of $\Gamma_h(\boldsymbol{K}_n)$ with support K_n such that $\|\sigma^n\|_{\boldsymbol{K}_n}$ is bounded with respect to n. Then, for every number m, the limit relations*

$$\lim_{n\to\infty} \sum_{M \in K_{nm}} |\sigma_1^n + \sigma_3^n|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_{M \in K_{nm}^*} |\sigma_1^{n*} + \sigma_3^{n*}|^2 = 0,$$

hold, where $\sigma_j^n = \sigma^n(a_j)$, $\sigma_j^{n*} = \sigma^{n*}(a_j)$ and $a_1,\dots, a_4$ is a cyclic sequence of edges of M.

PROOF. Without loss of generality we may assume that the number $m=0$. Now we fix an arbitrary 2-simplex M_1 of K_{10}. We can always find an increasing sequence $Q_0^3,\dots, Q_4^3$ of concentric quadrates of K_3 in the meaning defined in **3** such that $|Q_0^3| = |M_1|$.

Let Q_j^n ($j=0,\dots, 4$; $n=4, 5,\dots$) be the normal subdivision of Q_j^{n-1} for each n which is a subpolyhedron of K_n, and let $\boldsymbol{Q}_j^n = <Q_j^n, Q_j^{n*}>$ for each n. Then, by Lemma 7.2, we have

$$(3\cdot 2^{n-3} - 1)^2 (\|\sigma_x^n\|_{\boldsymbol{Q}_1^n}^2 + \|\sigma_y^n\|_{\boldsymbol{Q}_1^n}^2) \leqq \|\sigma^n\|_{\boldsymbol{Q}_4^n}^2 \qquad (n=3, 4,\dots),$$

where σ_x^n, σ_y^n are the differences defined in **3** for the present σ^n for each n. On the other hand, we can easily verify that

$$\sum_{M \in Q_0^n} |\sigma_1^n + \sigma_3^n|^2 + \sum_{M \in Q_1^{n*}} |\sigma_1^{n*} + \sigma_3^{n*}|^2 \leqq \frac{1}{2} (\|\sigma_x^n\|_{\boldsymbol{Q}_1^n}^2 + \|\sigma_y^n\|_{\boldsymbol{Q}_1^n}^2).$$

Hence we have

$$\sum_{M \in Q_0^n} |\sigma_1^n + \sigma_3^n|^2 + \sum_{M \in Q_1^{n*}} |\sigma_1^{n*} + \sigma_3^{n*}|^2 \leqq \frac{1}{2(3\cdot 2^{n-3} - 1)^2} \|\sigma^n\|_{\boldsymbol{Q}_4^n}^2.$$

Adding the last inequalities for all simplices M_1 of K_{10}, we have

$$\sum_{M\in K_{n0}} |\sigma_1^n+\sigma_3^n|^2+\sum_{M\in K_{n0}^*} |\sigma_1^{n*}+\sigma_3^{n*}|^2 \leqq \frac{9}{2(3\cdot 2^{n-3}-1)^2}\|\sigma^n\|_{\boldsymbol{K}_n}^2$$

$$(n=3, 4,\dots).$$

By (7.9) and Lemma 7.3, we obtain the following corollary.

COROLLARY 7.2. *Under the same assumption as Lemma 7.3, for every number m the following limit relation holds*

$$\lim_{n\to\infty} \|\sigma^{n*\#}-\sigma^{n\#*}\|_{W_m}^2=0.$$

5. Fundamental theorem.

THEOREM 7.1. *Let $\{\boldsymbol{K}_n=<K_n, K_n^*>\}_{n=0}^{\infty}$ be a sequence of normal complex polyhedra such that K_n is the normal subdivision of K_{n-1}. Let W be the Riemann surface based on the normal quadrangulation K_0. Let σ^n $(n=0, 1,\dots)$ be a difference of $\Gamma_h(\boldsymbol{K}_n)$ with support K_n. We suppose that $\{\sigma^n\}_{n=0}^{\infty}$ forms a Cauchy sequence, i.e.*

$$\lim_{m,n\to\infty} \|\sigma^{m\natural}-\sigma^n\|_{\boldsymbol{K}_n}=0 \qquad (n\geqq m). \tag{7.10}$$

Then the sequence $\{\sigma^{n\#}\}_{n=0}^{\infty}$ strongly converges to a harmonic differential $\omega\in\Gamma_h(W)$, i.e.

$$\lim_{n\to\infty} \|\sigma^{n\#}-\omega\|_W=0, \tag{7.11}$$

and the limit relations

$$\lim_{n\to\infty} \|\sigma\|_{\boldsymbol{K}_n}=\lim_{n\to\infty} \|\sigma^{n\#}\|_W=\|\omega\|_W \tag{7.12}$$

hold. Furthermore, the limit relation

$$\lim_{n\to\infty} \|\sigma^{n*\#}-\omega^*\|_{W_n}=0 \tag{7.13}$$

holds, where W_n is W minus carriers of 2-simplices of K_n^* which are not quadrangles for each n.

PROOF. We note that $\sigma^{m\#}-\sigma^{n\#}$ is the smooth extension of $\sigma^{m\natural}-\sigma^n$ to W. Then, (7.2) and (7.10) imply

$$\lim_{m,n\to\infty} \|\sigma^{m\#}-\sigma^{n\#}\|_W=0.$$

The last relation secures that there exists a differential $\omega\in\Gamma_c(W)$ satisfying (7.11) and the second equality of (7.12) holds.

We note that (7.10) implies the boundedness of $\|\sigma^n\|_{\boldsymbol{K}_n}$ with respect to n. We use the notations defined in **4**, and further we define the notations $\boldsymbol{K}_{nm}=<K_{nm}, K^*_{nm}>$ and $\boldsymbol{L}_{nm}=\boldsymbol{K}_n-\boldsymbol{K}_{nm}$. More precisely, $\boldsymbol{K}_{nm}$ and $\boldsymbol{L}_{nm}$ are the complex polyhedra defined as the sums of 2-simplices of $\boldsymbol{K}_n$ having their carriers on W_m and $\overline{W-W_m}$, respectively. By Lemmas 7.1 and 7.3, we see that

$$\lim_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_{nm}}=\lim_{n\to\infty}\|\sigma^{n\#}\|_{W_m} \qquad \text{for every } m.$$

If the limit relation

$$\lim_{m,n\to\infty}\|\sigma^n\|_{\boldsymbol{L}_{nm}}=0 \tag{7.14}$$

is shown, then by Lemma 7.1 the limit relation

$$\lim_{m,n\to\infty}\|\sigma^{n\#}\|_{W-W_m}=0 \qquad (n>m)$$

holds and hence we obtain the first equality of (7.12).

Noting the definition of the natural extension and by making use of Lemma 6.1, we can easily verify that

$$\|\sigma^{k\natural}\|_{\boldsymbol{L}_{nm}}\leqq\|\sigma^{k\natural}\|_{\boldsymbol{L}_{m+1,m}}\leqq\mu^{m-k}\|\sigma^{k\natural}\|_{\boldsymbol{L}_{k+1,k}}\leqq\mu^{m-k}\|\sigma^k\|_{\boldsymbol{K}_k} \qquad (n>m>k) \tag{7.15}$$

for every k, where $0<\mu\leqq 6/16$. Further,

$$\begin{aligned}\big|\|\sigma^n\|_{\boldsymbol{L}_{nm}}-\|\sigma^{k\natural}\|_{\boldsymbol{L}_{nm}}\big|&\leqq\|\sigma^n-\sigma^{k\natural}\|_{\boldsymbol{L}_{nm}}\\&\leqq\|\sigma^n-\sigma^{k\natural}\|_{\boldsymbol{K}_n}\to 0 \qquad (k,n\to\infty)\end{aligned} \tag{7.16}$$

holds. (7.15) and (7.16) imply (7.14).

By Corollary 7.2 and (7.11) we have that

$$\lim_{n\to\infty}\|\sigma^{n*\#}-\omega^*\|_{W_m}=0 \qquad \text{for every } m. \tag{7.17}$$

Clearly,

$$\lim_{m,n\to\infty}\|\omega^*\|_{W_n-W_m}=0 \qquad (n>m). \tag{7.18}$$

By (7.2) and (7.14) we see that

$$\|\sigma^{n*\#}\|_{W_n-W_m}\leqq\|\sigma^{n*}\|_{\boldsymbol{L}_{nm}}=\|\sigma^n\|_{\boldsymbol{L}_{nm}}\to 0 \qquad (m,n\to\infty). \tag{7.19}$$

(7.17), (7.18) and (7.19) imply (7.13). Since $\sigma^{n*\#}\in\Gamma_c(W_n)$ for every n, (7.13) implies that $\omega^*\in\Gamma_c(W')$ where $W'=\bigcup_{n=0}^{\infty}W_n$. Hence $\omega\in\Gamma_c(W')\cap\Gamma^*_c(W')=\Gamma_h(W')$. Further, since $\|\omega\|_W<\infty$ and $\Lambda_B=W-W'$ is a set of isolated points of W, each point of Λ_B is a removable harmonic singularity of ω and hence $\omega\in$

$\Gamma_h(W)$.

COROLLARY 7.3. *Under the same assumption as Theorem* 7.1, *the coefficients of* $\sigma^{n\#}$ *and* $\sigma^{n*\#}$ *uniformly converge to the coefficients of* ω *and* ω^* *respectively as* $n\to\infty$ *in each relatively compact subregion* Ω *of* W', *where* $W'=\bigcup_{n=0}^{\infty}W_n$.

PROOF. It is sufficient to prove the corollary in an arbitrarily fixed parameter disk D in W' with a local uniformizing parameter $z=x+iy$. Let $\sigma^{n\#}=\alpha_n(z)dx+\beta_n(z)dy$ and $\sigma^{n*\#}=\alpha'_n(z)dx+\beta'_n(z)dy$. By Courant-Friedrichs-Lewy's method (cf. pp. 48–52 of [4]), it is shown that the sequences of coefficients $\{\alpha_n\}_{n=0}^{\infty}$, etc. are uniformly bounded on D, and further are *equicontinuous* on D in the following sense: for any positive number $\varepsilon>0$, there exist an integer n_0 and a positive number $\delta>0$ such that $|\alpha_n(z_1)-\alpha_n(z_2)|<\varepsilon$, etc. provided $|z_1-z_2|<\delta$ and $n\geqq n_0$. By these consequences, the present corollary is immediately verified.

The following theorem stated in the analytic case immediately follows from Theorem 7.1.

THEOREM 7.2. *Let* ψ^n $(n=0, 1,\dots)$ *be a difference of* $\Gamma_a(\boldsymbol{K}_n)$. *We suppose that* $\{\psi^n_{K_n}\}_{n=0}^{\infty}$ *forms a Cauchy sequence, i.e.*

$$\lim_{m,n\to\infty}\|\psi^{m\natural}_{K_m}-\psi^n_{K_n}\|_{\boldsymbol{K}_n}=0 \qquad (n\geqq m), \tag{7.20}$$

where $\psi^n_{K_n}$ *is the restriction of* ψ^n *to* K_n *for each n. Then the sequence* $\{\psi^{n\#}\}_{n=0}^{\infty}$ *strongly converges to an analytic differential* $\phi\in\Gamma_a(W)$, *i.e.*

$$\lim_{n\to\infty}\|\psi^{n\#}-\phi\|_{W_n}=0, \tag{7.21}$$

where W_n *is* W *minus carriers of* 2-*simplices of* K_n^* *which are not quadrangles for each n, and the limit relations*

$$\lim_{n\to\infty}\|\psi^n\|_{\boldsymbol{K}_n}=\lim_{n\to\infty}\|\psi^{n\#}\|_{W_n}=\|\phi\|_W \tag{7.22}$$

hold.

6. The method of orthogonal projection.

THEOREM 7.3. *Let* $\{\boldsymbol{K}_n=<K_n, K_n^*>\}_{n=0}^{\infty}$ *be a sequence of normal complex polyhedra such that* K_n *is the normal subdivision of* K_{n-1} *for each n. Let* W *be the Riemann surface based on the normal quadrangulation* K_0. *Let* Θ *be an arbitrary difference of* $\Gamma_c(\boldsymbol{K}_0)$ *with support* K_0, *and let us suppose that* σ^n $(n=0, 1,\dots)$ *be the projection of* $\Theta^\natural$ *on* $\Gamma_h(\boldsymbol{K}_n)$, $\Gamma^*_{hm}(\boldsymbol{K}_n)$ *or* $\Gamma^*_{h0}(\boldsymbol{K}_n)\cap\Gamma_{hse}(\boldsymbol{K}_n)$. *Then, we obtain the same conclusion as Theorem* 7.1. *Furthermore,* *the inequalities*

(7.23) $$\|\sigma^n\|_{K_n} \geqq \|\sigma^{n\#}\|_W \geqq \|\omega\|_W \qquad \text{for every } n$$

and the monotone convergence of norms

(7.24) $$\|\sigma^n\|_{K_n} \searrow \|\omega\|_W \qquad (n \to \infty)$$

hold. And the limit differential ω *is the projection of* $\Theta^\#$ *on* $\Gamma_h(W)$, $\Gamma^*_{hm}(W)$ *and* $\Gamma^*_{h0}(W) \cap \Gamma_{hse}(W)$ *respectively, and hence* $\Theta^\# - \omega \in \Gamma_{e0}(W)$, $\Theta^\# - \omega \in \Gamma_{se}(W)$ *and* $\Theta^\#_{se} - \omega \in \Gamma_e(W)$ *respectively, where* Θ_{se} *is the projection of* Θ *on* $\Gamma_{se}(\boldsymbol{K}_0)$.

PROOF. The assumption of the theorem implies that

(7.25) $$\Theta^\natural = \sigma^n + \sigma^n_{e0}, \quad \sigma^n_{e0} \in \Gamma_{e0}(\boldsymbol{K}_n) \qquad (n=0, 1, \dots),$$

(7.26) $$\Theta^\natural = \sigma^n + \sigma^n_{se}, \quad \sigma^n_{se} \in \Gamma_{se}(\boldsymbol{K}_n) \qquad (n=0, 1, \dots)$$

and

(7.27) $$\Theta^\natural_{se} = \sigma^n + \sigma^n_e, \quad \sigma^n_e \in \Gamma_e(\boldsymbol{K}_n) \qquad (n=0, 1, \dots)$$

respectively. Hence we can easily see that

(7.28) $$\sigma^n - \sigma^{0\natural} = \sigma^{0\natural}_{e0} - \sigma^n_{e0} \in \Gamma_{e0}(\boldsymbol{K}_n) \qquad (n=0, 1, \dots),$$

(7.29) $$\sigma^n - \sigma^{0\natural} = \sigma^{0\natural}_{se} - \sigma^n_{se} \in \Gamma_{se}(\boldsymbol{K}_n) \qquad (n=0, 1, \dots)$$

and

(7.30) $$\sigma^n - \sigma^{0\natural} = \sigma^{0\natural}_e - \sigma^n_e \in \Gamma_e(\boldsymbol{K}_n) \qquad (n=0, 1, \dots)$$

respectively. Therefore, by Corollary 6.1, the assumption (7.10) of Theorem 7.1 is satisfied, and thus the same conclusion as Theorem 7.1 holds.

The left inequality of (7.23) follows from (7.2). If it has been shown that the limit differential ω is the projection of $\sigma^{n\#}$ on $\Gamma_h(W)$, $\Gamma^*_{hm}(W)$ and $\Gamma^*_{h0}(W) \cap \Gamma_{hse}(W)$ respectively for every n, then the right inequality is clear and (7.24) follows from Corollary 6.1 and (7.12).

The case where σ^n is the projection of $\Theta^\natural$ on $\Gamma_h(\boldsymbol{K}_n)$: (7.28) implies that $\sigma^{n\#} - \sigma^{0\#} \in \Gamma_{e0}(W)$ and thus by (7.11), $\omega - \sigma^{n\#} \in \Gamma_{e0}(W)$ for every n. Because of $\omega \in \Gamma_h(W)$, ω is the projection of $\sigma^{n\#}$ on $\Gamma_h(W)$ for every n, and thus by (7.25), ω is the projection of $\Theta^\#$ on $\Gamma_h(W)$.

The case where σ^n is the projection of $\Theta^\natural$ on $\Gamma^*_{hm}(\boldsymbol{K}_n)$: By (7.29) and (7.11), we find that $\omega - \sigma^{n\#} \in \Gamma_{se}(W)$ for every n. Let $\{\boldsymbol{K}_{nj}\}_{j=0}^\infty$ be a canonical exhaustion of $\boldsymbol{K}_n$ for each n such that K_{nj} $(n=0, 1, \dots)$ have a common carrier Ω_j for each j and such that $\partial\Omega_j \subset W'$ for every j where $W' = \bigcup_{n=0}^\infty W_n$. Let σ^{nj} $(j=0, 1, \dots)$ be the projection of σ^n, restricted to $\boldsymbol{K}_{nj}$, on $\Gamma^*_{hm}(\boldsymbol{K}_{nj})$ for each n. Then $\sigma^n -$

$\sigma^{nj} \in \Gamma_{hse}(\boldsymbol{K}_{nj})$. By the method of orthogonal projection, we can easily verify that

$$\lim_{j\to\infty} \|\sigma^n - \sigma^{nj}\|_{\boldsymbol{K}_{nj}} = 0 \qquad \text{for each } n.$$

We note that the smooth extension $\sigma^{nj*\#}$ of σ^{nj*} can be not defined only for each 2-simplex of K^*_{nj} but also for the conjugate left half 2-simplex of each 0-simplex of ∂K_{nj} w.r.t. ∂K_{nj} by a method similar to (7.1) such that the differential $\sigma^{nj*\#}$ is zero along $\partial\Omega_j$ because of $\sigma^{nj*} \in \Gamma_{hm}(\boldsymbol{K}_{nj})$. Then, by the method similar to Lemma 7.1 we can verify that

$$(7.31) \qquad \lim_{j\to\infty} \|\sigma^{n*\#} - \sigma^{nj*\#}\|_{W_n \cap \Omega_j} \leq \lim_{j\to\infty} \|\sigma^{n*} - \sigma^{nj*}\|_{\boldsymbol{K}_{nj}} = 0 \qquad \text{for each } n.$$

We shall omit the detailed argument. (7.13) and (7.31) imply that there exists an increasing sequence of numbers $j = j(n)$ $(n = 0, 1, \ldots)$ such that

$$(7.32) \qquad \lim_{n\to\infty} \|\sigma^{nj(n)*\#} - \omega^*\|_{W_n \cap \Omega_{j(n)}} = 0.$$

Let $\omega^{j(n)}$ be the differential of $\Gamma_{he}(\Omega_{j(n)})$ with the same boundary behavior as $\sigma^{nj(n)*\#}$ on $\partial\Omega_{j(n)}$ for each n. Because of $\omega^* \in \Gamma_{he}(W)$, (7.32) implies that

$$(7.33) \qquad \lim_{n\to\infty} \|\omega^{j(n)} - \omega^*\|_{\Omega_{j(n)}} = 0.$$

We note that $\sigma^{nj(n)*} \in \Gamma_{hm}(\boldsymbol{K}_{nj(n)})$ implies that $\omega^{j(n)} \in \Gamma_{hm}(\Omega_{j(n)})$. Then (7.33) implies that $\omega^* \in \Gamma_{hm}(W)$. $\omega - \sigma^{n\#} \in \Gamma_{se}(W)$ and $\omega \in \Gamma^*_{hm}(W)$ imply that ω is the projection of $\sigma^{n\#}$ on $\Gamma^*_{hm}(W)$ for every n, and thus by (7.26), ω is the projection of $\Theta^\#$ on $\Gamma^*_{hm}(W)$.

The case where σ^n is the projection of $\Theta^\natural$ on $\Gamma^*_{h0}(\boldsymbol{K}_n) \cap \Gamma_{hse}(\boldsymbol{K}_n)$, is analogous to the last case.

Corollary 7.4. *Under the same assumption as Theorem 7.3, let γ be a cycle on K^*_0, let $\chi^{n\gamma}$ and $\phi^{n\gamma}$ be the harmonic and the analytic reproducing differences respectively of γ on $\boldsymbol{K}_n$ defined in §4, and let χ^γ and ϕ^γ be the harmonic and the analytic reproducing differentials respectively of γ on W such that*

$$(\omega, (\chi^\gamma)^*)_W = \int_\gamma \omega \qquad \text{for every } \omega \in \Gamma_h(W),$$

$$(\phi, \phi^\gamma)_W = -i\int_\gamma \phi \qquad \text{for every } \phi \in \Gamma_a(W).$$

Then, the sequences $\{\chi^{n\gamma\#}\}_{n=0}^\infty$ and $\{\phi^{n\gamma\#}\}_{n=0}^\infty$ strongly converge to the differentials χ^γ and ϕ^γ respectively, i.e.

$$\lim_{n\to\infty} \|\chi^{n\gamma\#} - \chi^\gamma\| = 0,$$

$$\lim_{n\to\infty}\|\phi^{n\gamma\#}-\phi^{\gamma}\|=0,$$

the inequalities

$$\|\chi^{n\gamma}\|_{\boldsymbol{K}_n}\geqq\|\chi^{n\gamma\#}\|_W\geqq\|\chi^{\gamma}\|_W \qquad \text{for every } n,$$

$$\|\phi^{n\gamma}\|_{\boldsymbol{K}_n}\geqq\|\phi^{n\gamma\#}\|_W\geqq\|\phi^{\gamma}\|_W \qquad \text{for every } n$$

hold, and the monotone convergence of norms

$$\|\chi^{n\gamma}\|_{\boldsymbol{K}_n}\searrow\|\chi^{\gamma}\|_W \qquad (n\to\infty),$$

$$\|\phi^{n\gamma}\|_{\boldsymbol{K}_n}\searrow\|\phi^{\gamma}\|_W \qquad (n\to\infty)$$

holds.

PROOF. Choose the singular difference Θ^{γ} on $\boldsymbol{K}_0$ associated to γ defined in § 4 as the difference Θ of Theorem 7.3. Then the projection of $\Theta^{\gamma\natural}$ on $\Gamma_h(\boldsymbol{K}_n)$ gives the reproducing differential $\chi^{n\gamma}$.

7. Difference approximation of a differential. Let $\boldsymbol{K}=<K, K^*>$ be a normal complex polyhedron, and W be the Riemann surface based on the normal quadrangulation K. Let Θ be a closed differential on W, of class C^0. By a *difference approximation* τ *of* Θ on the normal polyhedron K, we mean the *closed difference* with support K defined by

$$\tau(a)=\int_a\Theta \qquad \text{for each } a\in K,$$

where the integration path a means the oriented arc with the carrier $|a|$ and with the same orientation as a.

THEOREM 7.4. *Let* $\{\boldsymbol{K}_n=<K_n, K_n^*>\}_{n=0}^{\infty}$ *be a sequence of normal complex polyhedra such that* K_n *is the normal subdivision of* K_{n-1} *for each* n*, and let* W *be the Riemann surface based on the normal quadrangulation* K_0*. Let* Θ *be a closed differential on* W*, of class* C^0*, and let* τ^n $(n=0, 1,\dots)$ *be the difference approximation of* Θ *on the normal polyhedron* K_n*. We suppose that*

(7.34) $$\lim_{j\to\infty}\|\tau^n\|_{\boldsymbol{K}_n-\boldsymbol{K}_{nj}}=0 \qquad \text{uniformly with respect to } n,$$

where $\{\boldsymbol{K}_{nj}\}_{j=0}^{\infty}$ *is an exhaustion of* $\boldsymbol{K}_n$ *for each* n *such that* K_{nj} $(n=0, 1,\dots)$ *have a common carrier* Ω_j *for each* j*. Then, we obtain that* $\tau^n\in\Gamma_c(\boldsymbol{K}_n)$ $(n=0, 1,\dots)$ *and* $\Theta\in\Gamma_c(W)$*, and for the sequence* $\{\sigma^n\}_{n=0}^{\infty}$ *of the projections of* τ^n *on* $\Gamma_h(\boldsymbol{K}_n)$ *for each* n*, we obtain the same conclusion as Theorem 7.1. And the* *limit differential ω of the sequence $\{\sigma^{n\#}\}_{n=0}^{\infty}$ is the projection of Θ on $\Gamma_h(W)$.*

Proof. Because the coefficients of $\tau^{n\#}$ uniformly converge to the coefficients of Θ as $n\to\infty$ on each Ω_j,

$$\lim_{n\to\infty}\|\tau^{n\#}-\Theta\|_{\Omega_j}=0 \qquad \text{for each } j. \tag{7.35}$$

By (7.3),

$$\frac{1}{3}\|\tau^{n}\|^2_{K_{nj}}\leqq\|\tau^{n\#}\|^2_{\Omega_j}\leqq\|\tau^{n\#}-\Theta\|^2_{\Omega_j}+\|\Theta\|^2_{\Omega_j}. \tag{7.36}$$

From (7.35), (7.36) and (7.34), it follows that $\|\tau^{n}\|_{K_n}$ is bounded with respect to n and hence $\tau^{n}\in\Gamma_c(\boldsymbol{K}_n)$.

By (7.35), $\lim_{m,n\to\infty}\|\tau^{m\#}-\tau^{n\#}\|_{\Omega_j}=0$ and hence, by (7.3),

$$\lim_{m,n\to\infty}\|\tau^{m\natural}-\tau^{n}\|_{K_{nj}}=0 \qquad \text{for each } j\ (n\geqq m). \tag{7.37}$$

(7.37) and (7.34) imply that

$$\lim_{m,n\to\infty}\|\tau^{m\natural}-\tau^{n}\|_{K_n}=0 \qquad (n\geqq m). \tag{7.38}$$

Hence, by (7.2),

$$\lim_{m,n\to\infty}\|\tau^{m\#}-\tau^{n\#}\|_{W}=0. \tag{7.39}$$

From (7.39) and (7.35), it follows that

$$\lim_{n\to\infty}\|\tau^{n\#}-\Theta\|_{W}=0 \tag{7.40}$$

and hence $\Theta\in\Gamma_c(W)$.

Let ω^{n} ($n=0, 1,\ldots$) be the projection of $\tau^{n\#}$ on $\Gamma_h(W)$, and let σ^{mn} ($m=0, 1,\ldots;\ n=m, m+1,\ldots$) be the projection of $\sigma^{m\natural}$ on $\Gamma_h(\boldsymbol{K}_n)$. By (7.38) and (7.40), we have that

$$\lim_{m,n\to\infty}\|\sigma^{mn\#}-\sigma^{n\#}\|_{W}=0 \qquad (n\geqq m), \tag{7.41}$$

$$\lim_{m\to\infty}\|\omega^{m}-\omega\|_{W}=0, \tag{7.42}$$

respectively. By Theorem 7.3,

$$\lim_{n\to\infty}\|\sigma^{mn\#}-\omega^{m}\|_{W}=0 \qquad \text{for each } m. \tag{7.43}$$

By (7.41), (7.42) and (7.43), we obtain that

$$\lim_{n\to\infty}\|\sigma^{n\#}-\omega\|_{W}=0. \tag{7.44}$$

(7.44) implies the assumption (7.10) of Theorem 7.1, and hence we obtain the present theorem.

§ 8. The approximation of a differential on a generic Riemann surface.

1. Convergence theorem.

THEOREM 8.1. *Let W be an arbitrary open Riemann surface, let us suppose that $\{K_n=K(\Omega_n, 4^n\Psi)\}_{n=0}^{\infty}$ is an increasing sequence of normal subdivisions by a quadratic differential Ψ which defines a regular exhaustion of W* (cf. Lemma 1.1), *and let $\boldsymbol{K}_n=<K_n, K_n^*>$ $(n=0, 1,\dots)$. Let K_{mn} be the subpolyhedron of K_n such that $|K_{mn}|=|K_m|$ and let $\boldsymbol{K}_{mn}=<K_{mn}, K_{mn}^*>$ $(m=0,\dots, n;\ n=0, 1,\dots)$. Let σ^n $(n=0, 1,\dots)$ be a difference of $\Gamma_h(\boldsymbol{K}_n)$ with support K_n. We suppose that $\{\sigma^n\}_{n=0}^{\infty}$ forms a Cauchy sequence, i.e.*

$$(8.1)\qquad \lim_{m,n\to\infty}\|\sigma^{m\natural}-\sigma^n\|_{\boldsymbol{K}_n}=0 \qquad (n\geqq m),$$

where we assume that the natural extension $\sigma^{m\natural}$ is defined on each $\boldsymbol{K}_n$ $(n=m, m+1,\dots)$ by setting $\sigma^{m\natural}=0$ on $\boldsymbol{K}_n-\boldsymbol{K}_{mn}$. Then the sequence $\{\sigma^{n\#}\}_{n=0}^{\infty}$ strongly converges to a harmonic differential $\omega\in\Gamma_h(W)$, i.e.

$$(8.2)\qquad \lim_{n\to\infty}\|\sigma^{n\#}-\omega\|_{\Omega_n}=0,$$

and the limit relations

$$(8.3)\qquad \lim_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_n}=\lim_{n\to\infty}\|\sigma^{n\#}\|_{\Omega_n}=\|\omega\|_W$$

hold. Furthermore, the limit relation

$$(8.4)\qquad \lim_{n\to\infty}\|\sigma^{n*\#}-\omega^*\|_{\Omega_n^{*\prime}}=0$$

holds, where Ω_n^* is the carrier of K_n^* and $\Omega_n^{*\prime}$ is Ω_n^* minus carriers of 2-*simplices of K_n^* which are not quadrangles for each n.*

PROOF. We assume that the smooth extension $\sigma^{n\#}$ $(n=0, 1,\dots)$ is defined on W by setting $\sigma^{n\#}=0$ on $W-\Omega_n$. By (7.2) and (8.1) we have that

$$\lim_{m,n\to\infty}\|\sigma^{m\#}-\sigma^{n\#}\|_W\leqq \lim_{m,n\to\infty}\|\sigma^{m\natural}-\sigma^n\|_{\boldsymbol{K}_n}=0.$$

The last relation secures that there exists a differential $\omega\in\Gamma_c(W)$ satisfying (8.2) and the second equality of (8.3) holds. Furthermore, by making use of Theorem 7.1 for a fixed Ω_n[1] we know that $\omega\in\Gamma_h(\Omega_n)$ for each n and thus $\omega\in\Gamma_h(W)$.

1) It is verified that Theorem 7. 1 holds also for the compact bordered case.

We note that (8.1) implies the boundedness of $\|\sigma^n\|_{\boldsymbol{K}_n}$ with respect to n. Then, by Theorem 7.1 we see that

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_{mn}}=\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n\#}\|_{\Omega_m}\leqq\lim_{n\to\infty}\|\sigma^{n\#}\|_{\Omega_n}. \tag{8.5}$$

The assumption (8.1) implies

$$\lim_{m,n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_n-\boldsymbol{K}_{mn}}=0 \qquad (n\geqq m). \tag{8.6}$$

Hence there exists the finite limit

$$\lim_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_n}=\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^n\|_{\boldsymbol{K}_{mn}}. \tag{8.7}$$

On the other hand, by Lemma 7.1

$$\|\sigma^n\|_{\boldsymbol{K}_n}\geqq\|\sigma^{n\#}\|_{\Omega_n} \qquad \text{for every } n. \tag{8.8}$$

(8.5), (8.7) and (8.8) imply the first equality of (8.3).

By Theorem 7.1 we see that

$$\lim_{n\to\infty}\|\sigma^{n*\#}-\omega^*\|_{\Omega_n^{*\prime}\cap\Omega_m^*}=0 \qquad \text{for every } m. \tag{8.9}$$

Further

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n*\#}-\omega^*\|_{\Omega_n^{*\prime}-\Omega_m^*}$$
$$\leqq\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n*\#}\|_{\Omega_n^{*\prime}-\Omega_m^*}+\lim_{m,n\to\infty}\|\omega^*\|_{\Omega_n^*-\Omega_m^*}.$$

Here $\omega\in\Gamma_h(W)$ implies that

$$\lim_{m,n\to\infty}\|\omega^*\|_{\Omega_n^*-\Omega_m^*}=0,$$

and by Lemma 7.1 and (8.6)

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n*\#}\|_{\Omega_n^{*\prime}-\Omega_m^*}\leqq\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n*}\|_{\boldsymbol{K}_n-\boldsymbol{K}_{mn}}=0.$$

Hence

$$\lim_{m\to\infty}\lim_{n\to\infty}\|\sigma^{n*\#}-\omega^*\|_{\Omega_n^{*\prime}-\Omega_m^*}=0. \tag{8.10}$$

(8.9) and (8.10) imply (8.4).

COROLLARY 8.1. *Let W be an arbitrary open Riemann surface, let us suppose that $\{K_n=K(\Omega_n, 4^n\Psi)\}_{n=0}^{\infty}$ is an increasing sequence of normal subdivisions by a quadratic differential Ψ which defines a regular exhaustion of W, and let $\boldsymbol{K}_n=\langle K_n, K_n^*\rangle$ $(n=0, 1,\dots)$. Let Θ be an arbitrary differential of*

$\Gamma_{c0}(W)$ with compact support contained in Ω_0, and let σ^n $(n=0, 1,\ldots)$ be the difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\Theta-\sigma^{n\#}\in\Gamma_{e0}(\Omega_n)$. Then, we obtain the same conclusion as Theorem 8.1. *Furthermore, the inequalities*

$$\|\sigma^n\|_{\boldsymbol{K}_n}\geqq\|\sigma^{n\#}\|_{\Omega_n}\geqq\|\omega\|_W \qquad \text{for every } n \tag{8.11}$$

hold and the monotone convergence of norms

$$\|\sigma^n\|_{\boldsymbol{K}_n}\searrow\|\omega\|_W \qquad (n\to\infty) \tag{8.12}$$

holds. And the limit differential ω is the projection of Θ on $\Gamma_{h0}(W)$.

PROOF. With the notation in Theorem 8.1, we assume that the natural extension $\sigma^{m\natural}$ is defined on each $\boldsymbol{K}_n$ $(n=m, m+1,\ldots)$ by setting $\sigma^{m\natural}=0$ on $\boldsymbol{K}_n-\boldsymbol{K}_{mn}$. Then $\sigma^{m\natural}\in\Gamma_{c0}(\boldsymbol{K}_n)$. Since $\sigma^n-\sigma^{m\natural}\in\Gamma_{e0}(\boldsymbol{K}_n)$ and $\sigma^n\in\Gamma_{h0}(\boldsymbol{K}_n)$, we have that

$$(\sigma^n-\sigma^{m\natural}, \sigma^n)_{\boldsymbol{K}_n}=0 \qquad (n\geqq m).$$

Hence

$$\|\sigma^n-\sigma^{m\natural}\|^2_{\boldsymbol{K}_n}\leqq\|\sigma^{m\natural}\|^2_{\boldsymbol{K}_n}-\|\sigma^n\|^2_{\boldsymbol{K}_n}\leqq\|\sigma^m\|^2_{\boldsymbol{K}_m}-\|\sigma^n\|^2_{\boldsymbol{K}_n}.$$

Therefore $\|\sigma^n\|_{\boldsymbol{K}_n}$ is monotone decreasing, and the assumption (8.1) of Theorem 8.1 is satisfied. Hence we obtain the same conclusion as Theorem 8.1. The proof of the remaining parts is easy.

COROLLARY 8.2. *Under the same assumption as Corollary* 8.1, *let γ be a cycle on K_0^*, let $\chi^{n\gamma}$ and $\phi^{n\gamma}$ be the harmonic and the analytic reproducing differences respectively of γ on $\boldsymbol{K}_n$ defined in* § 4, *and let χ^γ and ϕ^γ be the harmonic and the analytic reproducing differentials respectively of γ on W. Then, the sequences $\{\chi^{n\gamma\#}\}_{n=0}^\infty$ and $\{\phi^{n\gamma\#}\}_{n=0}^\infty$ strongly converge to the differentials χ^γ and ϕ^γ respectively, i.e.*

$$\lim_{n\to\infty}\|\chi^{n\gamma\#}-\chi^\gamma\|=0,$$

$$\lim_{n\to\infty}\|\phi^{n\gamma\#}-\phi^\gamma\|=0,$$

the inequalities

$$\|\chi^{n\gamma}\|_{\boldsymbol{K}_n}\geqq\|\chi^{n\gamma\#}\|_W\geqq\|\chi^\gamma\|_W \qquad \text{for every } n,$$

$$\|\phi^{n\gamma}\|_{\boldsymbol{K}_n}\geqq\|\phi^{n\gamma\#}\|_W\geqq\|\phi^\gamma\|_W \qquad \text{for every } n$$

hold, and the monotone convergence of norms

$$\|\chi^{n\gamma}\|_{\boldsymbol{K}_n}\searrow\|\chi^\gamma\|_W \qquad (n\to\infty),$$

$$\|\phi^{n\gamma}\|_{K_n} \searrow \|\phi^{\gamma}\|_W \qquad (n\to\infty)$$

holds.

PROOF. We may assume that the smooth extension $\chi^{0\gamma\#}$ is defined on W by setting $\chi^{0\gamma\#}=0$ on $W-\Omega_0$. Then $\chi^{0\gamma\#}\in\Gamma_{c0}(W)$. We can choose the differential $\chi^{0\gamma\#}$ as the differential Θ of Corollary 8.1, and we note that $\chi^{n\gamma}$ $(n=0, 1,\dots)$ is the difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\chi^{0\gamma\#}-\chi^{n\gamma\#}\in\Gamma_{e0}(\Omega_n)$ for each n.

§ 9. An application to numerical calculation.

1. Riemann's period matrix of a closed Riemann surface. Let W be a closed Riemann surface of genus g. Let $\{A_j, B_j\}_{j=1,\dots,g}$ be a canonical homology basis of W such that

$$A_j\times A_k=0, \quad B_j\times B_k=0, \quad A_j\times B_k=\delta_{jk} \qquad (j, k=1,\dots, g),$$

where by δ_{jk} we denote Kronecker's symbol. Then there exists a system of analytic differentials ψ_j on W $(j=1,\dots, g)$ such that

$$\int_{A_j}\psi_k=\delta_{jk} \qquad (j, k=1,\dots, g).$$

Then the matrix (τ_{jk}) determined by

$$\tau_{jk}=\int_{B_j}\psi_k \qquad (j, k=1,\dots, g)$$

is called *Riemann's period matrix*, which is an important conformal invariant determining the conformal structure of W. The matrix (τ_{jk}) is symmetric and the matrix $(\operatorname{Im}\tau_{jk})$ is negative definite.

For simplicity we set $C_{2j-1}=A_j$, $C_{2j}=B_j$ $(j=1,\dots, g)$. It is well known that there exists a unique system of harmonic differentials ω_j $(j=1,\dots, 2g)$ on W such that

$$\int_{C_{2j}}\omega_{2j-1}=-\int_{C_{2j-1}}\omega_{2j}=1 \qquad (j=1,\dots, g),$$

$$\int_{C_k}\omega_j=0 \qquad \text{for all other pairs } j, k.$$

We set

$$\alpha_{jk}=(\omega_j, \omega_k)_W=\int_{C_j}\omega_k^* \qquad (j, k=1,\dots, 2g).$$

Then the matrix (α_{jk}) is symmetric and positive definite. We can easily see that

Riemann's period matrix (τ_{jk}) can be calculated from (α_{jk}). Furthermore, to obtain (α_{jk}) is equivalent to obtain the system of quantities

$$\beta_{jk} = \|\omega_j + \omega_k\|^2 = \alpha_{jj} + 2\alpha_{jk} + \alpha_{kk} \qquad (j,\ k = 1, \ldots, 2g).$$

2. Determination of Riemann's period matrix. We shall continue from **1.** Let f be a meromorphic function of finite valence on W. Let Ψ be the quadratic differential $\Psi = df^2$ or the logarithmic quadratic differential $\Psi = dLf^2$ of the meromorphic function f. Let $\{K_n = K(\Omega_n, 4^n\Psi)\}_{n=0}^{\infty}$ be a sequence of normal subdivisions by the quadratic differential Ψ which defines a regular exhaustion of W', where $W' = W - e$ and e is the set of critical points of Ψ outside $|K_n|$ for all n.

We shall calculate approximately the quantities β_{jk} $(j,\ k = 1, \ldots, 2g)$. Let Θ_j $(j = 1, \ldots, 2g)$ be a differential of $\Gamma_{c0}(W')$ satisfying the following conditions:

(i) the support of Θ_j is contained in Ω_0;

(ii) Θ_j has the same period as ω_j respectively along each cycle;

(iii) an integral F_j of Θ_j on $\Omega_0 - C_j$ is constant on $\partial\Omega_0$.

Since $W' \in O_G$, O_G being the class of open Riemann surfaces not admitting a Green function, we find that ω_j $(j = 1, \ldots, 2g)$ is the projection of Θ_j on $\Gamma_{h0}(W') = \Gamma_h(W)$. We set $\Theta = \Theta_j + \Theta_k$ and apply Corollary 8.1 to the present Θ. Then we have that

$$\|\sigma^n\|_{K_n}^2 \geqq \|\sigma^{n\#}\|_{\Omega_n}^2 \geqq \|\omega_j + \omega_k\|_W^2 = \beta_{jk},$$

$$\|\sigma^n\|_{K_n}^2 \searrow \|\omega_j + \omega_k\|_W^2 = \beta_{jk} \qquad (n \to \infty),$$

where σ^n $(n = 0, 1, \ldots)$ is the difference of $\Gamma_h(K_n)$ with support K_n such that $\Theta - \sigma^{n\#} \in \Gamma_{e0}(\Omega_n)$. Hence we obtain a sequence of upper bounds of β_{jk} which converges monotonously to β_{jk}.

3. Numerical calculations. The following calculation method of the period matrix is applicable to an arbitrary closed Riemann surface. Here, in order to compare our computation results with the true values, we shall choose a Riemann surface W whose period matrix can be calculated by the complete elliptic integral. Let W be the two-sheeted covering surface with four branch points $-1/k$, -1, 1, $1/k$ $(0 < k < 1)$ over the whole z-plane. Then W is a closed Riemann surface of genus 1. We can choose a canonical homology basis A_1, B_1 on W so that their projections onto the z-plane are simple closed curves around -1 and 1, and 1 and $1/k$ respectively. In this case, we find that

$$\alpha_{12} = (\omega_1, \omega_2) = \int_{A_1} \omega_2^* = 0,$$

$$\frac{\beta_{11}}{4}=\|\omega_1\|^2=\frac{1}{\|\omega_2\|^2}=\frac{4}{\beta_{22}},$$

and

$$\tau_{11}=\int_{B_1}\psi_1=-i\int_{B_1}\omega_2^*=-i\|\omega_2\|^2=-\frac{i\beta_{22}}{4}.$$

We shall exhibit some results of numerical calculations of the quantity $i\tau_{11}$ for the following cases (i), (ii) and (iii).

(i) $1/k=2$.

Let f be the projection map of W onto the z-plane, let Ψ be the quadratic differential defined by $\Psi=(8df)^2$ and let Ω be the subregion of W lying on $\{-8<x<8,\ -8<y<8\}$. Then we can construct the normal quadrangulation $K_0=K(\Omega,\ \Psi)$ of Ω by Ψ. Let K_1 and K_2 be the normal subdivision of K_0 and K_1 respectively. Let σ^n $(n=0,\ 1,\ 2)$ be the unique difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\Theta_2-\sigma^{n\#}\in\Gamma_{e0}(\Omega_n)$. Then we obtained the numerical result in Table I. We can compare this with the value $i\tau_{11}=\|\omega_2\|^2=0.6396$ calculated by the

Table I

	$\|\sigma^n\|^2_{\boldsymbol{K}_n}$	$\|\sigma^{n\#}\|^2_{\Omega_n}$
$n=0$	0.6805	0.6719
$n=1$	0.6673	0.6630
$n=2$	0.6595	0.6578

complete elliptic integral.

(ii) $1/k=e^{\pi/4}$.

Let f be the projection map of W onto the z-plane, let Ψ be the quadratic differential defined by $\Psi=((128/2\pi)d\lg f)^2$ and let Ω be the subregion of W lying on $\{e^{-5\pi/8}<|z|<e^{7\pi/8}\}$. Then we can construct the normal quadrangulation $K_0=K(\Omega,\ \Psi)$ of Ω by Ψ. Let K_1 and K_2 be the normal subdivision of K_0 and K_1 respectively. Let σ^n $(n=0,\ 1,\ 2)$ be the unique difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\Theta_2-\sigma^{n\#}\in\Gamma_{e0}(\Omega_n)$ for each n. Then we obtained the numerical result in Table II. We can compare this with the value $i\tau_{11}=\|\omega_2\|^2=0.6731$ calculated by the complete elliptic integral.

(iii) $1/k=\sqrt{2}$.

Let f be the projection map of W onto the z-plane, let Ψ be the quadratic differential defined by $\Psi=(8df)^2$ and let Ω be the subregion of W lying on $\{-8<x<8,\ -8<y<8\}$. Then we can construct a sequence of normal subdivisions $K_n=K(\Omega_n,\ 4^n\Psi)$ $(n=0,\ 1,\dots)$ which is maximal under the condition $\Omega_n\subset\Omega$

Table II

	$\|\sigma^n\|^2_{K_n}$	$\|\sigma^{n\#}\|^2_{\Omega_n}$
$n=0$	0.6974	0.6931
$n=1$	0.6909	0.6887
$n=2$	0.6876	0.6866

respectively. Then $\lim_{n\to\infty}\Omega_n=\Omega-\{\sqrt{2}, -\sqrt{2}\}$. Let σ_n $(n=0, 1, 2)$ be the unique difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\Theta_2-\sigma^{n\#}\in\Gamma_{e0}(\Omega_n)$. Then we obtained the numerical result in table III. In the present case, we know that

Table III

	$\|\sigma^n\|^2_{K_n}$	$\|\sigma^{n\#}\|^2_{\Omega_n}$
$n=0$	0.6175	0.6070
$n=1$	0.5562	0.5508
$n=2$	0.5344	0.5319

the true value of $i\tau_{11}=\|\omega_2\|^2$ is 0.5.

Next, let Ψ be the quadratic differential defined by $\Psi=((128/2\pi)d\lg(2^{-1/4}f))^2$ and Ω be the subregion of W lying on $\{2^{1/4}e^{-\pi/2}<|z|<2^{1/4}e^{\pi/2}\}$. Then we can construct a sequence of normal subdivisions $K_n=K(\Omega_n, 4^n\Psi)$ $(n=0, 1,\dots)$ which is maximal under the condition $\Omega_n\subset\Omega$. Then $\lim_{n\to\infty}\Omega_n=\Omega-\{\sqrt{2}, -\sqrt{2}\}$. Let σ_n $(n=0, 1, 2)$ be a difference of $\Gamma_{h0}(\boldsymbol{K}_n)$ with support K_n such that $\Theta_2-\sigma^{n\#}\in\Gamma_{e0}(\Omega_n)$. Then we obtained the numerical result in Table IV.

Table IV

	$\|\sigma^n\|^2_{K_n}$	$\|\sigma^{n\#}\|^2_{\Omega_n}$
$n=0$	0.6260	0.6213
$n=1$	0.5882	0.5858
$n=2$	0.5579	0.5567

The present computations were carried out on the IBM 360/75 of the University of Illinois and the FACOM 230/60 of the computer center of Kyoto University.

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