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A Note on Hilbert's Nullstellensatz

Kazunori FUJITA

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In his paper [3], S. Lang generalized the famous Hilbert's Nullstellensatz to the polynomial ring in an arbitrary number of variables over an algebraically closed field; however it seems to the author that his method is based on a usual technique known for the polynomial ring in a finite number of variables. Also, a number of proofs of Hilbert's Nullstellensatz have been given by O. Zariski and others ([1], [4], [5]). The main purpose of this note is to introduce the notion of the property $J(\Lambda)$ for a ring, which leads to a new approach to the theorem, applicable to the generalized case. We discuss, in 2, the relationship between Hilbert's Nullstellensatz and a Hilbert ring.

Throughout this note, a ring means a commutative ring with identity element.

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1. Let R be a ring. We denote by $Ht_1(R)$ the set of prime ideals of height 1 in R and for any given subset D of R, we denote by $H_R(D)$ the set of prime ideals of height 1 in R which contain at least one element of D. Let A be an R-algebra and Λ be a set. A is said to be Λ-generated over R if there is an R-algebra homomorphism from a polynomial ring $R[\dots, X_\lambda, \dots]$, $\lambda \in \Lambda$, onto A. In what follows the set Λ will always be assumed to be infinite.

If a subset D of R satisfies the following conditions: (1) $\mathrm{card}(D) \leq \mathrm{card}(\Lambda)$ and (2) any element of D is not a zero divisor, then we say that D is a J-subsetof R.

DEFINITION. When $H_R(D)$ is properly contained in $Ht_1(R)$ for any J-subset D, we say that the ring R has the property $J(\Lambda)$.

LEMMA 1. *Let R be a unique factrization domain such that the cardinality of the set of prime elements of R is greater than that of the set Λ. Then R has the property $J(\Lambda)$. In particular if k is a field such that $\mathrm{card}(k) > \mathrm{card}(\Lambda)$. then any polynomial ring over k has the property $J(\Lambda)$.*

The proof is almost clear and is omitted.

LEMMA 2. *Let $R \subseteq A$ be integral domains such that A is integral over R. Then if R has the property $J(\Lambda)$, then so does A.*

PROOF. Let $D = \{b_\mu; \mu \in M\}$ be any J-subset of A; let $f(X) = X^{n_\mu} + \cdots +$

$d_\mu=0$ be an equation of integral dependence for b_μ of smallest degree. Then it is clear that $d_\mu \neq 0$ and the set $D'=\{d_\mu;\ \mu \in M\}$ is a J-subset of R. Since R has the property $J(\Lambda)$, $H_R(D')$ is properly contained in $Ht_1(R)$. Let $\mathfrak{p}$ be a prime ideal in $Ht_1(R)$ but not in $H_R(D)$; since A is integral over R, there is a prime ideal $\mathfrak{P}$ of height 1 in A lying over $\mathfrak{p}$. We can readily see that $\mathfrak{P} \notin H_A(D)$.

PROPOSITION 1. *Let A be an integral domain and let f: $R \to A$ be a ring homomorphism. If A has the property $J(\Lambda)$, then the quotient field $Q(A)$ of A is not Λ-generated over R.*

PROOF. Suppose that $Q(A)$ is Λ-generated, namely $Q(A)=f(R)[\dots, a_\lambda/b_\lambda,\dots]$, a_λ, $b_\lambda \in A$, $\lambda \in \Lambda$. Since A has the property $J(\Lambda)$, we can take a prime $\mathfrak{p}$ in $Ht_1(A)$ but not in $H_A(\{b_\lambda;\ \lambda \in \Lambda\})$. Let b a non-zero element of $\mathfrak{p}$. Since $1/b$ is an element of $Q(A)$, there is an element a of A such that ab is a product of b_λ's; this implies that $\mathfrak{p}$ must contain b_λ for some λ and therefore $\mathfrak{p} \in H_A(\{b_\lambda;\ \lambda \in \Lambda\})$. This is a contradiction.

COROLLARY 1. *Let k be a field such that card $(k)>$ card (Λ). Then, for any maximal ideal $\mathfrak{M}$ in $k[\dots, X_\lambda,\dots]$, $\lambda \in \Lambda$, the residue field $L=k[\dots, X_\lambda,\dots]/\mathfrak{M}$ is algebraic over k. In particular, if k is algebraically closed, then every maximal ideal $\mathfrak{M}$ is of the form $\mathfrak{M}=(\dots, X_\lambda-a_\lambda,\dots)$, $a_\lambda \in k$.*

PROOF. Suppose that L is not algebrakc over k; we let $\{t_\mu;\ \mu \in M\}$ be a transcendental basis for L over k. Let R be the integral closure of the ring $k[\dots, t_\mu,\dots]$ in L. Then the quotient field of R is L. Since $k[\dots, t_\mu,\dots]$ has the property $J(\Lambda)$ by Lemma 1, R also has the propetry $J(\Lambda)$ by Lemma 2. It follows from Proposition 1 that the quotient field L of R is not Λ-generated over k; this leads to a contradiction.

REMARK 1. When card (M) is finite, for any field k, a polynomial ring $k[X_1, X_2,\dots, X_r]$, $r \geq 1$, has the property $J(M)$. Therefore, a new proof of Hilbert's Nullstellensatz is obtained as in the proof of Corollary 1.

COROLLARY 2. *Let R be a ring such that card $(R/\mathfrak{m})>$ card (Λ) for every maximal ideal $\mathfrak{m}$ of R. Then, $\mathfrak{M}$ being any maximal ideal of $R[\dots, X_\lambda,\dots]$, the residue field $R[\dots, X_\lambda,\dots]/\mathfrak{M}$ is algebraic over $R/\mathfrak{p}$, where $\mathfrak{p}=R \cap \mathfrak{M}$.*

PROOF. It is easy to see that $R[\dots, X_\lambda,\dots]/\mathfrak{M} \simeq k(\mathfrak{p})[\dots, X_\lambda,\dots]/\mathfrak{M}k(\mathfrak{p})[\dots, X_\lambda,\dots]$, where $k(\mathfrak{p})=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, and that card $(k(\mathfrak{p}))>$ card (Λ). We see, by Corollary 1, that $R[\dots, X_\lambda,\dots]$ is algebraic over $k(\mathfrak{p})$.

REMARK 2. Let k be a field such that card $(k) \leq$ card (Λ). Then, for any cardinal number $\eta \leq$ card (Λ), we can find a maximal ideal $\mathfrak{M}$ of $k[\dots, X_\lambda,\dots]$ so that η is equal to the transcendental degree of $k[\dots, X_\lambda,\dots]/\mathfrak{M}$ over k. In fact, let Λ_1 be a subset of Λ such that card $(\Lambda_1)=\eta$ and card $(\Lambda-\Lambda_1)=$ card (Λ);

since the cardinality of the polynomial ring $k[\dots, X_\mu, \dots]$, $\mu \in \Lambda_1$, does not exceed the cardinality of Λ, there is a subset Λ_2 of $\Lambda - \Lambda_1$ such that card (Λ_2)=card $(k[\dots, X_\mu, \dots] - \{0\})$ and therefore there is a bijection $\varphi\colon \Lambda_2 \to k[\dots, X_\mu, \dots] - \{0\}$; for every element $\tau \in \Lambda_2$, we put $f_\tau = \varphi(\tau)$ and for every element $\lambda \in \Lambda - \Lambda_1 \cup \Lambda_2$ we put $f_\lambda = 1$. Let $\mathfrak{M}$ be the ideal in $k[\dots, X_\lambda, \dots]$, $\lambda \in \Lambda$, generated by $f_\lambda X_\lambda - 1$, $\lambda \in \Lambda - \Lambda_1$; then $k[\dots, X_\lambda, \dots]/\mathfrak{M} = k(\dots, X_\mu, \dots)$, $\mu \in \Lambda_1$, and therefore $\mathfrak{M}$ is a maximal ideal in $k[\dots, X_\lambda, \dots]$ and η is equal to the transcendental degree of $k[\dots, X_\lambda, \dots]/\mathfrak{M}$ over k.

2. The following Lemma 3 is due to O. Goldman (cf. [2]).

LEMMA 3. *A ring R is a Hilbert ring if and only if every maximal ideal in the polynomial ring $R[X]$ in a variable X contracts in R to a maximal ideal.*

LEMMA 4. *Let $R \subset A$ be rings such that A is integral over R. Then R is a Hilbert ring if and only if A is a Hilbert ring.*

PROOF. Suppose first that R is a Hilbert ring. Let $\mathfrak{M}$ be any maximal ideal in $A[X]$; then, since $A[X]$ is integral over $R[X]$, the contraction $\mathfrak{m} = R[X] \cap \mathfrak{M}$ is maximal in $R[X]$. The assumption and Lemma 3 imply that $\mathfrak{m} \cap R = (\mathfrak{M} \cap A) \cap R$ is maximal in R; therefore $\mathfrak{M} \cap A$ is maximal in A and again by Lemma 3, we see that A is a Hilbert ring.

Conversely suppose that A is a Hilbert ring. Let $\mathfrak{N}$ be any maximal ideal in $R[X]$; we take a maximal ideal $\mathfrak{M}$ in $A[X]$ lying over $\mathfrak{N}$. The assumption implies that $\mathfrak{M} \cap A$ is maximal in A; therefore $(\mathfrak{M} \cap A) \cap R = \mathfrak{N} \cap R$ is maximal in R. Now our assertion follows from Lemma 3.

LEMMA 5. *Let k be a field. Then the polynomial ring $k[\dots, X_\lambda, \dots]$, $\lambda \in \Lambda$, is a Hilbert ring if and only if card $(k) >$ card (Λ).*

PROOF. We put $R = k[\dots, X_\lambda, \dots]$, $\lambda \in \Lambda$. Firstly we suppose that card (k) $>$card(Λ). We see that, $\mathfrak{M}$ being any maximal ideal in $R[X]$, the residue field $R[X]/\mathfrak{M}$ is algebraic over k by Corollary 1 of Proposition 1. As $k \subset R/\mathfrak{M} \cap R \subset R[X]/\mathfrak{M}$, $R/\mathfrak{M} \cap R$ is a field; namely $\mathfrak{M} \cap R$ is maximal in R. Now Lemma 3 tells us that R is a Hilbert ring.

Conversely, we suppose that card$(k) \leq$ card(Λ). In order to show that R is not a Hilbert ring, by Lemma 4, we may assume that k is algebraically closed and by Lemma 3, it suffices to show that we can find a maximal ideal $\mathfrak{M}$ in $R[Y]$, Y being a variable, such that $\mathfrak{M}$ does not contract in R to a maximal ideal. Since card$(k) \leq$ card(Λ), there is a subset Λ_0 of Λ such that card(k)=card(Λ_0) and hence a bijection $\varphi\colon \Lambda_0 \to k$. For any element $\lambda \in \Lambda_0$ we put $a_\lambda = \varphi(\lambda)$ and, for every element $\lambda \in \Lambda - \Lambda_0$, we put $a_\lambda = 1$; let $\mathfrak{M}$ be the ideal in $R[Y]$ generated by $X_\lambda(Y - a_\lambda) - 1$, $\lambda \in \Lambda$; then $R[Y]/\mathfrak{M} \simeq k(Y)$, which implies that $\mathfrak{M}$ is maximal. Let λ_1 be the element of Λ_0 coresponding to 0 in k; we put $X = X_{\lambda_1}$. Let $\mathfrak{p}$ be the ideal in R generated by $X_\lambda(1-a_\lambda X)-X$, $\lambda \in \Lambda$; then $R/\mathfrak{p} \simeq k[X,\dots, X/1-a_\lambda X,\dots]$ and therefore $\mathfrak{p}$ is prime but not maximal in R. It is easy to see that $\mathfrak{M} \cap R = \mathfrak{p}$, which implies that $\mathfrak{M}$ is the desired ideal.

PROPOSITION 2. *Let k be a field. Then the following there conditions are equivalent*:
(1) *$card(k) > card(\Lambda)$.*
(2) *For every maximal ideal $\mathfrak{M}$ in $k[\dots, X_\lambda,\dots]$, $k[\dots, X_\lambda,\dots]/\mathfrak{M}$ is algebraic over k.*
(3) *$k[\dots, X_\lambda,\dots]$ is a Hilbert ring.*

PROOF. (1)$\Leftrightarrow$(2) follows from Corollary 1 of Proposition 1 and Remark 2. (1)$\Leftrightarrow$(3) follows from Lemma 5.

COROLLARY. *Let R be a ring.*
(1) *If for any maximal ideal $\mathfrak{m}$ in R, $card(R/\mathfrak{m}) > card(\Lambda)$ and every maximal ideal $\mathfrak{M}$ in $A = R[\dots, X_\lambda,\dots]$ contracts in R to a maximal ideal, then A is a Hilbert ring.*
(2) *If $R[\dots, X_\lambda,\dots]$ is a Hilbert ring, then for any maximal ideal $\mathfrak{m}$ in R, we have $card(R/\mathfrak{m}) > card(\Lambda)$.*

PROOF. (1) Let M be a set such that $\text{card}(M) = \text{card}(\Lambda)$. By assumption, any maximal ideal in $R[\dots, X_\mu,\dots]$, $\mu \in M$, contracts in R to a maximal ideal. Therefore, $\mathfrak{M}$ being any maximal ideal in $A[X]$, $\mathfrak{m} = \mathfrak{M} \cap R$ is maximal in R. Since $R/\mathfrak{m}[\dots, X_\lambda,\dots]$ is a Hilbert ring by Proposition 2, $A[X]/\mathfrak{M}$ is algebraic over $R/\mathfrak{m}$. As $R/\mathfrak{m} \subset A/\mathfrak{M} \cap A \subset A[X]/\mathfrak{M}$, $A/\mathfrak{M} \cap A$ is a field; namely $\mathfrak{M} \cap A$ is maximal in A. Hence, by Lemma 3, A is a Hilbert ring. (2) Since a homomorphic image of a Hilbert ring is also a Hilbert ring, $R/\mathfrak{m}[\dots, X_\lambda,\dots]$ is a Hilbert ring. Therefore, by Proposition 2, we have $\text{card}(R/\mathfrak{m}) > \text{card}(\Lambda)$.

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Department of Mathmatics,
Faculty of Science,
Hiroshima University