HIROSHIMA MATH. J.
4 (1974), 71–74

On Characterizations of Dedekind Domains

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(Received August 29, 1973)

Introduction

Let R be a commutative ring with unit element and M be an R-module. We consider the following two properties on M.

$(P)_R$: If $M=N_1+N_2$ where N_1 and N_2 are R-submodules of M, then $N_1=M$ or $N_2=M$.

$(Q)_R$: if N_1 and N_2 are R-submodules of M, then $N_1 \supset N_2$ or $N_2 \supset N_1$.

Clearly, the property $(Q)_R$ implies the property $(P)_R$ and the property $(P)_R$ implies that M is indecomposable.

In the case that R is a Dedekind domain, we shall exhibit all R-modules which satisfy $(P)_R$ and at once see that they satisfy also $(Q)_R$.

If we restrict ourselves to abelian groups, those groups which satisfy $(P)_Z$ are subgroups of $Z(p^\infty)$ for some prime p. In fact, this result came first and then it has been generalized to modules over a Dedekind domain R.

Next, suppose R to be a noetherian integral domain such that if $(P)_R$ is satisfied by an R-module M then so is $(Q)_R$. We shall show R must be a Dedekind domain if $R_\mathfrak{p}$ is analytically irreducible for any maximal ideal $\mathfrak{p}$. This gives us a new characterization of Dedekind domains.

Finally, in § 2, we shall discuss a relation between the notion of purity and that of essentiality and get another characterization of Dedekind domains.

§ 1. In the following we denote by $E_R(M)$ the injective envelope of an R-module M. First we determine all R-modules which satisfy the property $(P)_R$, or equivalently the property $(Q)_R$, when R is a Dedekind domain.

THEOREM 1. *Let R be a Dedekind domain, K be its quotient field and M be an R-module. Then the following statements are equivalent*:

(1) *M has $(P)_R$.*

(2) *M has $(Q)_R$.*

(3) *If R is not a discrete valuation ring, then M is isomorphic to a submodule of $E_R(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$ in R. If R is a discrete valuation ring, M is isomorphic to R, K or a submodule of $E_R(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$.*

PROOF. It follows immediately that (3)$\Rightarrow$(2)$\Rightarrow$(1). To show (1)$\Rightarrow$(3), we classify M into two cases; divisible or not divisible. In each case, M is either torsion or torsion-free.

Suppose for any maximal ideal $\mathfrak{p}$ of R, $\mathfrak{p}M=M$, that is, M is a divisible R-module. Then provided that M has $(P)_R$, M is either torsion or torsion-free. In fact since any divisible module over a Dedekind domain is injective, the torsion part of a divisible R-module is a direct summand of it. If M is a divisible torsion module and satisfies $(P)_R$, then M is an indecomposable and injective R-module and hence it is isomorphic to $E_R(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$ of R by Proposition 3.1 in [2]. If M is torsion-free, M must be a 1-dimensional vector space over K, that is, $M\cong K$. If R is not a discrete valuation ring, we see $K=R_{\mathfrak{p}}+R_{\mathfrak{q}}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are two defferent maximal ideals of R. Since $R_{\mathfrak{p}}\neq K$ and $R_{\mathfrak{q}}\neq K$, no torsion-free divisible modules have $(P)_R$ in this case.

Next suppose $\mathfrak{p}M\neq M$ for some maximal ideal $\mathfrak{p}$. Then clearly $M/\mathfrak{p}M$ has $(P)_R$ and $(P)_{R/\mathfrak{p}}$. On the other hand $M/\mathfrak{p}M$ is a vector space over $R/\mathfrak{p}$. Hence $\dim_{R/\mathfrak{p}}M/\mathfrak{p}M=1$, i.e. $M=Rx+\mathfrak{p}M$ for some x in M. Again making use of $(P)_R$ attached to M, we get $M=Rx$, since $\mathfrak{p}M\neq M$. If x is a torsion element, the order ideal $0(x)$ of x must be primary. In fact, if otherwise, $M=R/0(x)$ is decomposable by Chinese remainder theorem. This contradicts the assumption that M has $(P)_R$. Therefore $M\cong R/\mathfrak{p}^n$ for some maximal ideal $\mathfrak{p}$ in R. If x is torsion-free, then $M\cong R$. If R is not a discrete valuation ring, for two different maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$, $R=\mathfrak{p}+\mathfrak{q}$ holds. Hence, if R is not a discrete valuation ring, no torsion-free modules, divisible or not divisible, satisfy $(P)_R$. We have just covered all cases and our assertion has been proved.

Next we give a characterization of Dedekind domains in terms of $(P)_R$ and $(Q)_R$. For this purpose the following lemmas are necessary.

LEMMA 1. *Let R be a noetherian integral domain and $\mathfrak{p}$ be a maximal ideal of R. Let $\bar{R}_{\mathfrak{p}}$ be the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of the quotient ring $R_{\mathfrak{p}}$ of R. Then $E_R(R/\mathfrak{p})$ has a natural structure as an $\bar{R}_{\mathfrak{p}}$-module and any R-submodule of $E_R(R/\mathfrak{p})$ is an $\bar{R}_{\mathfrak{p}}$-submodule.*

PROOF. It is well known that $E_R(R/\mathfrak{p})$ is an $\bar{R}_{\mathfrak{p}}$-module in a natural way (cf. e.g., Theorem 3.6 in [2]). Let M be an R-submodule of $E_R(R/\mathfrak{p})$ and x be an element of M. Then there exists an integer i such that $\mathfrak{p}^i x=0$ by Theorem 3.4 in [2]. If s is an element of R not contained in $\mathfrak{p}$, there exist a and b in R such that $as+bc=1$ for some c in $\mathfrak{p}^i$. If we put $y=ax$, we see $x=sy$ and hence $x/s=y\in M$. From this it follows easily that M is an $\bar{R}_{\mathfrak{p}}$-module.

LEMMA 2. *Let R and $\mathfrak{p}$ be the same as in Lemma 1. Assume that $R_{\mathfrak{p}}$ is analytically irreducible. Then $E_R(R/\mathfrak{p})$ has the property $(P)_R$.*

PROOF. By our assumption, $\bar{R}_{\mathfrak{p}}$ is also an integral domain. If $E_R(R/\mathfrak{p})$ has not $(P)_R$, there exist two proper R-submodules M and N of $E_R(R/\mathfrak{p})$ such that $E_R(R/\mathfrak{p})=M+N$. By Lemma 1, M and N are also $\bar{R}_{\mathfrak{p}}$-submodules of $E_R(R/\mathfrak{p})$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the ideals of $\bar{R}_{\mathfrak{p}}$ corresponding to M and N in the sense of Theorem 4.2 in [2] respectively. Then by this theorem $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero ideals and $\mathfrak{a}\cap\mathfrak{b}=(0)$. But this contradicts the assumption that $\bar{R}_{\mathfrak{p}}$ is an integral domain. This proves our lemma.

LEMMA 3. *Let R be a noetherian integral domain which is not a Dedekind domain. Then there exists a maximal ideal $\mathfrak{p}$ such that $E_R(R/\mathfrak{p})$ does not satisfy $(Q)_R$.*

PROOF. There is a maximal ideal $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is not a discrete valuation ring, because, if otherwise, R must be a Dedekind domain. Then $\bar{R}_{\mathfrak{p}}$ is not a discrete valuation ring. Therefore there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\bar{R}_{\mathfrak{p}}$ that neither one of them contains the other. From Theorem 4.2 of [2], it follows that there are two $\bar{R}_{\mathfrak{p}}$-submodules of $E_R(R/\mathfrak{p})$ such that neither one of them contains the other. These $\bar{R}_{\mathfrak{p}}$-submodules are clearly R-submodules. This means that $E_R(R/\mathfrak{p})$ does not satisfy $(Q)_R$.

The following theorem is an immediate consequence of Lemmas 2 and 3.

THEOREM 2. *Let R be a noetherian integral domain such that $R_{\mathfrak{p}}$ is analytically irreducible for any maximal ideal $\mathfrak{p}$ of R. Assume that any R-module with $(P)_R$ satisfies $(Q)_R$. Then R is a Dedekind domain.*

§2. Let M be an R-module and N its submodule. When $rN=N\cap rM$ for every $r\in R$, N is said to be pure in M. We denote by $0(x)$ the order ideal of $x\in M$ and by $0(\bar{x})$ the order ideal of x modulo N, namely $0(\bar{x})=N:x=\{r\in R; rx\in N\}$; we observe that $0(\bar{x})$ does not depend on the choice of a representative of the coset $x+N$. We can readily see that the above definition of purity is equivalent to saying that, for any x in M, $0(\bar{x})$ is the set-theoretical union of $0(x+n)$, $n\in N$. Thus we give the first definition of purity as follows:

($P1$) For any x in M, $0(\bar{x})=\cup\, 0(y)$, where in the right hand side the union means the set-theoretical one and y runs over elements of the coset $\bar{x}=x+N$.

Now it is natural to introduce the second definition of purity in a stronger form[1]:

($P2$) For any x in M, $0(\bar{x})=0(x)$ for some representative x of the coset $\bar{x}$.

If R is a Dedekind domain, then ($P2$) follows from ($P1$) (Kaplansky [1], Lem-

1) This definition is suggested by Kaplansky in [1].

ma 4), namely two definitions coincide. In what follows, we shall show the converse, i.e. an integral domain R for which ($P1$) means ($P2$) must be a Dedekind domain.

First we remark that the notion of essentiality is opposite to that of purity.

LEMMA 4. *M is essential over a submodule N if and only if* $0(\bar{x})\supsetneqq 0(x)$ *for every $x\neq 0$ in M.*

PROOF. M is essential over a submodule N if and only if $N\cap Rx\neq 0$ for every non-zero element x of M. Our assertion follows immediately from this fact.

COROLLAY. Let N be a pure submodule of M in the sense of ($P2$). If M is essential over N, then $M=N$.

Let now R be an integral domain for which two definitions ($P1$) and ($P2$) coincide. Let M be any divisible module over R. Then it is easy to see that M is pure in the sense of ($P1$) in the injective envelope $E_R(M)$ of M; and therefore pure in the sense of ($P2$). The corollary to Lemma 4 implies that $M=E_R(M)$, namely M is injective. A domain R is a Dedekind domain if and only if every divisible module is injective, and therefore we can obtain the following

THEOREM 3. *Let R be an integral domain. Then R is a Dedekind domain if and only if the definitions* ($P1$) *and* ($P2$) *coincide.*

References

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