## On the l-Number of 1-Cycles on an Abelian Variety

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## Introduction

For any divisor D on an abelian variety X of dimension r, we denote by  $\phi_D$  the homomorphism from X to the Picard variety  $\hat{X}$  of X associated with the divisorial correspondence  $m^*\mathcal{O}_X(D) \otimes p_1^*\mathcal{O}_X(-D) \otimes p_2^*\mathcal{O}_X(-D)$  on  $X \times X$ , where  $m: X \times X \to X$  is the addition morphism of X and  $p_i: X \times X \to X$  means the projection to the *i*-th component for i=1, 2. By Riemann-Roch theorem, the number of Euler-Poincaré  $\chi(\mathcal{O}_X(D))$  is related with the self-intersection number  $(D^r)$  and with deg  $\phi_D$  as follows;

$$\chi(\mathscr{O}_{\chi}(D)) = \frac{(D^r)}{r!}$$
 and  $\deg \phi_D = \{\chi(\mathscr{O}_{\chi}(D))\}^2$ .

In particular, if D is ample, then  $\chi(\mathcal{O}_X(D)) = \dim H^{\circ}(X, \mathcal{O}_X(D)) = l(D)$ , where l(D) is the dimension of the linear system L(D), and deg  $\phi_D$  becomes  $l(D)^2$ .

On the other hand, for 1-cycles on an abelian variety, an analogy to the relation deg  $\phi_D = 1(D)^2$  was given by K. Toki [6], in the following way; that is, for a positive 1-cycle C on an r-dimensional abelian variety X over the complex number field C, the 1-number l(C), defined by  $l(C) = \text{deg}(r_*C)/r!$  where  $r_*C$ means the r-times Pontrjagin product of C, satisfies the equality

$$\deg \phi_C = l(C)^2.$$

Here  $\phi_c$  is the homomorphism from Picard variety  $\hat{X}$  of X to X defined by  $\phi_c(\hat{x}) = S(C \cdot (P|_{\{\hat{x}\} \times X}))$  for any  $\hat{x} \in \hat{X}$ , where P is the Poincaré divisor on  $\hat{X} \times X$  and S means the sum on X. He proved this equality replacing the intersection and the Pontrjagin products by cohomological languages.

The purpose in this paper is to prove the same equality with no restriction on the characteristic of the ground field, by means of Jacobian varieties of curves.

In  $n^{\circ}1$ , we shall review some results on abelian varieties and curves as preliminaries for our main theorem which will be proved in  $n^{\circ}2$ .

My hearty thanks are due to Prof. Shoji Koizumi for his critisism, and his valuable advice.

1. Let k be an algebraically closed field of any characteristic, which will be fixed throughout the paper. Let K, X and Y be three abelian varieties over k. If these abelian varieties form an exact sequence  $0 \longrightarrow K \xrightarrow{\iota} X \xrightarrow{\pi} Y \longrightarrow 0$  in the

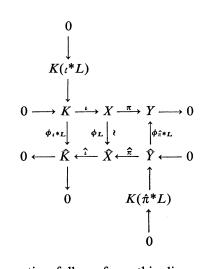
category of commutative group schemes of finite type over the field k, then the dual varieties of them also make the exact sequence  $0 \longrightarrow \hat{Y} \stackrel{\hat{\pi}}{\longrightarrow} \hat{X} \stackrel{\hat{\iota}}{\longrightarrow} \hat{K} \longrightarrow 0$ .

For an invertible sheaf L on an abelian variety X, as stated in the introduction we denote by  $\phi_L$  the homomorphism associated with the divisorial correspondence  $m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$  on  $X \times X$ , where  $m: X \times X \to X$  is the addition morphism of X and  $p_i: X \times X \to X$  means the projection to the *i*-th component for i=1, 2. We denote by K(L) the scheme theoretic kernel of  $\phi_L$ . Under this situation we obtain

**PROPOSITION 1.** Let K, X, Y and  $\iota$ ,  $\pi$  be as above. Let L be a non-degenerate invertible sheaf on X with  $|\chi(L)| = 1$ . Then  $K(\iota^*L) \cong K(\pi^*L)$ , in particular  $\deg(\phi_{\iota^*L}) = \deg(\phi_{\pi^*L})$ .

In fact, let  $P_X$  and  $P_Y$  be the Poincaré invertible sheaves of  $\hat{X} \times X$  and  $\hat{Y} \times Y$  respectively. We identify X and  $\hat{\hat{X}}$  (resp. Y and  $\hat{\hat{Y}}$ ) by the divisorial correspondence  $P_X$  (resp.  $P_Y$ ). Moreover we identify X and  $\hat{X}$  by the isomorphism  $\phi_L$ .

Then under these identifications,  $\phi_L^{-1} = \phi_{(\phi_L)\star L}$  and we obtain the commutative diagram:



Then obviously our assertion follows from this diagram.

Moreover we quote a result of T. Matsusaka [2], which will paly an essential part in the proof of our main theorem in next section.

Let  $J^g$  be the Jacobian variety of a complete non-singular curve C,  $\varphi$  be the canonical inclusion map of C into J, and  $\Theta$  be the corresponding canonical divisor on J. For any integer r such that  $1 \le r \le g$ , we put

 $\varphi^{(r)} = \varphi \circ p_1 + \dots + \varphi \circ p_r$ :  $C^r = \overbrace{C \times \dots \times C}^r \to J$ , and  $W^r$  = the set theoretic image of  $\varphi^{(r)}$ , where  $p_i: \overbrace{C \times \dots \times C}^r \to C$  means the projection to the *i*-th component for

i = 1, ..., n.

PROPOSITION 2. deg $(\varphi^{(r)}: C^r \rightarrow W^r) = r!$ . (cf. [1], p. 37)

**PROPOSITION 3.** (T. Matsusaka) Let  $u_1, ..., u_{g-r}$  be g-r independent generic points of J. Then  $\Theta_{u_1} ... \Theta_{u_{g-r}}$  is numerically equivalent to  $(g-r)!W^r$ . (cf. [2], appendix.)

2. Let X be an abelian variety of dimension r over the field k. For an irreducible curve C on X, and its inclusion morphism f, we define a morphism  $f^{(s)}$  by

$$f^{(s)} = f \circ p_1 + \dots + f \circ p_s \colon C^s = \overbrace{C \times \dots \times C}^s \to X$$

for any positive integer s, where  $p_i$  means as above. In particular for s=0, we mean by  $C^{\circ}$  a point and by  $f^{(\circ)}$  a constant map  $C^{\circ} \hookrightarrow X$  with value origin O.

For a given positive integer n, let  $C_i(i=1, ..., n)$  be irreducible curves on Xand  $f_i$  be the inclusion morphism  $C_i \hookrightarrow X$  for each i. Let  $r_1, ..., r_n$  be n nonnegative integers such that  $r_1 + \cdots + r_n = r$ . We denote by  $F^{(r_1, ..., r_n)}$  the morphism

$$f_1^{(r_1)} \circ p_1^{(r_1)} + \cdots + f_n^{(r_n)} \circ p_n^{(r_n)} \colon C_1^{r_1} \times \cdots \times C_n^{r_n} \longrightarrow X,$$

and we put  $d(C_1^{r_1}, ..., C_n^{r_n}) = \deg F^{(r_1, ..., r_n)}$ , where  $p_i^{(r_i)}: C_1^{r_1} \times \cdots \times C_n^{r_n} \to C_i^{r_i}$ means the projection to the component  $C_i^{r_i}$  for each *i*. Obviously, if  $r_i > \text{genus}$  of  $C_i$  for some *i*, then  $d(C_1^{r_1}, \cdots, C_n^{r_n}) = 0$ .

For any 1-cycle  $C = \sum_{i=1}^{n} m_i C_i \in \mathscr{Z}^1(X)$ , we define the Pontrjagin product  ${}^*C$  and its degree by

$${}_{*}^{r}C = \sum_{\substack{r_{1}+\cdots+r_{n}=r\\r_{1}\geq 0}} \frac{r!}{r_{1}!\cdots r_{n}!} m_{1}^{r_{1}}\cdots m_{n}^{r_{n}}d(C_{1}^{r_{1}},\ldots,C_{n}^{r_{n}}) \cdot X$$

and

deg 
$$\binom{r}{*}C$$
 =  $\sum_{\substack{r_1+\dots+r_n=r\\r_1\geq 0}} \frac{r!}{r_1!\cdots r_n!} m_1^{r_1} \dots m_n^{r_n} d(C_1^{r_1}, \dots, C_n^{r_n})$ , respectively.

Obviously these definitions do not depend on the expression  $\sum_{i=1}^{n} m_i C_i$  of C. Then from the above notice, we obtain

(1) 
$$\deg ({}^{r}_{*}C) = \sum_{\substack{r_{1}+\cdots+r_{n}=r\\genus of c_{i}\geq r_{i}\geq 0}} \frac{r!}{r_{1}!\ldots r_{n}!} m_{1}^{r_{1}}\ldots m_{n}^{r_{n}}d(C_{1}^{r_{1}},\ldots,C_{n}^{r_{n}}).$$

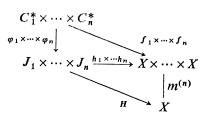
According to K. Toki, we define the *l*-number of a positive 1-cycle C by

 $l(C) = \deg({}^{*}_{*}C)/r!$ . Moreover for a 1-cycle C on X, we define the homomorphism  $\phi_{C}$  from  $\hat{X}$  to X by  $\phi_{C}(\hat{x}) = S(C \cdot (P|_{\{\hat{x}\} \times X}))$  for any  $\hat{x} \in \hat{X}$ , where P is the Poincaré divisor on  $\hat{X} \times X$ , and S means the sum on X.

From now on we denote by C a positive 1-cycle on X, which we write  $\sum_{i=1}^{n} C_i$ (C<sub>i</sub>'s may not be distinct.), and we assume that  $C_i$  contains the origin, for each *i*. Let  $C_i^*$  be a non-singular model of  $C_i$  and  $J_i$  the Jacobian variety of it. Let  $\varphi_i$  denote the canonical inclusion map of  $C_i^*$  into  $J_i$ , and we may assume that for each *i*,  $\varphi_i(C_i^*)$  contains the origin. Then there exists a canonical homomorphism  $h_i$  such that

 $\begin{array}{ccc} C_i^* & \xrightarrow{f} & X \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & &$ 

Therefore we obtain the commutative diagram:



where H and  $m^{(n)}$  mean the morphism  $h_1 \circ p_1 + \cdots + h_n \circ p_n$  and the addition morphism respectively. Moreover let  $\Theta_i$  be a canonical divisor on  $J_i$  corresponding to  $\varphi_i(C_i^*)$  for each i=1, ..., n. Then we obtain

PROPOSITION 4.  $\phi_C = H \circ (\phi_{\Theta_1}^{-1} \times \cdots \times \phi_{\Theta_n}^{-1}) \circ \hat{H}$ , i.e.,  $\phi_C = \phi_{H^*\Theta}$ , where  $\Theta = \sum_{i=1}^n J_1 \times \cdots \times J_{i-1} \times \Theta_i \times J_{i+1} \times \cdots \times J_n$ .

PROOF. Obviously  $m^{(n)} = \Delta_{\hat{X}} : \hat{X} \hookrightarrow \hat{X} \times \cdots \times \hat{X}$ ; the diagnoal map. Therefore by the diagram (2),

$$H = m^{(n)} \circ (h_1 \times \cdots \times h_n)$$

and

(2)

$$\hat{H} = (\hat{h}_1 \times \dots \times \hat{h}_n) \circ m^{(n)}$$
$$= (\hat{\varphi}_1^{-1} \times \dots \times \hat{\varphi}_n^{-1}) \circ (\hat{f}_1 \times \dots \times \hat{f}_n) \circ \Delta_{\hat{X}}$$
$$= (\phi_{\theta_1} \times \dots \times \phi_{\theta_n}) \circ (\hat{f}_1 \times \dots \times \hat{f}_n) \circ \Delta_{\hat{X}}$$

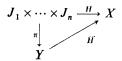
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Hence

$$H_{\circ}(\phi_{\Theta_{1}}^{-1} \times \cdots \times \phi_{\Theta_{n}}^{-1}) \circ \hat{H} = H_{\circ}(\hat{f}_{1} \times \cdots \times \hat{f}_{n}) \circ \Delta_{\hat{X}} = \phi_{C}.$$
q.e.d.

MAIN THEOREM. Let X be an abelian variety of dimension r. Then for any positive 1-cycle  $C = \sum_{i=1}^{n} C_i$  on X,  $\deg(\phi_c) = l(C)^2$ .

**PROOF.** Under the same notations as above, we have only to consider the case where H is surjective. Let K be the connected component at 0 of  $(\ker H)_{red}$ , and we put Y=the quotient variety  $J_1 \times \cdots \times J_n/K$ . Let  $\pi$  be the canonical surjection from  $J_1 \times \cdots \times J_n$  to Y. Then there exists an isogeny  $\overline{H}$  satisfying the commutative diagram



and dualizing this diagram, we have

$$\hat{J}_1 \times \cdots \times \hat{J}_n \xleftarrow{\hat{I}} \hat{X}$$

$$\hat{\hat{\pi}} \qquad \qquad \hat{\hat{\pi}} \qquad \qquad \hat{$$

Therefore from these diagrams and Proposition 4,

(3) 
$$\phi_{C} = \overline{H} \circ \pi \circ (\phi_{\Theta_{1}}^{-1} \times \cdots \times \phi_{\Theta_{n}}^{-1}) \circ \hat{\pi} \circ \widehat{H} = \overline{H} \circ \phi_{\widehat{\pi}}^{*} \circ \widehat{H}.$$

While, by Proposition 1,  $\deg(\phi_{\theta|\hat{Y}}) = \deg(\phi_{\theta|K})$ . From the Riemann-Roch theorem on K, we have  $\deg(\phi_{\theta|K}) = \{K \cdot \Theta^{g_1 + \dots + g_n - r})/(g_1 + \dots + g_n - r)!\}^2$ , where  $g_i$  is the genus of  $C_i$  for each *i*. On the other hand, by Proposition 3,

$$(K \cdot \Theta^{g_1 + \dots + g_n - r}) = \sum_{\substack{s_1 + \dots + s_n = g_1 + \dots + g_n - r \\ 0 \le s_i \le g_i}} \frac{(g_1 + \dots + g_n - r)!}{s_1! \cdots s_n!} (K \cdot \Theta^{s_1}_1 \times \dots \times \Theta^{s_n}_n)$$

$$= \sum_{\substack{s_1 + \dots + s_n = g_1 + \dots + g_n - r \\ 0 \le s_i \le g_i}} (g_1 + \dots + g_n - r)! (K \cdot W^{g_1 - s_1}_1 \times \dots \times W^{g_n - s_n}_n),$$

$$= (g_1 + \dots + g_n - r)! \sum_{\substack{r_1 + \dots + r_n = r \\ 0 \le r_i \le g_i}} (K \cdot W^{r_1}_1 \times \dots \times W^{r_n}_n),$$

where  $r_i = g_i - s_i$ . Therefore

$$\deg \left( \phi_{\theta \mid \hat{Y}} \right) = \left\{ \sum_{\substack{r_1 + \dots + r_n = r \\ 0 \le r_i \le g_i}} \left( K \cdot W_1^{r_1} \times \dots \times W_n^{r_n} \right) \right\}^2,$$

and from the equality (3),

(4) 
$$\deg \phi_{C} = (\deg \overline{H})^{2} \cdot \deg (\phi_{\Theta}|_{\widehat{Y}})$$
$$= \{ \sum_{\substack{r_{1}+\cdots+r_{n}=r\\0 \leq r_{1} \leq g_{1}}} \deg \overline{H} \cdot (K \cdot W_{1}^{r_{1}} \times \cdots \times W_{n}^{r_{n}}) \}^{2}.$$

On the other hand, we have the commutative diagram

$$C_1^{*r_1} \times \cdots \times C_n^{*r_n}$$

$$\varphi_1^{(r_1) \times \cdots \times \varphi_n^{(r_n)}} \downarrow \qquad F^{(r_1, \dots, r_n)}$$

$$J_1 \times \cdots \times J_n \xrightarrow{II} X.$$

Therefore

$$d(C_1^{r_1}, \dots, C_n^{r_n}) = \deg F^{(r_1, \dots, r_n)}$$
  
= deg  $\overline{H}$ ·deg  $(\varphi_1^{(r_1)} \times \dots \times \varphi_n^{(r_n)})$ :  $C_1^{r_1} \times \dots \times C_n^{r_n}$   
 $\longrightarrow W_1^{r_1} \times \dots \times W_n^{r_n})$ ·deg $(\pi |_{W_1^{r_1} \times \dots \times W_n^{r_n}})$ .

Applying Proposition 2, we obtain

$$d(C_1^{r_1},\ldots,C_n^{r_n}) = \deg \overline{H} \cdot (r_1 ! \ldots r_n !) (K \cdot W_1 \times \cdots \times W_n).$$

Therefore from (4) and (1), we obtain

deg 
$$\phi_C = \left\{ \frac{\deg({}^{*}C)}{r!} \right\}^2 = l(C)^2.$$
 q.e.d.

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