Some Remarks on the Cauchy Problem for $p$-Parabolic Equations

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In his paper [11] S. Mizohata gave a semi-group theoretic treatment of the Cauchy problem for a regularly $p$-parabolic equation. This was successfully done with the aid of an operator matrix $H_q(t)=H_q(x, t, D_x)$ introduced therein. Recently D. Ellis [2] developed a Hilbert space approach to the Cauchy problem for a uniformly $p$-parabolic equation, following in rough outline the method explored by S. Kaplan [9] in his treatment of the Cauchy problem for a parabolic operator $-\frac{\partial}{\partial t} - L(t)$, where $L(t)$ is uniformly strongly elliptic. Generally, in such an approach, special attention has been paid to find out energy estimates appropriate to the problem. As for the Cauchy problem for a specified parabolic system (§ 6 in [7]), the present author, in collaboration with K. Yoshida, has tried a generalization of Kaplan’s treatment indicated above by introducing a certain type of energy estimates.

The main purpose of this paper is to investigate the uniqueness and existence theorems of a solution to the Cauchy problem for a regularly $p$-parabolic equation from a Hilbert space approach as done by D. Ellis [2], relying upon another type of energy estimates which will be established with the aid of a prescribed operator matrix $H_q(t)$, and following the same arguments as in our treatment (§ 6 in [7]) of a parabolic system.

By the Cauchy problem we shall always mean a fine Cauchy problem as described in paper [7]. With this in mind, in Section 1, some notations and functional spaces are introduced with a precise formulation of such a Cauchy problem for a regularly $p$-parabolic equation, where the notions of the $D^k$-boundary value and the $D^k$-canonical extension of a distribution are discussed in some detail. In Section 2 the energy inequalities (cf. Theorems 1 and 2 below) for a regularly $p$-parabolic operator and for its dual operator are derived by making use of the operator matrix $H_q(t)$, which was introduced by S. Mizohata [11]. The former estimate will be of a type very similar to the one obtained in [7, Theorem 8]. These estimates enable us to apply a Hilbert space approach to our problem. Finally in Section 3 the uniqueness and existence theorems for our problem are discussed along this line of thought. We improve some of the results obtained by D. Ellis [2]. Combining Corollary 4 with Proposition 5 below, we have a
refinement of Theorem 9 in his paper [2]. This, in a sense, is a generalization of a result of S. Mizohata [11, Proposition 5]. We add here that the improvement itself has been announced in his paper [2] without proof.

1. Preliminaries

We denote by $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ an $(n+1)$-dimensional Euclidean space with a generic point $(x, t) = (x_1, \ldots, x_n, t)$ and write $D^\alpha = (D^\alpha_1, \ldots, D^\alpha_n)$, $D^\alpha_1, \ldots, D^\alpha_n$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$. For a point $\xi = (\xi_1, \ldots, \xi_n)$ of the dual Euclidean space $\Xi^n$, we write $|\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$ and $\xi^\alpha = (\xi_1^\alpha, \ldots, \xi_n^\alpha)$.

Let $p$ be a positive integer and let $a_{\alpha,j}(x, t)$ for $|\alpha| < p$, $j = 1, 2, \ldots, m$, where by $\mathcal{A}(H)$ we mean the space of $C^\infty$ functions $a$ on $H = \mathbb{R}^n \times [0, T]$ such that $a$ is bounded with its derivatives of every order. In the present paper we shall consider the differential operator

$$P = D^m_t + \sum_{j=1}^m \sum_{|\alpha| = j} a_{\alpha,j}(x, t) D^\alpha_x D_t^{m-j}, \quad (m \geq 1)$$

satisfying the following condition: for every root $\tau = \tau(x, t, \xi)$, $\xi \in \Xi^n$, of the polynomial

$$P_0(x, t, \xi, \tau) = \tau^m + \sum_{j=1}^m \sum_{|\alpha| = j} a_{\alpha,j}(x, t) \xi^\alpha \tau^{m-j}$$

in $\tau$ there exists a positive constant $\delta$, independent of $x$, $t$ and $\xi$ but depending on $T$, such that $\Im \tau \geq \delta$ for $(x, t) \in H$ and $\xi \in \Xi^n$ with $|\xi| = 1$. Then $P$ is called a regularly $p$-parabolic operator in $0 \leq t < T[11, \text{p. 269}]$. Let $P = D^m_t + \sum_{j=1}^m \sum_{|\alpha| = j} a_{\alpha,j}(x, t) D^\alpha_x D_t^{m-j}$, where $a_{\alpha,j} \in C^\infty(\bar{R}^n_{+1})$, $R^+_n = \mathbb{R}^n \times (0, \infty)$, and their restrictions $a_{\alpha,j}|_{H_T} = \mathbb{R}^n \times [0, T]$, belong to the space $\mathcal{A}(H)$ for any $T > 0$. If for any $T > 0$ there exists a positive constant $\delta_T$ such that $\Im \tau \geq \delta_T$ for $(x, t) \in H_T$ and $\xi \in \Xi^n$ with $|\xi| = 1$, then $P$ is called a regularly $p$-parabolic operator in $0 \leq t < \infty$. It is known that $p$ must be a positive even integer. In what follows, we write $p = 2p'$.

By $\mathcal{D}'((\mathcal{D}_{L^2})_x)$ we mean the $\varepsilon$-product $\mathcal{D}'(\mathcal{D}_{L^2})_x$ and by $\mathcal{D}'((\mathcal{D}_{L^2})_x)(H)$ the space of distributions $\in \mathcal{D}'(H)$ which can be extended to distributions $\in \mathcal{D}'((\mathcal{D}_{L^2})_x)$. The quotient topology is introduced in $\mathcal{D}'((\mathcal{D}_{L^2})_x)(H)$. Similarly the space $\mathcal{D}'(\mathcal{D}_{L^2})_x(R_n \times (-\infty, T])$ will be defined.

Let $u \in \mathcal{D}'((\mathcal{D}_{L^2})_x)(H)$ and suppose $u(x, \varepsilon t)$ converges in $\mathcal{D}'((\mathcal{D}_{L^2})_x)(H)$ to a distribution $v$ as $\varepsilon \downarrow 0$. Then we see that $v$ is independent of $t$ and it can be written in the form $a_0 \otimes Y_t$, where $a_0 \in (\mathcal{D}_{L^2})_x$ and $Y_t$ is a Heaviside function [6, p. 375]. $a_0$ is called the $\mathcal{D}_{L^2}$-boundary value of $u$ and denoted by $\lim_{t \uparrow 0} u$ [6, p. 375].
Let \( \phi \in \mathcal{D}(\mathbb{R}^+) \) be such that \( \phi \geq 0 \) and \( \int_0^{\infty} \phi dt = 1 \), and put \( \rho = Y^* \phi \). Consider a \( u \in \mathcal{D}'((\mathcal{D}'_L)_x)(H) \). Then \( \rho(t/\varepsilon)u \) may be regarded as an element of \( \mathcal{D}'((\mathcal{D}'_L)_x)(R_n \times (-\infty, T]) \) for any \( \varepsilon > 0 \). If \( \rho(t/\varepsilon)u \) converges in \( \mathcal{D}'((\mathcal{D}'_L)_x)(R_n \times (-\infty, T]) \) to \( v_\phi \) as \( \varepsilon \downarrow 0 \), then \( v_\phi \) does not depend on the choice of \( \phi \). The limit element is called the \( \mathcal{D}'_L \)-canonical extension of \( u \) over \( t=0 \).

The \( \mathcal{D}'_L \)-canonical extension \( u_\ldots \) exists whenever \( \mathcal{D}'_L \lim u \) exists.

In the present paper we shall consider the fine Cauchy problem

\[
\begin{cases}
Pu = f & \text{in } \hat{H} \\
\left. u_0 \equiv \mathcal{D}'_L \lim_{t \to 0} (u, D_i u, \ldots, D^n_t u) \right|_{t=0} = x
\end{cases}
\]

for preassigned \( f \in \mathcal{D}'((\mathcal{D}'_L)_x)(H) \) and \( x = (x_0, \ldots, x_{m-1}) \), \( x_j \in (\mathcal{D}'_L)_x \). Suppose there exists a solution \( u \in \mathcal{D}'((\mathcal{D}'_L)_x)(H) \) of (1). Then \( f \) and \( u \) must have the \( \mathcal{D}'_L \)-canonical extensions \( f_\ldots \) and \( u_\ldots \) over \( t=0 \) [5, p. 82; 7, p. 404].

If we put \( F = (0, \ldots, 0, f)' \) and \( U = (u_1, \ldots, u_m)' \) with \( u_j = D^j_{t=0} u \), where \( V' \) means the transposed vector of \( V \), we can rewrite (1) in vector form as

\[
\begin{cases}
LU \equiv D_t U - A(t) U = F & \text{in } \hat{H}, \\
\mathcal{D}'_L \lim_{t \to 0} U = x
\end{cases}
\]

with

\[
A(x, t, D_x) = \begin{pmatrix}
0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-a_m & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},
\]

\[
a_j = \sum_{|x| \leq j \rho} a_{x, j}(x, t) D_x^x.
\]

We shall write by \( M(x, t, \xi) \) the matrix \( A(x, t, \xi) \) with \( a_j(x, t, \xi) \) replaced by \( a^\xi_j(x, t, \xi) = \sum_{|x|=j \rho} a_{x, j}(x, t) \xi^x \).

We shall next introduce some spaces. Let \( \sigma, s \) be any real numbers. By \( \mathcal{K}_s = \mathcal{K}_s(R_n) \) [8, p. 45] we mean the set of all distributions \( u \in \mathcal{K}'(R_n) \) such that \( \hat{u} \) is a function and

\[
\|u\|_{s}^2 = \int \|\hat{u}(\xi)\|^2 (1 + |\xi|^2)^s d\xi < \infty,
\]

and by \( \mathcal{K}_{\sigma,s} = \mathcal{K}_{\sigma,s}(R_{n+1}) \) [9, p. 172] the space of all distributions \( u \in \mathcal{K}'(R_{n+1}) \) such that \( \hat{u} \) is a function and

\[
\|u\|_{\sigma,s}^2 = \int \|\hat{u}(\xi, \tau)\|^2 (1 + |\xi|^2)^s (1 + |\tau|^2)^{\sigma/p} d\xi d\tau < \infty.
\]
In what follows, we shall use the notations
\[ D_s(R_n) = \mathcal{H}_s(R_n) \times \mathcal{H}_{s-p}(R_n) \times \cdots \times \mathcal{H}_{s-(m-1)p}(R_n), \]
\[ D_{s,d}(R_{n+1}) = \mathcal{H}_{s,d}(R_{n+1}) \times \mathcal{H}_{s,d-p}(R_{n+1}) \times \cdots \times \mathcal{H}_{s,d-(m-1)p}(R_{n+1}), \]
where the norms \( \| \cdot \| \) and \( \| \cdot \| \) are defined by \( \{ \| \cdot \|^2 + \cdots + \| \cdot \|^2 \}_{s-(m-1)p}^{1/2} \) and \( \{ \| \cdot \|^2_{s,s} + \cdots + \| \cdot \|^2_{s,s-(m-1)p} \}_{s}^{1/2} \) respectively. We shall denote by \( D_{s,d}(R_n) \) and \( D_{s,d}(R_{n+1}) \) the dual spaces of \( D_s(R_n) \) and \( D_{s,d}(R_{n+1}) \) respectively. By \( J_{p,s}^{k_p}(H) \) we mean the set of all \( u \in \mathcal{D}'(\hat{H}) \) such that there exists a distribution \( v \in \mathcal{N}_{p,s}(R_{n+1}) \) with \( u = v \) in \( \hat{H} \). The norm of \( u \) is defined by \( \| u \|_{s,s} = \inf \| v \|_{s,s} \), the infimum being taken over all such \( v \). Especially, the space \( J_{p,s}^{k_p}(H) \), \( k \) being a non-negative integer, has the equivalent norm
\[ \left( \sum_{j=0}^{k} \int_0^T \| D^j u(\cdot, t) \|_{s}^{2} dt \right)^{1/2}, \]
which will also be denoted by the symbol \( \| u \|_{p,s} \). We shall consider the space \( \mathcal{H}_{s,d}(H) \), the space of all \( u \in \mathcal{H}_{s,d}(R_{n+1}) \) with supp \( u \subset H \). Then \( \mathcal{H}_{s,d}(H) \) and \( \mathcal{H}_{s,d,p}(H) \) are anti-dual Hilbert spaces with respect to an extension of the sesquilinear form \( \int_{R_n} u \overline{v} dx dt, u \in C_0^\infty(H), v \in C_0^\infty(\hat{H}) \) [7, p. 51]. The spaces \( D_{s,d}(H), D_{s,d}(H) \) and the like are similarly defined.

Consider the space \( \mathcal{H}_{p,s}(H) \). The \( \mathcal{D}'_1\)-boundary value \( \mathcal{D}'_1\lim_{t \uparrow 0} u \) exists for every \( u \in \mathcal{H}_{p,s}(H) \) if and only if \( \sigma > \sigma' \). If this is the case, \( \mathcal{D}'_1\lim_{t \uparrow 0} u \) must belong to the space \( \mathcal{H}_{s+s-p}(R_n) \). The \( \mathcal{D}'_1\)-canonical extension \( u \) exists for every \( u \in \mathcal{H}_{s,s}(H) \) if and only if \( \sigma > \sigma' \). It is also noticed that \( \mathcal{H}_{s,s}(H) \) and \( \mathcal{H}_{s,s}(H) \) may be identified for \( |\sigma| < \sigma' \) [4, p. 416]. Let \( k \) be a positive integer such that \( |\sigma - k| < \sigma' \). Then \( u_\ast \in \mathcal{H}_{s,s}(R_n \times (-\infty, T]) \) for every \( u \in \mathcal{H}_{s,s}(H) \) if and only if \( \mathcal{D}'_1\lim_{t \uparrow 0} u = \mathcal{D}'_1\lim_{t \uparrow 0} D_t u = \cdots = \mathcal{D}'_1\lim_{t \uparrow 0} D_{t-1} u = 0 \) [4, p. 419], where the \( \mathcal{D}'_1\)-boundary value coincides with the distributional boundary value [3, p. 12].

2. Energy inequalities

Let \( P \) be a regularly \( p \)-parabolic operator in \( 0 \leq t \leq T \). We shall derive energy inequalities for \( P \) and for its dual operator \( P^* \) by making use of the operator matrix \( H^q(t) \), which was constructed by S. Mizohata [11]. He starts for the construction of \( H^q(t) \) with the following consideration.

Let \( P_0(\tau) = \tau^m + \sum_{j=1}^{m} a_j(\tau, t) \tau^{m-j} \) and consider the symmetric polynomial in \( \tau \) and \( \tau' \):
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\[ K(P_0; \tau, \tau') = \frac{P_0(\tau)P_0^*(\tau') - P_0(\tau')P_0^*(\tau)}{\tau - \tau'} = \sum_{h,k=1}^{m} A_{hk} \tau^{-1} \tau'^{-1}, \]

where $P_0^*(\tau)$ stands for $\overline{P_0(\tau)}$. Then $-i A_{hk}$ is real and coincides with $-i A_{kh}$. Since all roots of the polynomial $P_0(\tau)$ lie in the half-plane $\text{Im} \tau \geq \delta_T > 0$ for $(x, t) \in H = R_n \times [0, T]$ and $\xi \in \mathbb{Z}_n$ with $|\xi| = 1$, the Hermitian form

\[ H(P_0; u_1, \ldots, u_m) = -i \sum_{h,k=1}^{m} A_{hk} u_h \bar{u}_k \]

is positive definite \([1, \text{p. 64}]. \) Let $B$ be the real symmetric matrix $(b_{hk})$ with $b_{hk} = -i A_{hk}$. Then it follows that $-i(BM - (BM)^*) \geq 0$ for $M = M(x, t, \xi)$ stated in Section 1, where $(x, t) \in H$ and $\xi \in \mathbb{Z}_n$ with $|\xi| = 1$. On the basis of these facts and applying the method of J. Leray \([10, \text{pp. 121-127}] \) in connection with hyperbolic operators to the parabolic case, S. Mizohata has obtained the proposition below.

Let us denote by $E_s = E_s(D_x)$ the operator matrix $(e_{hk})$, $e_{hk} = S^{2s-2(h-1)p}$, $h=1, \ldots, m$ and $e_{hk} = 0$ otherwise, where $S$ is a pseudo-differential operator with symbol $\lambda(\xi) = (1 + |\xi|^2)^{1/2}$. For two Hermitian matrices $C_1(x, t, D_x)$ and $C_2(x, t, D_x)$ whose components are differential operators with coefficients in $\mathcal{A}(H)$, the inequality $C_1(x, t, D_x) \leq C_2(x, t, D_x)$ means that $(C_1(x, t, D_x) \phi, \phi) \leq (C_2(x, t, D_x) \phi, \phi)$ for any $\phi \in L^2_x$ and $t \in [0, T]$, where $(, )$ means the inner product in $L^2_x$. For the system of operators $L = D_t - A(t)$ stated in Section 1 we have

**Proposition 1** (S. Mizohata). Let $q$ be any integer. Then there exists an Hermitian matrix $H_q(t) = H_q(x, t, D_x)$ such that

\[ \alpha E_q \leq H_q(t) \leq \alpha_q E_q, \]

\[ -i(H_q(t)A(t) - (H_q(t)A(t))^*) \geq \frac{\epsilon}{2} E_{q+p'} - \gamma_q E_q \]

with positive constants $\epsilon$, $\alpha$, $\alpha_q$ and $\gamma_q$, which are independent of $(x, t) \in H$, and $H_q(t)$ is an $\mathcal{L}(D_{s}, D_{s-2q})$-valued $C^\infty$ function of $t \in [0, T]$ for any real $s$.

We shall give an energy inequality for $L$. We need the following lemma (cf. Lemma 3 in \([7, \text{p. 405}] \)).

**Lemma 1.** Let $r(t)$ and $\rho(t)$ be two real-valued functions defined in the interval $0 \leq t \leq T$ and assume that $r$ is continuous and $\rho$ is non-decreasing. Then the inequality

\[ r(t) \leq C(\rho(t) + \int_0^t r(t')dt') \quad (C > 0 \text{ is a constant}) \]
implies \( r(t) \leq Ce^{ct} \rho(t) \).

**Theorem 1.** Let \( s \) be any real number. Then there exists a constant \( C_T \), independent of \( U \) and \( t_0, t_1 \) but depending on \( s \), such that

\[
(E_s) \| U(t) \|_{\bar{B}_{s+p}} + \int_{t_0}^{t_1} \| U(t) \|_{\bar{B}_{s+p}} \, dt \leq C_T (\| U(t_0) \|_{\bar{B}_{s+p}} + \int_{t_0}^{t_1} \| LU(t) \|_{\bar{B}_{s}} \, dt)
\]

for any \( t_0, t_1 \) with \( 0 \leq t_0 < t_1 \leq T \) and any \( U = (u_1, \ldots, u_m)' \), \( u_j \in C_0^\infty (\mathbb{R}_{n+1}) \).

**Proof.** Let \( U = (u_1, \ldots, u_m)' \) with \( u_j \in C_0^\infty (\mathbb{R}_{n+1}) \) and put \( F = LU \) and \( h^2(t) = (H_0(t)U(t), U(t)) \). Then we have

\[
\frac{d}{dt} h^2(t) = i(H_0(t)D_sU(t), U(t)) - i(H_0(t)U(t), D_sU(t)) + \left( \frac{d}{dt} H_0(t) \cdot U(t), U(t) \right)
\]

\[
= i(H_0(t)A(t)U(t), U(t)) - i(H_0(t)U(t), A(t)U(t)) + i(H_0(t)F(t), U(t)) - i(H_0(t)U(t), F(t)) + \left( \frac{d}{dt} H_0(t) \cdot U(t), U(t) \right)
\]

\[
\leq -\frac{\epsilon}{2} (E_p U(t), U(t)) + \gamma_0 (E_0 U(t), U(t)) + 2 |\text{Im}(H_0(t)F(t), U(t))| + \left| \left( \frac{d}{dt} H_0(t) \cdot U(t), U(t) \right) \right|
\]

and therefore

\[
h^2(t_1) - h^2(t_0) \leq -\frac{\epsilon}{2} \int_{t_0}^{t_1} \| U(t) \|_{\bar{B}_{s+p}} \, dt + (\gamma_0 + \gamma_0') \int_{t_0}^{t_1} \| U(t) \|_{\bar{B}_{s}} \, dt + \left( \frac{d}{dt} H_0(t) \cdot U(t), U(t) \right) dt
\]

with a constant \( \gamma_0' \) such that \( \frac{d}{dt} H_0(t) \leq \gamma_0 E_0, 0 \leq t \leq T \). Put \( V = S^{-s-p'} U \). Then each component \( v_j \) of \( V \) is a \((\mathcal{D}_{L^2}, \mathcal{X})\)-valued \( C^\infty \) function of \( t \in [0, T] \) and

\[
\alpha \| V(t_1) \|_{\bar{B}_{s+p}} - \alpha_0 \| V(t_0) \|_{\bar{B}_{s+p}} \leq -\frac{\epsilon}{2} \int_{t_0}^{t_1} \| V(t) \|_{\bar{B}_{s+p}} \, dt + \gamma_0 \int_{t_0}^{t_1} \| V(t) \|_{\bar{B}_{s+p}} \, dt + 2 \int_{t_0}^{t_1} |(H_0 L S^{s+p'} V(t), S^{s+p'} V(t))| dt,
\]

\[
\left| (H_0 L S^{s+p'} V(t), S^{s+p'} V(t)) \right| \leq \| (H_0 S^{s+p'} L V, S^{s+p'} V) \| + \| H_0 (A S^{s+p'} - S^{s+p'} A) V, S^{s+p'} V \|.
\]

Since \( H_0(t) \) is a continuous operator from \( D_s \) into \( D_s^0 \) for each \( t \in [0, T] \) and
$D_{p'}^*$ is the dual space of $D_{p'}$, we have the following estimates:

$$
|(H_0 S^{s+p'} LV, S^{s+p'} V)| = |(H_0 S^{s+p'} LV, S^{s+p'} V)_{D_{p'}, D_{p'}}|
$$

$$
\leq \|H_0 S^{s+p'} LV\|_{D_{p'}, D_{p'}} \|S^{s+p'} V\|_{D_{p'}}
$$

$$
\leq C_1 \|S^{s+p'} LV\|_{D_{-p}, D_{s+p}} \|V\|_{D_{s+p}}
$$

$$
= C_1 \|LV\|_{D_{s}} \|V\|_{D_{s+p}}
$$

$$
|(H_0 (AS^{s+p'} - S^{s+p'} A) V, S^{s+p'} V)|
$$

$$
= |(H_0 (AS^{s+p'} - S^{s+p'} A) V, S^{s+p'} V)_{D_{p'}, D_{p'}}|
$$

$$
\leq \|H_0 (AS^{s+p'} - S^{s+p'} A) V\|_{D_{p'}, D_{p'}} \|V\|_{D_{s+p}}
$$

$$
\leq C_2 \|(AS^{s+p'} - S^{s+p'} A) V\|_{D_{-p}, D_{s+p}} \|V\|_{D_{s+p}}
$$

with constants $C_1$ and $C_2$. Here the operator matrix $AS^{s+p'} - S^{s+p'} A$ has the form $(\alpha_h(t))$ with $\alpha_{hk}(t) = 0$ for $h \neq m$. In virtue of Proposition 15 in [6, p. 387] we see that $\alpha_{mk}(t)$ is the operator of order $\leq (m-k+1)p+s+p'-1$. Thus there exists a constant $C_3$ such that

$$
\|(AS^{s+p'} - S^{s+p'} A) V\|_{D_{-p}, D_{s+p}} \leq C_3 \|V\|_{D_{s+p-1}}
$$

and therefore we have

$$
|(H_0 LS^{s+p'} V(t), S^{s+p'} V(t))| \leq C_1 \|LV(t)\|_{D_{s}} \|V(t)\|_{D_{s+p}} + C_2 C_3 \|V(t)\|_{D_{s+p-1}} \|V(t)\|_{D_{p'}}
$$

Let $\epsilon'$ be any positive number. Then there exists a constant $C_4(\epsilon')$ such that

$$
\|V\|_{D_{s+p-1}} \leq \epsilon' \|V\|_{D_{s+p}} + C_4(\epsilon') \|V\|_{D_{s}}
$$

and we have the inequalities

$$
\|V\|_{D_{s+p-1}} \|V\|_{D_{s+p}} \leq (\epsilon' \|V\|_{D_{s+p}} + C_4(\epsilon') \|V\|_{D_{s}}) \|V\|_{D_{s+p}}
$$

$$
\leq 2\epsilon' \|V\|_{D_{s+p}} + C_5(\epsilon') \|V\|_{D_{s}}
$$

with a constant $C_5(\epsilon')$ and consequently

$$
\int_{t_0}^{t_1} (H_0 LS^{s+p'} V(t), S^{s+p'} V(t)) dt \leq \epsilon' (1 + 2C_2 C_3) \int_{t_0}^{t_1} \|V(t)\|_{D_{s+p}}^2 dt + C_6(\epsilon') \int_{t_0}^{t_1} \|LV(t)\|_{D_s}^2 dt + C_7(\epsilon') \int_{t_0}^{t_1} \|V(t)\|_{D_{s}}^2 dt
$$
Lemma 1 we obtain \((E_s)\) for \(V\). Thus the proof is complete.

For the regularly \(p\)-parabolic operator \(P\) we have the following energy inequality.

**Corollary 1.** Let \(s\) be any real number. Then there exists a constant \(C_T\), independent of \(u\) and \(t_0\), \(t_1\) but depending on \(s\), such that

\[
\sum_{j=0}^{m-1} \| D_j^s u(\cdot, t_1) \|_{L_2<s'}^2 + \sum_{j=0}^{m-1} \int_{t_0}^{t_1} \| D_j^s u(\cdot, t) \|_{L_2<s'-j}^2 dt \\
\leq C_T \left( \sum_{j=0}^{m-1} \| D_j^s u(\cdot, t_0) \|_{L_2<s'}^2 + \int_{t_0}^{t_1} \| Pu(\cdot, t) \|_{L_2<s'-j}^2 dt \right)
\]

for any \(t_0, t_1\) with \(0 \leq t_0 < t_1 \leq T\) and any \(u \in C_0^0(R_{n+1})\).

Let us consider the formal adjoint operator of \(P\):

\[
P^* = D_t^m + \sum_{j=1}^m D_t^{m-j} a_j(x, t, D_x) = D_t^m + \sum_{j=1}^m c_j(x, t, D_x) D_t^{m-j}
\]

where

\[
a_j(x, t, D_x) = \sum_{|a| \leq j} D^a a_j, \
c_j(x, t, D_x) = \sum_{|a| \leq j} a_j D^a + \sum_{|a| < j} c_{a,j} D^a
\]

Let \(v \in C_0^0(R_{n+1})\) and put \(g = P^* v\). If we write \(V = (v_1, \ldots, v_m)'\) with \(v_j = D_t^{j-1} v, j = 1, 2, \ldots, m\) and \(G = (0, \ldots, 0, g)'\), then \(P^* v = g\) can be rewritten in the vector form

\[
\hat{L} V = D_t V - C(t) V = G,
\]

where \(C(t) = C(x, t, D_x)\) is the operator matrix \(A(x, t, D_x)\) with \(a_j(x, t, D_x)\) replaced by \(c_j(x, t, D_x)\). Following the method of construction of \(H_q(t) = H_q(x, t, D_a)\) obtained by S. Mizohata with the matrix \(M(x, t, \xi)\) replaced by \(\hat{M}(x, t, \xi)\), we can find an operator matrix \(\hat{H}_q(t)\), \(q\) being any integer, such that

\[
\beta E_q \leq \hat{H}_q(t) \leq \beta E_q
\]

with positive constants \(\varepsilon, \beta, \beta_q\) and \(\gamma_q\), which are independent of \((x, t) \in H\). \(\hat{H}_q(t)\) is an \(L(D_q, D_{q-1}^*)\)-valued \(C^\infty\) function of \(t \in [0, T]\) for any real \(s\).

We shall derive the following energy inequality for \(\hat{L}\) by making use of \(\hat{H}_q(t)\).

**Theorem 2.** Let \(q\) be any integer. Then there exists a constant \(C_T\), independent of \(u\) and \(t_0, t_2\) but depending on \(q\), such that

\[
\| V(t_0) \|_{D_q} \leq C_T (\| V(t_1) \|_{D_q} + \int_{t_0}^{t_1} \| \hat{L} V(t) \|_{D_q} dt)
\]
for any $t_0, t_1$ with $0 \leq t_0 < t_1 \leq T$ and any $V=(v_1, \ldots, v_m)'$, $v_j \in C_0^\infty(R_{n+1})$.

**Proof.** Let $V=(v_1, \ldots, v_m)'$ with any $v_j \in C_0^\infty(R_{n+1})$ and put $G=L^*V$ and $h^2(t)=(\tilde{H}_q(t)V(t), V(t))$. There exists a positive constant $\beta_q$, independent of $(x, t) \in H$, such that $\frac{d}{dt}\tilde{H}_q(t) \leq \beta_q E_q$, $0 \leq t \leq T$. In the same way as in the proof of Theorem 1 we have

$$\frac{d}{dt} h^2(t) \geq \varepsilon (E_q, V, V) - (\gamma_q + \beta_q')(E_q V, V) - 2\|G\|_{D_q} \|\tilde{H}_q V\|_{D_q}^2 \quad \geq -2C_1 h^2(t) - 2C_2 \|G\|_{D_q} h(t)$$

with $C_1=(\gamma_q + \beta_q')/(2\beta)$ and a positive constant $C_2$, which implies

$$\frac{d}{dt} (e^{ct} h(t)) \geq -C_2 e^{ct} \|G(t)\|_{D_q}.$$ 

Thus we obtain

$$h(t_0) \leq e^{c(t_1-t_0)} h(t_1) + C_2 \int_{t_0}^{t_1} e^{c(t-t_0)} \|G(t)\|_{D_q} dt.$$ 

Since we have the inequalities $\beta\|V(t)\|_{L_q}^2 \leq h^2(t) \leq \beta_q \|V(t)\|_{L_q}^2$, our proof is complete.

For the formal adjoint operator $P^*$ we have the following

**Corollary 2.** Let $q$ be any integer. Then there exists a constant $C_T$, independent of $v$ and $t_0, t_1$ but depending on $q$, such that

$$\sum_{j=0}^{m-1} \|D^j v(\cdot, t_0)\|_{q_{-j}p} \leq C_T \left( \sum_{j=0}^{m-1} \|D^j v(\cdot, t_1)\|_{q_{-j}p} + \int_{t_0}^{t_1} \|P^* v(\cdot, t)\|_{q_{-(m-1)p}} dt \right)$$

for any $t_0, t_1$ with $0 \leq t_0 < t_1 \leq T$ and any $v \in C_0^\infty(R_{n+1})$.

### 3. Cauchy problem

Let us consider the fine Cauchy problem (1):

$$\begin{cases}
P u \equiv D_{n}^{m}u + \sum_{j=1}^{m} a_j (x, t, D_{x}) D_{n-j} u = f & \text{in } \dot{H},

u_{0} \equiv \mathcal{D}_{t}^{1} \lim_{t \downarrow 0} (u, D_{x}u, \ldots, D_{n-1}u) = \alpha
\end{cases}$$

for preassigned $f \in \mathcal{D}_{x}^{1} ((\mathcal{D}_{x}^{1})_{x})(H)$ and $\alpha=(\alpha_0, \ldots, \alpha_{m-1})$, $a_j \in (\mathcal{D}_{x}^{1})_{x}$, where $a_j (x, t, D_{x}) = \sum |\alpha| \leq j \alpha_{\alpha} D_{x}^{\alpha}$ with $\alpha_{\alpha} \in \mathcal{B}(H)$. As noted in [5, p. 78], $a_{\alpha}$ can be extended to a function $\in \mathcal{B}(R_{n+1})$. We assume that $a_{\alpha} \in \mathcal{B}(R_{n+1})$.

Suppose there exists a solution $u \in \mathcal{D}_{x}^{1} ((\mathcal{D}_{x}^{1})_{x})(H)$ of (1). Then $f, u$ have
the $\mathcal{D}_{L^2}$-canonical extensions $u_\sim, f_\sim$ as noted in Section 1. In addition, $u_\sim$ and $f_\sim$ satisfy the equation
\[ P(u_\sim) = f_\sim + \sum_{k=0}^{m-1} D^k_t \delta \otimes \gamma_k(0) \quad \text{in } R_n \times (-\infty, T], \]
where
\[ \gamma_k(t) = -i \sum_{j=k+1}^{m} \sum_{l=1}^{j-k} (-1)^j \binom{j-l}{k} D^l_t a_{m-j}(x, t, D_x) \gamma_{j-l}. \]

For example, $\gamma_{m-1} = -ix_0$, $\gamma_{m-2} = -ia_1 x_0 - ia_1$, $\gamma_{m-3} = -i(a_2 - (m-2)D_t a_1) x_0 - ia_1 x_0 - ia_2$, \ldots [5, p. 82]. In what follows, we shall use the notation $\Gamma_t(\alpha) = (\gamma_0(t), \ldots, \gamma_{m-1}(t))$. Then $\Gamma_t$ is an isomorphism of $D_s$ onto $D^s_{(m-1)p}$ for any real $s$.

Conversely, if $v \in \mathcal{D}'_i((\mathcal{D}_{L^2})_x)(R_n \times (-\infty, T])$ with support in $R_n \times [0, T]$ is a solution of the equation
\[ Pv = f_\sim + \sum_{k=0}^{m-1} D^k_t \delta \otimes \gamma_k(0), \]
that is,
\[ ((v, P*w)) = ((f_\sim, w)) + (\Gamma_0(\alpha), w_0), \quad w \in C_0^0(R_n \times (-\infty, T)), \]
where by $(\ , \ )$ we mean the scalar product between $\mathcal{D}'_i((\mathcal{D}_{L^2})_x)(R_n \times (-\infty, T])$ and $\mathcal{D}((-\infty, T)) \otimes_{i} (\mathcal{D}_{L^2})_x$, then the restriction $u = v|_{H}$ is a solution of the Cauchy problem (1) and $v = u_\sim$. The equation (3) implies Green's formula:
\[ (((Pu)_\sim, w)) - ((u_\sim, P*w)) = - (\Gamma_0(u_0), w_0). \]

Similarly we have the equation
\[ (((Pu)_\sim, w)) - ((u_\sim, P*w)) = (\Gamma_T(u_T), w_T) - (\Gamma_0(u_0), w_0), \]
where $w_T = \mathcal{D}_{L^2}$-lim$_{t \to T}$ $(w, D_t w, \ldots, D^{m-1}_t w)$, $u_\sim$ is the $\mathcal{D}_{L^2}$-canonical extension of $u$ over $t=T$ and $(\ , \ )$ means the scalar product between $\mathcal{D}'_i((\mathcal{D}_{L^2})_x)$ and $\mathcal{D}((-\infty, T)) \otimes_{i} (\mathcal{D}_{L^2})_x$.

Let $s$ be any real number and let $L, \tilde{L}$ be the differential operator systems that correspond to the operators $P, P^*$ respectively, which are defined in Section 1. Then we have

**PROPOSITION 2.** If $U \in D_{0,s+p}(H)$, $LU = F \in D_{0,s}(H)$ and $\mathcal{D}_{L^2}$-lim$_{t \to 0}$ $U = x \in D_{s+p}(R_n)$, then $U \in D_{p,s}(H)$ and $U$ satisfies the inequality
\[ \|U(t)\|_{\mathcal{B}_{s+p}}^p + \int_0^t \|U(t)\|_{\mathcal{B}_{s+p}}^p dt \leq C_T(\|x\|_{\mathcal{B}_{s+p}}^p + \int_0^t \|F(t)\|_{\mathcal{B}_{s}}^p dt), \quad 0 \leq t \leq T. \]
In particular, if $F=0$ and $\alpha=0$, then $U=0$.

**Proof.** From the relation $D_t U = F + A(t) U \in D_{p,s}(H)$ we see that $U \in D_{p,s}(H)$. There exists a sequence $\{\Phi_k\}$, $\Phi_k \in C_0^\infty(R_{n+1}) \times \cdots \times C_0^\infty(R_{n+1})$, such that $\{\Phi_k\}$ converges in $D_{p,s}(H)$ to $U$. The sequences $\{\Phi_k(\cdot, 0)\}$ and $\{L\Phi_k\}$ converge in $D_{s+p}$ and $D_{0,s}(H)$ to $\alpha$ and $F$ respectively. Owing to the energy inequality (Eq.), we see that $\{\Phi_k\}$ is a Cauchy sequence in $D_{0,s+p}(H)$. Let $V$ be the limit of $\{\Phi_k\}$. Clearly $V$ coincides with $U$ as a distribution and $U$ satisfies the above inequality.

**Theorem 3.** If $U \in \mathcal{D}_t'(\mathcal{D}_{L^2}(\mathcal{X})) \times \cdots \times \mathcal{D}_t'(\mathcal{D}_{L^2}(\mathcal{X}))(H)$, $LU=0$ in $\mathring{H}$ and $\mathcal{D}_{L^2}-\lim_{t \uparrow 0} U=0$, then $U=0$ in $\mathring{H}$.

**Proof.** There exist two integers $k, l$ such that $U \in D_{k,l}(H)$. Suppose $k<p$. From the relation $D_t U = A(t) U \in D_{k,l-p}(H)$ it follows that $U \in D_{k+p,l-p}(H)$. Repeating the procedure, we see that $U \in D_{p+k+1-l-p}(H)$. Thus Proposition 2 implies $U=0$.

Let us denote by $\mathcal{E}_p^t(D_s)$ (resp. $\mathcal{E}_p^t(D_s)$), $0 \leq t < T$, the space of $\mathcal{X}_s(R_n)$-valued (resp. $D_s(R_n)$-valued) continuous functions of $t \in [0, T)$. Along the same line as in the proof of the preceding theorem, we have

**Proposition 3.** If $V \in \mathcal{D}_t'(\mathcal{D}_{L^2}(\mathcal{X}))(H) \times \cdots \times \mathcal{D}_t'(\mathcal{D}_{L^2}(\mathcal{X}))(H)$, $LV=0$ in $\mathring{H}$ and $\mathcal{D}_{L^2}-\lim_{t \uparrow T} V=0$, then $V=0$ in $\mathring{H}$.

**Proof.** We can find a real $s$ such that $V \in D_{p,s}(H)$. There exists a sequence $\{\Phi_k\}$, $\Phi_k \in C_0^\infty(R_{n+1}) \times \cdots \times C_0^\infty(R_{n+1})$, such that $\{\Phi_k\}$ converges in $D_{p,s}(H)$ to $V$. The sequence $\{\Phi_k(\cdot, T)\}$ converges in $D_{s+p}$ to 0 and therefore it converges in $D_s$ to 0. On the other hand the sequence $\{L\Phi_k\}$ converges in $D_{0,s}(H)$ to 0. In virtue of Theorem 2 we have

$$\|\Phi_k(\cdot, t)\|_{D_s} \leq C_T(\|\Phi_k(\cdot, T)\|_{D_s} + \int_0^T \|L\Phi_k(t)\|_{D_s} dt)$$

and therefore $\{\Phi_k\}$ converges in $\mathcal{E}_p^t(D_s)$, $0 \leq t < T$, to 0. Thus $V$ vanishes in $\mathring{H}$.

**Corollary 3.** If $v \in \mathcal{D}_t'(\mathcal{D}_{L^2}(\mathcal{X}))(H)$, $P^*v=0$ in $\mathring{H}$ and $\mathcal{D}_{L^2}-\lim_{t \uparrow T} (v, D_tv, \ldots, D_t^{n-1}v)=0$, then $v=0$ in $\mathring{H}$.

**Theorem 4.** For any $f \in \mathcal{X}_{0,s}(H)$ and $\alpha \in D_{s+(m-1)p+p'}$ there exists a unique solution $u \in \mathcal{X}_{mp,s}(H)$ of the Cauchy problem (1) and $u$ satisfies the inequality

$$\sum_{j=0}^{m-1} \int_0^t \|D_j u(\cdot, t)\|_{I_{s+(m-j-1)p+p'}}^2 + \sum_{j=0}^{m-1} \int_0^t \|D_j u(\cdot, t)\|_{I_{s+(m-j)p}}^2 dt$$
\[ \leq C_T(\|\alpha\|_{L^2_r}^2 + \sum_{j=0}^{m-1} \sum_{j=0}^{\ell} \|D^j\phi_k(\cdot, t)\|_{L^2_r}^2 + \sum_{j=0}^{\ell} \|P\phi_k(\cdot, t)\|_{L^2_r}^2) \]

with a constant \( C_T \).

PROOF. We shall first show that \( A = \{(P\phi, \Gamma_0(\phi_0)) : \phi \in \mathcal{C}_0^r(R_{n+1})\} \) is dense in \( \mathcal{K}_{0, s}(H) \times D^{s+p}_r(R_a) \). Let \( w \in \mathcal{K}_{0, s}(H) \) and \( \beta \in D^{s-p}_r(R_a) \) such that

\[ \int_0^T (P\phi(\cdot, t), w(\cdot, t))dt + (\Gamma_0(\phi_0), \beta) = 0 \]

for any \( \phi \in C_0^r(R_{n+1}) \). If we take \( \phi \in C_0^r(H) \), then the relation is reduced to

\[ \int_0^T (P\phi(\cdot, t), w(\cdot, t))dt = 0, \]

which means \( P^*w = 0 \) in \( \tilde{H} \). If we take \( \phi \in C_0^r(H) \) such that \( \phi = 0 \) near \( t = 0 \), then

\[ 0 = \int_0^T (P\phi(\cdot, t), w(\cdot, t))dt = (\Gamma_T(\phi_T), w_T), \]

where \( \phi_T = (\phi(\cdot, T), D\phi(\cdot, T), \ldots, D^{n-1}\phi(\cdot, T)) \). Since \( \Gamma_T(\phi_T) \) may be arbitrarily taken, it follows that \( w_T = 0 \). By Corollary 3 \( w \) must vanish in \( \tilde{H} \) and therefore \( (\Gamma_0(\phi_0), \beta) = 0 \) for any \( \phi \in C_0^r(H) \), which implies \( \beta = 0 \).

For any given \( f \in \mathcal{K}_{0, s}(H) \) and \( \alpha \in D^{s+(m-1)p+p'}_r \) there exists a sequence \( \{\phi_k\} \), \( \phi_k \in C_0^r(H) \), such that \( (\phi_k(\cdot, 0), \ldots, D^{n-1}\phi_k(\cdot, 0)) \) converges in \( D^{s+(m-1)p+p'}_r \) to \( \alpha \) and \( \{P\phi_k\} \) converges in \( \mathcal{K}_{0, s}(H) \) to \( f \). In virtue of the energy inequality

\[ \sum_{j=0}^{m-1} \|D^j\phi_k(\cdot, t)\|_{L^2_r}^2 + \sum_{j=0}^{n-1} \|D^j\phi_k(\cdot, t)\|_{L^2_r}^2 + \sum_{j=0}^{\ell} \|P\phi_k(\cdot, t)\|_{L^2_r}^2 \leq C_T(\|P\phi_k(0, 0)\|_{L^2_r}^2 + \|P\phi_k(\cdot, t)\|_{L^2_r}^2 + \|P\phi_k(\cdot, t)\|_{L^2_r}^2) \]

we see that \( (\phi_k, \ldots, D^{n-1}\phi_k) \) is a Cauchy sequence in \( D_{0,s+mp}(H) \). Let \( (v_1, \ldots, v_m) \) be the limit. From the fact that \( D^j\phi_k \) converges in \( \mathcal{K}_{-j,p,a+mp}(H) \) to \( D^jv_1 \) and the space \( \mathcal{K}_{0,s+(m-j)p} \) belongs to the space \( \mathcal{K}_{-j,p,a+mp}(H) \) it follows that \( v_{j+1} = D^jv_1, j = 1, \ldots, m-1 \), and \( P^*v_1 = f \) in \( \tilde{H} \) with \( (v_1) = \alpha \). Since \( (v_1, D^jv_1, \ldots, D^{n-1}v_1) \in D_{0,s+mp}(H) \) and \( D^{n-1}v_1 = -\sum_{j=1}^{m} a_j(x, t, D^jv_1) \in \mathcal{K}_{0,a}(H) \) we see that \( (v_1, D^jv_1, \ldots, D^{n-1}v_1) \in D_{p,a+mp}(H) \) and therefore \( v_1 \in \mathcal{K}_{mp,a}(H) \), which is a unique solution of the Cauchy problem (1) (Theorem 3) and satisfies the above inequality (4).

REMARK. Theorem 4 is in a sense a generalization of a result of S. Mizohata [11, Proposition 5].

PROPOSITION 4. Let \( k \) be any non-negative integer. For any \( f \in \mathcal{K}_{kp,a}(H) \)
and \( \alpha \in D_{s+(m+k)p} \); there exists a unique solution \( u \in \mathcal{X}_{(m+k)p}(H) \) of the Cauchy problem (1) and \( u \) satisfies the inequality

\[
\sum_{j=0}^{m+k-1} \| D_ju(\cdot, t) \|^2_{s+(m+k-j)p-p'} + \sum_{j=0}^{m+k-1} \int_0^t \| D_ju(\cdot, t) \|^2_{s+(m+k-j)p} dt \\
\leq C_T(\| \alpha \|^2_{2_s+(m+k)p-p'} + \| D_jf(\cdot, 0) \|^2_{s+(k-j)p-p'} + \| f(\cdot, 0) \|^2_{s+p'} + \sum_{j=0}^{m+k-1} \int_0^t \| D_jf(\cdot, t) \|^2_{s+(k-j)p} dt), \quad 0 \leq t \leq T,
\]

with a constant \( C_T \).

**Proof.** In the case where \( k=0 \), the statement coincides with Theorem 4. Let us consider the case \( k \geq 1 \). Since \( f \in \mathcal{X}_{kp}(H) \subset \mathcal{X}_{0,s+kp}(H) \) it follows from Theorem 4 that there exists a unique solution \( u \in \mathcal{X}_{mp+kp}(H) \) of (1). For any \( \alpha \in D_{s+(m+k)p} \), the unique solution satisfies

\[
\| u \|^2_{D_{s+(m+k)p}} + \int_0^t \| u \|^2_{\mathcal{A}_{s+(m+1)p}} dt \leq C_T(\| \alpha \|^2_{D_{s+(m+k)p}} + \int_0^t \| f(t) \|^2_{s+p} dt)
\]

with a constant \( C_T \). Put \( V = DU \). Then \( V \in D_{0,s+mp}(H) \), \( D_t V - A(t)V = D_tF + D_tA(t)U \in D_{0,s+mp}(H) \), \( D_{t+1}V \rightarrow \lim_{t \rightarrow 1} V \in D_{s+(m-1)p+mp} \) and therefore \( V \) satisfies

\[
\| V(t) \|^2_{D_{s+(m-1)p+mp}} + \int_0^t \| V(t) \|^2_{D_{s+mp}} dt \\
\leq C_T(\| V(0) \|^2_{D_{s+(m-1)p+mp}} + \int_0^t \| D_tV(t) \|^2_{s+p} dt + \int_0^t \| U(t) \|^2_{D_{s+mp}} dt)
\]

with constant \( C_T \), where

\[
\| V(0) \|^2_{D_{s+(m-1)p+mp}} \leq C_1(\| \alpha \|^2_{D_{s+mp+p'}} + C_2(\| f(\cdot, 0) \|^2_{s+p'} + \sum_{j=0}^{m+k-1} \int_0^t \| D_jf(\cdot, t) \|^2_{s+(k-j)p} dt)
\]

with constants \( C_1 \) and \( C_2 \). Summing (6) and (7) and applying Lemma 1 to the result, we have

\[
\int_0^t \| D_ju(\cdot, t) \|^2_{s+(m-j)p+p'} + \sum_{j=0}^{m+k-1} \int_0^t \| D_ju(\cdot, t) \|^2_{s+(m-j+1)p} dt \\
\leq C_T(\| \alpha \|^2_{D_{s+mp+p'}} + \| f(\cdot, 0) \|^2_{s+p'} + \sum_{j=0}^{m+k-1} \int_0^t \| D_jf(\cdot, t) \|^2_{s+(1-j)p} dt)
\]
with a constant \( C \). Repeating this procedure we obtain (5).

Let \( k \) be a positive integer and put \( \eta_0 = \mathcal{X}_{\sigma,0}(H) \), \( \eta_1 = \mathcal{X}_{k\sigma,0}(H) \). Then \( \eta_1 \) is dense in \( \eta_0 \) and \( ||u||_{\eta_0} \leq ||u||_{\eta_1} \) for any \( u \in \eta_1 \) and therefore there exists an unbounded self-adjoint operator \( J \) in \( \eta_0 \) with domain \( \eta_1 \), which generates a Hilbert scale \( \{ \eta_\lambda \}_{-\infty < \lambda < \infty} \). In the same way as in the proof of Corollary 4 in [5, p. 97] we see that \( \eta_\lambda = \mathcal{X}_{k\lambda,0}(H) \) within the equivalent norms. From the preceding proposition the map \( (f, \alpha) \mapsto u \) which assigns a unique solution \( u \) to the data \( (f, \alpha) \) is continuous from \( \mathcal{X}_{\sigma,0}(H) \times D_{s+mp-p'} \) into \( \mathcal{X}_{mp,0}(H) \) and from \( \mathcal{X}_{k\sigma,0}(H) \times D_{s+(m+k)p-p'} \) into \( \mathcal{X}_{(m+k)p,0}(H) \). By the interpolation theorem we obtain

**Corollary 4.** Let \( \sigma \) be any non-negative number. For any \( f \in \mathcal{X}_{\sigma,0}(H) \) and \( \alpha \in D_{s+mp-p'} \) there exists a unique solution \( u \in \mathcal{X}_{\sigma+mp,0}(H) \) of the Cauchy problem (1) and \( (f, \alpha) \mapsto u \) is a continuous map from \( \mathcal{X}_{\sigma,0}(H) \times D_{s+mp-p'} \) into \( \mathcal{X}_{\sigma+mp,0}(H) \).

We shall denote by \( \mathcal{X}_{\sigma,0}(H_-) \) the space which is a restriction of the space \( \mathcal{X}_{\sigma,0}(\mathbb{R}_{+1}) \) to \( \mathbb{R} \times (-\infty, T) \) and similarly \( \tilde{D}_{\sigma,0}(H_-) \) is defined.

**Proposition 5.** Let \( \sigma \) be a real number with \( -p' < \sigma < 0 \). For any \( f \in \mathcal{X}_{\sigma,0}(H) \) and \( \alpha \in D_{s+mp-p'} \) there exists a unique solution \( u \in \mathcal{X}_{\sigma+mp,0}(H) \) of the Cauchy problem (1) and \( (f, \alpha) \mapsto u \) is a continuous map from \( \mathcal{X}_{\sigma,0}(H) \times D_{s+mp-p'} \) into \( \mathcal{X}_{\sigma+mp,0}(H) \).

**Proof.** Let \( f \in \mathcal{X}_{\sigma,0}(H) \) and \( \alpha \in D_{s+mp-p'} \). Since \( -p' < \sigma < 0 \) the \( \mathcal{D}_L \)-canonical extension \( f_- \) belongs to the space \( \mathcal{X}_{\sigma,0}(H_-) \). Let \( g \in \mathcal{X}_{\sigma+mp,0}(H_-) \) be such that \( (D_\lambda - i\lambda \rho(D_\lambda))m g = f_- \), where \( \lambda(D_\lambda) \) is the operator with symbol \( \lambda(\xi) = (1 + |\xi|^2)^{1/2} \). Then it follows from Corollary 3 in [6, p. 393] that \( \mathcal{D}_L \)-limit \( (g, D_\lambda g, \ldots, D_\lambda^{m-1} g) = 0 \). The Cauchy problem (1) is reduced to

\[
\left\{
\begin{array}{l}
P(D)(u-g) = \sum_{j=1}^{m} ((-i)^j \left( \begin{array}{c}
\lambda \\
j
\end{array} \right) \lambda^j \rho(D_\lambda) - a_j(x, t, D_\lambda)) D_\rho^{m-j} g \quad \text{in } \mathcal{H} \\
\mathcal{D}_L \text{-lim}_{t \to 0} ((u-g), D_t(u-g), \ldots, D_t^{m-1}(u-g)) = \alpha,
\end{array}
\right.
\]

where \( \sum_{j=1}^{m} ((-i)^j \lambda^j \rho(D_\lambda) - a_j(x, t, D_\lambda)) D_\rho^{m-j} g \in \mathcal{X}_{\sigma+mp,s-p}(H_-) \) with \( \sigma + p > p' \). It follows from Corollary 4 that there exists a unique solution \( v \in \mathcal{X}_{\sigma+(m+1)p,s-p}(H_-) \) of the Cauchy problem (8). Thus \( u = v + g \in \mathcal{X}_{\sigma+mp,0}(H) \) is a unique solution of the Cauchy problem (1). In view of the closed graph theorem it follows that \( (f, \alpha) \mapsto u \) is a continuous map from \( \mathcal{X}_{\sigma,0}(H) \times D_{s+mp-p'} \) into \( \mathcal{X}_{\sigma+mp,0}(H) \).

Let \( \sigma, s \) be any real numbers and write \( \sigma = kp + \sigma' \) with integer \( k \) and \( -p' < \sigma' \leq p' \). Then we have the following

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**Theorem 5.** For any $\alpha \in \mathcal{D}_{\sigma+\alpha+mp}(\mathbb{H}_-)$ and $f \in \mathcal{K}_{\sigma+mp}(\mathbb{H})$ there exists a unique solution $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H})$ of the Cauchy problem (1). In particular, if $\alpha = 0$ then $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$. 

**Proof.** Consider the case where $k \geq 0$. By Proposition 5 and Corollary 4 it suffices to show that $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$ for $\alpha = 0$. Suppose $\alpha = 0$, that is, $\mathcal{D}_{L_2} \lim (u, \ldots, D_t^{\sigma-1}u) = 0$. If $k > 0$ then $f \in \mathcal{K}_{kp+\alpha}(\mathbb{H}_-)$. From the equation $P(D)u = f$ we obtain $\mathcal{D}_{L_2} \lim (u, \ldots, D_t^{\sigma-1}u) = 0$. If $\sigma' < p'$ then $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$. If $\sigma' = p'$ then $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$ and $\mathcal{D}_{L_2} \lim (u, \ldots, D_t^{\sigma-1}u) = 0$. Thus there exists a unique solution $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$. Consequently, $u = \mathcal{D}_{L_2} \lim (u, \ldots, D_t^{\sigma-1}u) = 0$. 

**Proposition 6.** For any $h \in \mathcal{K}_{\sigma+\alpha}(\mathbb{H}_-)$ there exists a unique solution $v \in \mathcal{K}_{\sigma+\alpha}(\mathbb{H}_-)$ of $Pv = h$. 

**Proof.** In the case where $\sigma > p'$, the problem to find a solution $v$ of $Pv = h$ is equivalent to the problem to find a solution $u$ of the Cauchy problem $Pu = h$ with $\mathcal{D}_{L_2} \lim (u, \ldots, D_t^{\sigma-1}u) = 0$. Thus there exists a unique solution $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H})$ and $u \in \mathcal{K}_{\sigma+mp}(\mathbb{H}_-)$. 

In the case where $\sigma \leq -p'$, our assertion will follow in the same way as in the proof of Theorem 5. 

Let $P$ be a regularly $p$-parabolic operator in $0 \leq T < \infty$ and consider the Cauchy problem 

\[
\begin{align*}
Pu = f & \quad \text{in } R^n_{+1}, \\
\mathcal{D}_{L_2} \lim (u, D_t u, \ldots, D_t^{\sigma-1}u) = & \quad \alpha
\end{align*}
\]

for given $\alpha \in (\mathcal{D}_{L_2})_1 \times \cdots \times (\mathcal{D}_{L_2})_9$ and $f \in \mathcal{D}(R^n_{+1})(R^i_+) = \mathcal{D}(R^i_+)\mathcal{D}(\mathcal{D}_{L_2})_9$ which has the $\mathcal{D}_{L_2}$-canonical extension $f_\alpha$. From the fact that Theorem 3 holds true of any $H_T = R_\alpha \times [0, T]$, the Cauchy problem (9) is unique in $\mathcal{D}(\mathcal{D}_{L_2}_9(R^n_{+1})$. 

The spaces $\mathcal{X}_{\sigma,s}(R^+_{n+1})$ and $\mathcal{X}_{\sigma,s}'(R^+_{n+1})$ are defined in the same way as $\mathcal{X}_{\sigma,s}(H)$ and $\mathcal{X}_{\sigma,s}'(H)$. By $\mathcal{X}_{\sigma,s}(R^+_{n+1})$ we mean the space of $u \in \mathcal{D}'(R^+_{n+1})$ such that $\phi u \in \mathcal{X}_{\sigma,s}(R^+_{n+1})$ for any $\phi \in \mathcal{D}(R)$ and the topology is defined by the semi-norms $u \to \|\phi u\|_{\sigma,s}$. Along the same way as in the proof of Theorem 5 and Proposition 6 we have the following

**Theorem 5'**. For any $\alpha \in D_{\sigma + s + m + p'}$ and $f \in \mathcal{X}_{\sigma,s}''(R^+_{n+1})$ with $f_\triangleleft \in \mathcal{X}_{\sigma,s}(R^+_{n+1})$ there exists a unique solution $u \in \mathcal{X}_{\sigma + m + p,s}(R^+_{n+1})$ of the Cauchy problem (9). In particular, if $\alpha = 0$ then $u_\triangleleft \in \mathcal{X}_{\sigma + m + s}(R^+_{n+1})$.

**Proposition 6'**. For any $h \in \mathcal{X}_{\sigma,s}''(R^+_{n+1})$ there exists a unique solution $v \in \mathcal{X}_{\sigma + m + s}(R^+_{n+1})$ of $Pv = h$.

Let us denote by $\mathcal{D}'_t$ the subspace of $\mathcal{D}'$ which consists of all one-dimension- al distributions with support contained in $[0, \infty)$ and by $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ the $\epsilon$-product $\mathcal{D}'_t \times (\mathcal{D}'_t)_+$, which is a reflexive, ultrabornological Souslin space [6, p. 372]. In the same way as in the proof of Theorem 5 [7, p. 415] we have

**Theorem 6**. For any $h \in (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ there exists a unique solution $v \in (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ of $Pv = h$ and $h \to v$ is a continuous map from $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ onto itself.

**Proof**. Take a sequence $\{t_j\}$ of real numbers such that $t_0 < 0 < t_1 < t_2 < \ldots$, $\lim_{j \to \infty} t_j = \infty$ and put $U_j = (t_j, t_{j+2})$. Let $\{\phi_j\}$ be a partition of unity subordinate to the covering $\{U_j\}_{j=0,1,\ldots}$ of $(0, \infty)$ and consider the equations $P\phi_j = \phi_j f_j$ for $j = 0, 1, \ldots$, where $\phi_j f \in \mathcal{X}_{\sigma + s_j}(R^+_{n+1})$. In virtue of Proposition 6' there exists a unique solution $v_j \in \mathcal{X}_{\sigma + m + s_j}(R^+_{n+1}) \subseteq (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$. By our energy inequality (E) we see that $v_j = 0$ for $t < t_j$. Thus $v = \sum v_j$ is well defined in $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ and $v$ is unique in $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$.

Consider the map

$$l: (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+ \ni v \to Pv \in (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+,$$

which is linear, continuous and onto. Since the space $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ is ultrabornological and Souslin it follows from the open mapping theorem that $l$ is an epimorphism. Thus the proof is complete.

As a consequence of Theorem 6 we can state the following

**Theorem 7**. For any $\alpha \in (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$ with $f_\triangleleft \in (\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$, the fine Cauchy problem (9) has a unique solution $u \in \mathcal{D}'(R^+_{n+1})$ and $(f_\triangleleft, \alpha) \to u_\triangleleft$ is a continuous map under the topology of $(\mathcal{D}'_t)_+(\mathcal{D}'_t)_+$.

We shall close this paper with some remarks on the Cauchy problem (2):
Some Remarks on the Cauchy Problem for $\frac{d}{dt}$-Parabolic Equations

$$\begin{aligned}
\begin{cases}
LU = D_t U - A(t) U = F & \text{in } \dot{H}, \\
\mathcal{D}_x^{\frac{d}{dt}} \lim_{t \to 0} U = \alpha
\end{cases}
\end{aligned}$$

for preassigned $F \in \mathcal{D}'((\mathcal{D}'_x)_x)(H) \times \cdots \times \mathcal{D}'((\mathcal{D}'_x)_x)(H)$ with $\mathcal{D}_x^{\frac{d}{dt}}$-canonical extension $F_{\infty}$ and $\alpha \in (\mathcal{D}_x^{\frac{d}{dt}})_x \times \cdots \times (\mathcal{D}_x^{\frac{d}{dt}})_x$. As shown in Theorem 1 the energy inequality $(E_\alpha)$ holds true for any $U = (u_1, \ldots, u_m)'$, $u_j \in C_0^\infty(R_{n+1})$.

Let $s$ be any real number. If for any $F \in D_{0,s}(H)$ and $\alpha \in D_{s+p}^\prime(R_n)$ there exists a solution $U \in D_{0,s+p}(H)$ of the Cauchy problem (2) we shall say that $(CP)_s$ holds for $L$. As shown in Theorem 3, $U$ is uniquely defined if it exists. In the same way as in the proof of Proposition 7' in [7, p. 434] we have

**Proposition 7.** $(CP)_s$ holds for $L$ if and only if the conditions that $W \in D_{0,-s}(H)$, $L^* W = 0$ in $\dot{H}$ and $\mathcal{D}_x^{\frac{d}{dt}} \lim W = 0$ imply $W = 0$ in $\dot{H}$.

**Lemma 2.** Suppose $(CP)_s$ holds for some $s$. Then, for any $F \in C_0^\infty(H) \times \cdots \times C_0^\infty(H)$ and $\alpha \in C_0^\infty(R_n) \times \cdots \times C_0^\infty(R_n)$ a unique solution $U$ of the Cauchy problem (2) belongs to the space $D_{0,s}(H)$ for any $s'$.

**Proof.** From our assumption it follows that $U \in D_{0,s+p}(H)$. If we put $V_1 = \lambda(D_x) U$, then

$$\begin{aligned}
\begin{cases}
D_t V_1 + A(t) V_1 = \lambda(D_x) F + (A(t) \lambda(D_x) - \lambda(D_x) A(t)) U & \text{in } \dot{H}, \\
\mathcal{D}_x^{\frac{d}{dt}} \lim_{t \to 0} V_1 = \lambda(D_x) \alpha
\end{cases}
\end{aligned}$$

where $\lambda(D_x) F \in \mathcal{S}(H)$, $\lambda(D_x) \alpha \in \mathcal{S}(R_n)$ and $(A(t) \lambda(D_x) - \lambda(D_x) A(t)) U \in D_{0,s}(H)$ [6, p. 387]. From our assumption it follows that $V_1 = \lambda(D_x) U \in D_{0,s+p}(H)$ and therefore $U \in D_{0,s+p+1}(H)$.

If we put $V_2 = \lambda^2(D_x) U$, then

$$\begin{aligned}
\begin{cases}
D_t V_2 + A(t) V_2 = \lambda^2(D_x) F + (A(t) \lambda^2(D_x) - \lambda^2(D_x) A(t)) U & \in D_{0,s}(H), \\
\mathcal{D}_x^{\frac{d}{dt}} \lim_{t \to 0} V_2 = \lambda^2(D_x) \alpha \in \mathcal{S}(R_n).
\end{cases}
\end{aligned}$$

Thus $V_2 = \lambda^2(D_x) U \in D_{0,s+p}(H)$ and therefore $U \in D_{0,s+p+2}(H)$. Repeating this procedure, we see that $U \in \cap_s D_{0,s}(H)$.

**Proposition 8.** If $(CP)_s$ holds for some $s$, then it does also for any $s'$.

**Proof.** For any given $F \in D_{0,s}(H)$ and $\alpha \in D_{s+p}^\prime(R_n)$ there exist two sequences $\{F_j\}$, $F_j \in C_0^\infty(H) \times \cdots \times C_0^\infty(H)$ and $\{\alpha_j\}$, $\alpha_j \in C_0^\infty(R_n) \times \cdots \times C_0^\infty(R_n)$ such that $\{F_j\}$ and $\{\alpha_j\}$ converge in $D_{0,s}(H)$ and $D_{s+p}^\prime(R_n)$ respectively. Let $U_j$ be a unique solution of the Cauchy problem (2) for $L$ associated with $F_j$ and $\alpha_j$. 


Then $U_j$ belongs to the space $\bigcap D_{0,s}(H)$ and it satisfies the energy inequality

$$\|U_j(t)\|_{H^{s+1}} + \int_0^T \|F_j(t)\|_{H^s}^2 dt \leq C_T(\|x_j\|_{H^{s+1}} + \int_0^T \|F_j(t)\|_{H^{s+1}}^2 dt), \quad 0 \leq t \leq T,$$

with a constant $C_T$, which implies that $\{U_j\}$ is a Cauchy sequence in $D_{0,s'+p}(H)$. By the relation $D_t U_j = F_j - A(t)U_j \in D_{0,s}(H)$ we see that $\{U_j\}$ is also a Cauchy sequence in $D_{p,s'}(H)$. Let $U$ be the limit of $U_j$ in $D_{p,s'}(H)$. Then $U \in D_{p,s'}(H)$ satisfies $LU = F$ in $\bar{H}$ and $D'_{L,t}-\lim U = \alpha$, which means that $(CP)_{\epsilon}$ holds true.

From the energy inequality $(E_\alpha)$ and Proposition 8 we can prove the following proposition in the same arguments as used in [7, Proposition 6].

**Proposition 9.** If for any $F \in D_{0,s}(H)$ and $\alpha \in D_{s+p}^\prime (R_n)$ the Cauchy problem (2) has a solution $U \in D'_{s+(\alpha \in D_{s+1}(\alpha))(H) \times \cdots \times D'_{s+(\alpha \in D_{s+1}(\alpha))(H) \times H^s}$, Hiroshima Math. J. 1 (1971), 405–425.

If we suppose $(CP)_{\alpha}$ for $L$, then our discussions on the Cauchy problem for a specified parabolic system given in Section 6 of [7] can be applied also to the Cauchy problem for $L$.

### References


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