HIROSHIMA MATH. J.
4 (1974), 743–763

The Gradient of a Convex Function on a Regular Functional Space and its Potential Theoretic Properties

Nobuyuki KENMOCHI and Yoshihiro MIZUTA

(Received May 20, 1974)

Introduction

This work was motivated by the recent papers of Calvert [7, 8] who discussed potential theoretic properties of nonlinear monotone operators and extended the Dirichlet space theory (cf. Beurling-Deny [1, 2], Deny [9, 10], Itô [13, 14], Bliedtner [3, 4]) to the nonlinear case in Sobolev spaces. He treated nonlinear analogues of the modulus contraction, the unit contraction, the principle of lower envelope, the domination principle and the complete maximum principle, etc. In his argument, however, there is no notion of potentials. In this paper, restricting our arguments to a class of gradients of convex functions, we introduce a notion of potential with respect to a given convex function and show that Calvert's arguments are valid in a more general space, namely, in a regular functional space. Moreover, by introducing a notion of capacity with respect to a given convex function, we discuss the refinement of functions in a regular functional space.

§ 1. Preliminaries

Let X be a locally compact Hausdorff space with a countale base and ξ be a positive (Radon) measure on X. Let $\mathscr{X}=\mathscr{X}(X;\xi)$ be a real reflexive Banach space whose elements are real-valued locally ξ-summable functions defined ξ–*a.e.* on X (hereafter we write simply "*a.e.*" for "ξ–*a.e.*"). We denote by $\mathscr{X}^*$ the dual space of $\mathscr{X}$, by $\|u\|$ (resp. $\|u^*\|$) the norm of $u\in\mathscr{X}$ (resp. $u^*\in\mathscr{X}^*$) and by $<u^*, u>$ the value of $u^*\in\mathscr{X}^*$ at $u\in\mathscr{X}$. We denote the strong (resp. weak) convergence by "$\xrightarrow{s}$" (resp. "$\xrightarrow{w}$"). For functions $u, v\in L^1_{loc}=L^1_{loc}(X;\xi)$, we write $u\vee v$ and $u\wedge v$ for $\max(u, v)$ and $\min(u, v)$, respectively. Especially, we write u^+ and u^- for $u\vee 0$ and $-(u\wedge 0)$, respectively.

Throughout this paper, let $1<p<\infty$ and Φ be a strictly convex function on $\mathscr{X}$ such that

$$(1.1)\qquad \begin{cases}\Phi(0)=0,\\ \Phi(u)\geq C\|u\|^p \qquad \text{for any } u\in\mathscr{X},\end{cases}$$

where C is a positive constant. Suppose further that Φ is bounded on each bounded subset of $\mathscr{X}$ and is everywhere differentiable in the sense of Gateaux, that is, there is an operator $G: \mathscr{X} \to \mathscr{X}^*$ such that for any $u, v \in \mathscr{X}$

$$<Gu, v> = \lim_{t \downarrow 0} \frac{\Phi(u+tv) - \Phi(u)}{t}.$$

This operator G is called the gradient of Φ and denoted by $\nabla\Phi$.

Now, we state basic properties of $\nabla\Phi$ without proof:

(a) Let $u \in \mathscr{X}$ and $u^* \in \mathscr{X}^*$. Then $u^* = \nabla\Phi(u)$ if and only if $<u^*, v-u> \leq \Phi(v) - \Phi(u)$ for all $v \in \mathscr{X}$.

(b) $\nabla\Phi$ is strictly monotone, i.e., $<\nabla\Phi(u) - \nabla\Phi(v), u-v> > 0$ for any $u, v \in \mathscr{X}$ such that $u \neq v$.

(c) For each $u \in \mathscr{X}$, $<\nabla\Phi(v), v-u>/\|v\| \to \infty$ as $\|v\| \to \infty$.

(d) $\nabla\Phi$ is demicontinuous,i.e., if $u_n \xrightarrow{s} u$ in $\mathscr{X}$ as $n \to \infty$, then $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in $\mathscr{X}^*$ as $n \to \infty$.

(e) For any $u, v \in \mathscr{X}$,

$$\Phi(u) - \Phi(v) = \int_0^1 < \nabla\Phi(v + t(u-v)), u-v > \mathrm{dt}.$$

LEMMA 1.1. *$\nabla\Phi$ is one to one and onto.*

PROOF. As remarked above, $\nabla\Phi$ is a monotone[1)], demicontinuous and coercive[1)] mapping of $\mathscr{X}$ into $\mathscr{X}^*$. Hence, in view of a result of Browder [5; Theorem 3], the range of $\nabla\Phi$ is all of $\mathscr{X}^*$. The fact that it is one to one follows directly from property (b). q.e.d.

DEFINITION 1.1. (cf. [2], [4], [9], [10], [13]) *$\mathscr{X} = \mathscr{X}(X; \xi)$ is called a functional space if the following axiom is satisfied*:

Axiom (a) For each compact subset K of X, there is a constant $M(K) > 0$ such that

$$\int_K |u| d\xi \leq M(K)\|u\| \qquad \textit{for all} \quad u \in \mathscr{X}.$$

Next, denote by $\mathscr{M}$ the space of all bounded ξ-measurable functions on X with compact support. Then Axiom [a] implies that for each $f \in \mathscr{M}$, the functional L_f given by

$$L_f(v) = \int f v d\xi$$

1) For words "monotone" and "coercive", see [6].

belongs to $\mathscr{X}^*$. Therefore, from Lemma 1.1 we obtain

THEOREM 1.1. *Let $\mathscr{X}$ be a functional space. Then for each $f \in \mathscr{M}$, there is a unique function $u_f \in \mathscr{X}$ such that*

$$<\nabla \Phi(u_f), v> = \int f v d\xi \qquad \textit{for all} \quad v \in \mathscr{X}.$$

From now on, we denote by $\mathscr{C}$ the space of all continuous functions on X with compact support.

DEFINITION 1.2. (cf. [2], [4], [9], [13]) *A functional space $\mathscr{X} = \mathscr{X}(X; \xi)$ is called regular if the following axiom is satisfied:*
Axiom (b) *$\mathscr{C} \cap \mathscr{X}$ is dense both in $\mathscr{C}$ and in $\mathscr{X}$.*

DEFINITION 1.3. *Let $\mathscr{X}$ be a regular functional space. A function $u \in \mathscr{X}$ is called a potential (with respect to Φ) if there is a (signed) measure μ on X such that*

$$<\nabla \Phi(u), v> = \int v d\mu \qquad \textit{for all} \quad v \in \mathscr{C} \cap \mathscr{X}. \tag{1.2}$$

If such a μ exists, then it is unique and called the associated measure of u, and we write $u = u_\mu$. In particular, a potential u_μ is called pure if μ is positive.

We note that the function u_f obtained in Theorem 1.1 is a potential whose associated measure is $f d\xi$.

The lemma below will be needed in the proofs of our main thoerems in § 3 and 4.

Let Ω be an open subset of X and denote by $\mathscr{C}(\Omega)$ the space of all continuous functions on Ω with compact support.

LEMMA 1.2. *Let $\mathscr{C}_1$ be a dense linear subspace of $\mathscr{C}(\Omega)$ such that $v^+ \in \mathscr{C}_1$ for any $v \in \mathscr{C}_1$. Let L be a positive linear functional on $\mathscr{C}_1$. Then L can be extended to a positive linear functional on $\mathscr{C}(\Omega)$ and hence there is a positive measure σ on Ω such that*

$$L(v) = \int_\Omega v d\sigma \qquad \textit{for all} \quad v \in \mathscr{C}_1.$$

The assertion of this lemma was shown in a more general form in [20; Corollary 2.3].

§ 2. Normalized contractions

In this section, let $\mathscr{X}$ be a functional space.

A mapping $T: (-\infty, \infty) \rightarrow (-\infty, \infty)$ is called a normalized contraction on $(-\infty, \infty)$, if it has the following properties: $T0 = 0$ and $|Tr - Tr'| \leq |r - r'|$. Such a mapping T can be considered as a mapping from L^1_{loc} into itself by putting $(Tu)(x) = Tu(x)$ for $x \in X$ at which $|u(x)| < \infty$.

LEMMA 2.1. *Let T be a normalized contraction on $(-\infty, \infty)$. Let $\{u_n\}$ be a sequence in $\mathscr{X}$ such that $Tu_n \in \mathscr{X}$ for all n. If $u_n \xrightarrow{s} u$ in $\mathscr{X}$ and $\{Tu_n\}$ is bounded in $\mathscr{X}$, then $Tu \in \mathscr{X}$ and $Tu_n \xrightarrow{w} Tu$ in $\mathscr{X}$.*

PROOF. Since $\mathscr{X}$ is reflexive, $\{Tu_n\}$ is weakly relatively compact in $\mathscr{X}$. Let $\{Tu_{n'}\}$ be any subsequence of $\{Tu_n\}$ weakly convergent and v be the weak limit. Then we have

$$(2.1) \qquad \left|\int (Tu - v) f\mathrm{d}\right| \xi \leq \int |Tu - Tu_{n'}| \, |f| \mathrm{d}\xi + \left|\int (Tu_{n'} - v) f \mathrm{d}\xi\right|$$

for any $f \in \mathscr{M}$ and n'. By Theorem 1.1,

$$\int (Tu_{n'} - v) f \mathrm{d}\xi = \langle \nabla \Phi(u_f), Tu_{n'} - v \rangle \longrightarrow 0 \qquad as \quad n' \longrightarrow \infty .$$

By using Axiom (a) we have

$$\int |Tu_{n'} - Tu| \, |f| \mathrm{d}\xi \leq \int |u_{n'} - u| \, |f| \mathrm{d}\xi \leq (\sup |f|) M \|u_{n'} - u\| \longrightarrow 0$$

as $n' \rightarrow \infty$, where M is a positive constant which depends on the support of f. Hence by (2.1) we have

$$\int (Tu - v) f \mathrm{d}\xi = 0 \qquad \text{for any} \quad f \in \mathscr{M} .$$

This implies that $v = Tu$ in $\mathscr{X}$ and $Tu_n \xrightarrow{w} Tu$ in $\mathscr{X}$ as $n \rightarrow \infty$. q.e.d.

In the Dirichlet space theory, there is an important class of normalized contractions on $(-\infty, \infty)$ of the following type:

DEFINITION 2.1. *Let k be a non-negative number or ∞, and define a mapping $T_k: (-\infty, \infty) \rightarrow [0, \infty)$ by $T_k r = \min\{\max(r, 0), k\}$. Then we say that T_k operates in $\mathscr{X}$ (with respect ot Φ), if the following two conditions are satisfied:*

(C_k) $\quad T_k u \in \mathscr{X} \quad$ *for any* $\quad u \in \mathscr{X}$.

(ΦC_k) $\quad \Phi(u) + \Phi(v) \geq \Phi(u + T_k(v - u)) + \Phi(v - T_k(v - u))$

for any $\quad u, v \in \mathscr{X}$.

We now give a necessary and sufficient condition for condition (ΦC_k).

PROPOSITION 2.1. *Let* $k \in [0, \infty]$. *Under condition* (C_k), *condition* (ΦC_k) *is equivalent to the following*:

$$(\Phi C_k)' \quad < \nabla \Phi(u+T_k v) - \nabla \Phi(u), v - T_k v > \geq 0 \qquad \text{for any} \quad u, v \in \mathscr{X}.$$

PROOF. Clearly, (ΦC_k) is equivalent to

$$\Phi(z) + \Phi(z+w) \geq \Phi(z+T_k w) + \Phi(z+w-T_k w) \qquad \text{for any } z, w \in \mathscr{X}. \tag{2.2}$$

First assume (2.2). Let u and v be any functions in $\mathscr{X}$ and t be any positive number, and take $w = T_k v + t(v - T_k v)$ and $z = u$ in (2.2). Then, noting that $T_k w = T_k v$, we have

$$\Phi(u+T_k v + t(v-T_k v)) + \Phi(u) \geq \Phi(u+T_k v) + \Phi(u+t(v-T_k v)).$$

Hence

$$< \nabla \Phi(u+T_k v) - \nabla \Phi(u), v - T_k v>$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \{\Phi(u+T_k v + t(v-T_k v)) - \Phi(u+T_k v) - \Phi(u+t(v-T_k v)) + \Phi(u)\}$$

$$\geq 0.$$

Conversely assume $(\Phi C_k)'$. For any $u, v \in \mathscr{X}$, we have by property (e) and $(\Phi C_k)'$

$$\begin{aligned} \Phi(u+v) - \Phi(u+T_k v) &= \int_0^1 < \nabla \Phi(u+T_k v) + t(v-T_k v)), v - T_k v > dt \\ &\geq \int_0^1 < \nabla \Phi(u+t(v-T_k v)), v - T_k v > dt \\ &= \Phi(u+v-T_k v) - \Phi(u). \end{aligned}$$

Thus (2.2) is proved. q.e.d.

REMARK. In particular, when T_∞ operates in $\mathscr{X}$, we often say that the modulus contraction operates in $\mathscr{X}$. Calvert [7: §2] gave the definitions of the contractions onto $[0, k]$ in the form $(\Phi C_k)'$ for a general class of nonlinear monotone operators in a Sobolev space. Also, compare them with the definitions in Deny [9] and Itô [13].

LEMMA 2.2. *Suppose that condition* (C_{k_0}) *is satisfied for some* k_0 *such that* $0 < k_0 < \infty$. *Then for every* k, $0 \leq k \leq \infty$, *the following holds*: $T_k \phi \in \mathscr{X}$ *for any*

$\phi \in \mathscr{C} \cap \mathscr{X}$.

PROOF. The lemma is trivial for $k=0$. Let ϕ be any function in $\mathscr{C} \cap \mathscr{X}$. In case $0<k<\infty$, we observe that

$$T_k\phi = (k/k_0)T_{k_0}(k_0\phi/k) \in \mathscr{X}.$$

Next, let us consider the case where $k=\infty$. Taking a number M which is larger than $\sup|\phi|$, we have by the above fact

$$T_\infty\phi = \phi^+ = T_M\phi \in \mathscr{X}. \qquad \text{q.e.d.}$$

Using this lemma we prove

LEMMA 2.3. *Suppose that $\mathscr{X}$ is regular and that condition (C_k) is satisfied for some k such that $0<k<\infty$. Then for any open set G in X, $\{\phi \in \mathscr{C} \cap \mathscr{X}; \operatorname{supp}\phi \subset G\}$ is dense in $\mathscr{C}_G = \{\phi \in \mathscr{C}; \operatorname{supp}\phi \subset G\}$ with respect to the topology of $\mathscr{C}$.*

PROOF. Let ϕ be any non-negative function in $\mathscr{C}_G$. Given $\varepsilon>0$, there exists a function $\psi \in \mathscr{C} \cap \mathscr{X}$ such that $|\phi-\psi|<\varepsilon$ on X on account of Axiom (b). Set $\psi_\varepsilon = (\psi - T_\varepsilon\psi)^+$. Then, by Lemma 2.2, $\psi_\varepsilon \in \mathscr{C} \cap \mathscr{X}$ and we see that $\psi_\varepsilon(x)=0$ if $\phi(x)=0$, so that $\psi_\varepsilon \in \mathscr{C}_G \cap \mathscr{X}$ and $|\psi_\varepsilon - \phi|<2\varepsilon$ on X. Thus the lemma is proved. q.e.d.

PROPOSITION 2.2. *Suppose that $\mathscr{X}$ is regular and let r be a positive number. In addition to assumptions on Φ made in § 1, suppose that $\Phi(\lambda u) = |\lambda|^r\Phi(u)$ for any $u \in \mathscr{X}$ and any real number λ. If T_{k_0} operates in $\mathscr{X}$ for some k_0 such that $0<k_0<\infty$, then all T_k, $0 \le k \le \infty$, operate in $\mathscr{X}$.*

PROOF. Clearly, T_0 operates in $\mathscr{X}$. Now, assume that $0<k<\infty$. For an arbitrary $\phi \in \mathscr{C} \cap \mathscr{X}$, we have $T_k\phi \in \mathscr{X}$ by Lemma 2.2. Moreover, by our assumption and (ΦC_{k_0}),

$$\begin{aligned}\Phi(T_k\phi) &= \Phi((k/k_0)T_{k_0}(k_0\phi/k)) \\ &= (k/k_0)^r\Phi(T_{k_0}(k_0\phi/k)) \\ &\le (k/k_0)^r\Phi(k_0\phi/k) \\ &= \Phi(\phi).\end{aligned}$$

This implies that the mapping: $\phi \to T_k\phi$ from $\mathscr{C} \cap \mathscr{X}$ into $\mathscr{X}$ is bounded on each bounded subset of $\mathscr{C} \cap \mathscr{X}$ in the topology of $\mathscr{X}$. Therefore, by Axiom (b) and Lemma 2.1, condition (C_k) holds. Next, notice that

$$\nabla\Phi(\lambda u) = |\lambda|^{r-1}\operatorname{sign}(\lambda)\nabla\Phi(u), \qquad \lambda \neq 0. \tag{2.3}$$

Let ϕ, ψ be any functions in $\mathscr{C}\cap\mathscr{X}$. Then, by (2.3) and $(\Phi C_{k_0})'$

$$\begin{aligned}
&<\nabla\Phi(\phi+T_k\psi)-\nabla\Phi(\phi), \psi-T_k\psi> \\
&\quad = <\nabla\Phi(\phi+(k/k_0)T_{k_0}(k_0\psi/k))-\nabla\Phi(\phi), \psi-(k/k_0)T_{k_0}(k_0\psi/k)> \\
&\quad = (k/k_0)^r<\nabla\Phi(k_0\phi/k+T_{k_0}(k_0\psi/k))-\nabla\Phi(k_0\phi/k), k_0\psi/k \\
&\qquad -T_{k_0}(k_0\psi/k)> \geq 0.
\end{aligned}$$

Hence, just as in the proof of Proposition 2.1, we obtain

$$\Phi(\phi+T_k(\psi-\phi))+\Phi(\psi-T_k(\psi-\phi)) \leq \Phi(\phi)+\Phi(\psi)$$
$$\text{for any } \phi,\ \psi\in\mathscr{C}\cap\mathscr{X}.$$

Taking account of the strong continuity and weak sequential lower semicontinuity of Φ, we consequently see that (ΦC_k) holds. Thus T_k operates in $\mathscr{X}$.

Finally, we can show (C_∞) and (ΦC_∞) in a way similar to the above.

q.e.d.

§ 3. Potentials with respect to Φ

For a ξ-measurable set A in X and v, w in L^1_{loc}, we simply write "$v\geq w$ (resp. $v=w$) on A" for "$v\geq w$ (resp. $v=w$) a.e. on A". Especially, we write "$v\geq w$ (resp. $v=w$)" for "$v\geq w$ (resp. $v=w$) on X".

In this section, let $\mathscr{X}$ be a regular functional space.

The following theorem is a nonlinear version of a result of Beurling-Deny [2; Lemma 2].

THEOTEM 3.1. *Suppose that $v^+\in\mathscr{X}$ for any $v\in\mathscr{X}$ and the mapping: $v\to v^+$ from $\mathscr{X}$ into $\mathscr{X}$ is bounded on bounded subsets of $\mathscr{X}$. Let $u\in\mathscr{X}$. Then the following three statements are equivalent to each other:*

(i) *u is a pure potential.*

(ii) $\Phi(u+v)\geq\Phi(u)$ *for all $v\in\mathscr{X}$ such that $v\geq 0$.*

(iii) $<\nabla\Phi(u), v>\ \geq 0$ *for all $v\in\mathscr{X}$ such that $v\geq 0$.*

PROOF. The equivalence between (ii) and (iii) is easily obtained from the definition of $\nabla\Phi$ and property (a). Next assume (i). Then there exists a positive measure μ on X such that (1.2) holds. Let v be an arbitrary non-negative function in $\mathscr{X}$. In view of Axiom (b) and Lemma 2.1 for $T=T_\infty$, we find a sequence $\{v_n\}$

in $\mathscr{C} \cap \mathscr{X}$ such that $v_n \geq 0$ for all n and $v_n \xrightarrow{w} v$ in $\mathscr{X}$ as $n \to \infty$. Therefore we see that

$$<\nabla \Phi(u), v> = \lim_{n\to\infty} <\nabla \Phi(u), v_n> = \lim_{n\to\infty} \int v_n \mathrm{d}\mu \geq 0\,,$$

so we have (iii). Conversely, assume (iii) and consider the functional:

$$v \in \mathscr{C} \cap \mathscr{X} \longrightarrow <\nabla \Phi(u), v> .$$

From Lemma 1.2 and Axiom (b) it follows that there exists a positive measure μ on X such that

$$<\nabla \Phi(u), v> = \int v \mathrm{d}\mu \qquad \text{for all} \quad v \in \mathscr{C} \cap \mathscr{X}. \qquad \text{q.e.d.}$$

COROLLARY. *Suppose that $v^+ \in \mathscr{X}$ and $\Phi(v^+) \leq \Phi(v)$ for any $v \in \mathscr{X}$. Then any pure potential is non-negative.*

PROOF. Let $u \in \mathscr{X}$ be a pure potential, and let us consider $\mathscr{K} = \{v \in \mathscr{X}; v \geq u\}$. Then $\inf\{\Phi(v); v \in \mathscr{K}\}$ is attained at a unique function of $\mathscr{K}$, because $\mathscr{K}$ is closed and convex in $\mathscr{X}$ and Φ is strictly convex. By Theorem 3.1, u is the minimizing function. On the other hand, by our assumption, $u^+ \in \mathscr{K}$ and $\Phi(u^+) \leq \Phi(u)$. Hence $u = u^+$, i.e., $u \geq 0$. q.e.d.

Throughout the rest of this section, we assume that T_∞ operates in $\mathscr{X}$. In this case, the following holds:

$$(3.1) \qquad \Phi(u^+) + \Phi(-u^-) \leq \Phi(u) \qquad \text{for any} \quad u \in \mathscr{X}.$$

Moreover, Theorem 3.1 is valid, since it follows from (3.1) that the mapping: $v \to T_\infty v = v^+$ from $\mathscr{X}$ into itself is bounded on each bounded set in $\mathscr{X}$.

LEMMA 3.1. *Let μ and ν be associated measures of potentials. If σ is a measure such that $\mu \leq \sigma \leq \nu$, then σ is the associated measuer of a potential.*

PROOF. Let ϕ be a non-negative function in $\mathscr{C} \cap \mathscr{X}$. Then we have

$$\int \phi \mathrm{d}\sigma \leq \int \phi \mathrm{d}\nu = <\nabla \Phi(u_\nu), \phi>$$

and

$$\int \phi \mathrm{d}\sigma \geq \int \phi \mathrm{d}\mu = <\nabla \Phi(u_\mu), \phi> .$$

Therefore for any $\psi \in \mathscr{C} \cap \mathscr{X}$ we write $\psi = \psi^+ - \psi^-$ and have by (1.1) and (3.1)

$$\int \psi d\sigma \leq <\nabla \Phi(u_\upsilon),\ \psi^+> + <\nabla \Phi(u_\mu),\ -\psi^-> $$
$$\leq (\|\nabla \Phi(u_\mu)\| + \|\nabla \Phi(u_\upsilon)\|)\Big(\frac{\Phi(\psi)}{C}\Big)^{1/p}.$$

Hence

$$\Big|\int \psi d\sigma\Big| \leq (\|\nabla \Phi(u_\mu)\| + \|\nabla \Phi(u_\upsilon)\|)\Big(\frac{\Phi(\psi)}{C}\Big)^{1/p}.$$

This means that the functional: $\psi \in \mathscr{C} \cap \mathscr{X} \to \int \psi d\sigma$ is continuous with respect to the topology of $\mathscr{X}$, so there exists a $u^* \in \mathscr{X}^*$ such that $<u^*, \psi> = \int \psi d\sigma$ for any $\psi \in \mathscr{C} \cap \mathscr{X}$. Lemma 1.1 shows that $u^* = \nabla \Phi(u)$ for some $u \in \mathscr{X}$. This u is a potential u_σ by definition. q.e.d.

THEOREM 3.2. *Let μ and ν be associated measures of potentials. If there is an associated measure σ of a potential such that $\mu \geq \sigma$ and $\nu \geq \sigma$, then $u_\mu \wedge u_\nu$ is a potential. Moreover, if we denote by η the associated measure of $u_\mu \wedge u_\nu$, then we have $\eta \geq \sigma$.*

PROOF. By the above lemma $\mu \wedge \nu$ is the associated measure of a potential. Let $\mathscr{K} = \{v \in \mathscr{X};\, v \geq u_\mu \wedge u_\nu\}$ and define an operator B: $\mathscr{X} \to \mathscr{X}^*$ by $Bv = \nabla \Phi(v) - \nabla \Phi(u_{\mu \wedge \upsilon})$. Then B is a monotone demicontinuous operator from $\mathscr{X}$ into $\mathscr{X}^*$ such that for each $w \in \mathscr{X}$, $<Bv, v-w>/\|v\| \to \infty$ as $\|v\| \to \infty$. By virtue of a result of Browder [6; Theorem 3] there is $v_0 \in \mathscr{K}$ such that $<Bv_0, v-v_0> \geq 0$ for any $v \in \mathscr{K}$. For any $v \in \mathscr{X}$ such that $v \geq 0$, we have $v_0 + v \in \mathscr{K}$ and hence $<Bv_0, v> \geq 0$. Therefore by Lemma 1.1 and Theorem 3.1 there exists a pure potential u_{η_0} such that $Bv_0 = \nabla \Phi(u_{\eta_0})$. This means that v_0 is a potential whose associated measure is $\eta_0 + \mu \wedge \nu$. If it is shown that $v_0 = u_\mu \wedge u_\nu$, then the proof of the theorem is completed.

First we see that $<Bv_0, v_0 - v_0 \wedge u_\mu> \leq 0$, because $v_0 \wedge u_\mu \in \mathscr{K}$. Next, by $(\Phi C_\infty)'$,

$$<B(v_0 \wedge u_\mu), v_0 - v_0 \wedge u_\mu>$$
$$= <B(u_\mu - (u_\mu - v_0)^+), (u_\mu - v_0)^->$$
$$\geq <Bu_\mu, (u - v_0)^-> .$$

For a non-negative function $\phi \in \mathscr{C} \cap \mathscr{X}$ we have

$$<Bu_\mu, \phi> = \int \phi d\mu - \int \phi d(\mu \wedge \nu) \geq 0.$$

By Lemma 2.1 we see that $<Bu_\mu, v> \geq 0$ for any non-negative $v \in \mathscr{X}$. Hence, $<Bu_\mu, (u_\mu - v_0)^-> \geq 0$. Therefore

$$<\nabla\Phi(v_0) - \nabla\Phi(v_0 \wedge u_\mu), v_0 - v_0 \wedge u_\mu>$$

$$= <Bv_0 - B(v_0 \wedge u_\mu), v_0 - v_0 \wedge u_\mu> \leq 0.$$

Property (b) implies that $v_0 = v_0 \wedge u_\mu$. Similarly we obtain $v_0 = v_0 \wedge u_\mu$. Since $v_0 \in \mathscr{K}$, $v_0 = u_\mu \wedge u_\nu$. q.e.d.

COROLLARY. *If u, v are pure potentials, then $u \wedge v$ is also a pure potential.*

THEOREM3.3. *Let μ and ν be associated measures of potentials. If there exists an associated measure σ of a potential such that $\mu \geq \sigma$, $\nu \geq \sigma$ and $<\nabla\Phi(u_\mu) - \nabla\Phi(u_\sigma), (u_\mu - u_\nu)^+> = 0$, then $u_\mu \leq u_\nu$.*

PROOF. By Theorem 3.2, $u_\mu \wedge u_\nu$ is a potential whose associated measure η is $\geq \sigma$. As in the proof of Theorem 3.1 we see that $<\nabla\Phi(u_\eta) - \nabla\Phi(u_\sigma), v> \geq 0$ for any non-negative $v \in \mathscr{X}$. Hence we have

$$<\nabla\Phi(u_\eta) - \nabla\Phi(u_\sigma), (u_\mu - u_\nu)^+> \geq 0.$$

Since $u_\mu - u_\eta = u_\mu - u_\mu \wedge u_\nu = (u_\mu - u_\nu)^+$, we have by using our assumption

$$<\nabla\Phi(u_\mu) - \nabla\Phi(u_\eta), u_\mu - u_\eta> \leq 0,$$

which implies $u_\mu = u_\eta$ (cf. property (b)). Then $u_\mu \leq u_\nu$. q.e.d.

COROLLARY 1. *Let μ and ν be associated measures of potentials. If $\mu \leq \nu$, then $u_\mu \leq u_\nu$.*

PROOF. Take $\sigma = \mu$ in the theorem. q.e.d.

COROLLARY 2. *Let f be a non-negative function in $\mathscr{M}$ and u_μ be a pure potential. If $u_f \leq u_\mu$ on the set $\{x \in X; f(x) > 0\}$, then $u_f \leq u_\mu$.*

In fact, take $\sigma = 0$ in the theorem and note that $<\nabla\Phi(u_f), (u_f - u_\mu)^+> = 0$.

REMARK. Ideas in the proofs of Theorems 3.2 and 3.3 are found in Calvert [7; §2]. Theorems 3.2 and 3.3 are nonlinear analogues of the principle of lower envelope and the domination principle in the Dirichlet space, respectively.

THEOREM 3.4. *Let k be a positive number and suppose that T_k operates in $\mathscr{X}$. Let μ and ν be associated measures of potentials such that $\mu \geq \sigma$ and $\nu \geq \sigma$ for some associated measure σ of a potential. Then $u_\mu \wedge (u_\nu + k)$ is a potential whose associated measure is $\geq \sigma$.*

PROOF. We note that $u \wedge (v+k)$ belongs to $\mathscr{X}$ for any $u, v \in \mathscr{X}$, because $u \wedge (v+k) = u \wedge v + T_k(u-v)$. Set $w = u_\mu \wedge (u_\nu + k)$, $\mathscr{K} = \{v \in \mathscr{X}; v \geq w\}$ and define the operator B from $\mathscr{X}$ into $\mathscr{X}^*$ by $Bv = \nabla\Phi(v) - \nabla\Phi(u_{\mu\wedge\nu})$. Then, just as in the proof of Theorem 3.2, we see that there exists $v_0 \in \mathscr{K}$ such that $<Bv_0, v - v_0> \geq 0$ for all $v \in \mathscr{K}$ and that v_0 is a potential whose associated measure is $\geq \mu \wedge \nu$ and $v_0 = v_0 \wedge u_\mu$. Therefore we have only to show that $v_0 = v_0 \wedge (u_\nu + k)$. For this, first note that $<Bv_0, v_0 - v_0 \wedge (u_\nu + k)> \leq 0$. Next, by $(\Phi C_k)'$,

$$
\begin{aligned}
&<B(v_0 \wedge (u_\nu + k)), v_0 - v_0 \wedge (u_\nu + k)> \\
&= <B(v_0 \wedge u_\nu + T_k(v_0 - u_\nu)^+), (v_0 - u_\nu)^+ - T_k(v_0 - u_\nu)^+> \\
&\geq <B(v_0 \wedge u_\nu), v_0 - v_0 \wedge (u_\nu + k)> .
\end{aligned}
$$

By Theorem 3.2, $v_0 \wedge u_\nu$ is also a potential whose associated measure is $\geq \mu \wedge \nu$. Therefore, the right hand side of the above inequality is non-negative, so that

$$
\begin{aligned}
&<\nabla\Phi(v_0) - \nabla\Phi(v_0 \wedge (u_\nu + k)), v_0 - v_0 \wedge (u_\nu + k)> \\
&= <B(v_0) - B(v_0 \wedge (u_\nu + k)), v_0 - v_0 \wedge (u_\nu + k)> \leq 0 .
\end{aligned}
$$

By property (b) we have $v_0 = v_0 \wedge (u_\nu + k)$. q.e.d.

COROLLARY. *Let k be a positive number and suppose that T_k operates in $\mathscr{X}$. If u_μ and u_ν are pure potentials, then $u_\mu \wedge (u_\nu + k)$ is also a pure potential.*

THEOREM 3.5. *Let k be a positive number and suppose that T_k operates in $\mathscr{X}$. Let μ and ν be associated measures of potentials. If there exists an associated measure σ of a potential such that $\mu \geq \sigma$, $\nu \geq \sigma$ and $<\nabla\Phi(u_\mu) - \nabla\Phi(u_\sigma), (u_\mu - u_\nu - k)^+> = 0$, then $u_\mu \leq u_\nu + k$.*

PROOF. By using Theorem 3.4 and Property (b), we can obtain the theorem in the same way as in the proof of Theorem 3.3. q.e.d.

COROLLARY. *Let k be a positive number and assume that T_k operates in $\mathscr{X}$. Let f be a non-negative function in $\mathscr{M}$ and u_μ be a pure potential. If $u_f \leq u_\mu + k$ on the set $\{x \in X; f(x) > 0\}$, then $u_f \leq u_\mu + k$.*

REMARK. Theorems 3.4 and 3.5 are nonlinear versions of the strong principle of lower envelope and the complete maximum principle in the Dirichlet space, respectively.

§ 4. The condenser and balayage principles

In this section, let $\mathscr{X}$ be a regular functional space in which T_∞ and some

T_k with $0<k<\infty$ operate.

THEOREM 4.1. (*Condenser principle*) *Let* G_0 *and* G_1 *be two open sets in* X *with disjoint closures,* G_1 *being relatively compact. Then there exists a potential* u_μ *such that*

(1) $0 \leq u_\mu \leq k$;

(2) $u_\mu = 0$ on G_0 and $= k$ on G_1;

(3) μ^+ is supported by $\bar{G}_1$ and μ^- by $\bar{G}_0$.

PROOF. Define a closed and convex set $\mathscr{K}$ in $\mathscr{X}$ by

$$\mathscr{K} = \{v \in \mathscr{X}; v \geq 0 \text{ on } G_1 \text{ and } v \leq 0 \text{ on } G_0\}.$$

It is easily seen that $\mathscr{K}$ is non-empty. Let us consider $\alpha=\inf\{\Phi(v); v\in\mathscr{K}\}$. Then there exists a unique function u such that $\alpha=\Phi(u)$. Besides, we see that

$$<\nabla\Phi(u), v-u> \geq 0 \qquad \text{for any} \quad v\in\mathscr{K},$$

and that $u=T_k u$, since $T_k u\in\mathscr{K}$ and $\Phi(T_k u)\leq\Phi(u)$ by (ΦC_k). Thus u satisfies (1) and (2). Next, we shall show that u is a potential whose associated measure satisfies (3). Set

$$\mathscr{U} = \{\phi \in \mathscr{C} \cap \mathscr{X}; \phi \geq 0 \quad on\, G_1 \; and \; \leq 0 \; on\, G_0\},$$

$$\mathscr{V} = \{\phi \in \mathscr{C} \cap \mathscr{X}; \phi \geq 0 \quad on\, G_1 \; and \; \mathrm{supp}\, \phi \subset X-\bar{G}_0\}$$

and

$$\mathscr{W} = \{\phi \in \mathscr{C} \cap \mathscr{X}; \phi \geq 0 \quad on\, G_0 \; and \; \mathrm{supp}\, \phi \subset X-\bar{G}_1\}.$$

Since $u+\phi\in\mathscr{K}$ for any $\phi\in\mathscr{U}$, we have

$$<\nabla\Phi(u), \phi> \geq 0 \qquad \text{for any} \quad \phi\in\mathscr{U}. \tag{4.1}$$

Noting Lemma 2.3 and applying Lemma 1.2 for $\Omega=X-\bar{G}_0$, $\mathscr{C}_1=\{\phi\in\mathscr{C}\cap\mathscr{X};$ $\mathrm{supp}\,\phi\subset X-\bar{G}_0\}$ and L: $\phi\in\mathscr{C}_1\to<\nabla\Phi(u), \phi>$ which is non-negative on $\mathscr{C}_1^+=\{\phi\in\mathscr{C}_1; \phi\geq 0$ on $X-\bar{G}_0\}$ and vanishes on $\{\phi\in\mathscr{C}_1; (\mathrm{supp}\,\phi)\cap\bar{G}_1=\emptyset\}$ on account of (4.1), we can find a positive measure ν on X such that

$$<\nabla\Phi(u), \phi> = \int\phi d\nu \qquad \text{for any} \quad \phi\in\mathscr{V}$$

and $\mathrm{supp}\,\nu\subset\bar{G}_1$. Similarly, we find a positive measure τ on X with support in $\bar{G}_0$ such that

$$<\nabla\Phi(u), \phi> = -\int\phi d\tau \qquad \text{for any} \quad \phi\in\mathscr{W}.$$

In the same way as in Deny [9; Théorème 1], Bliedtner [4; Theorem 13.2] and Fowler [11; Lemma 3.2], for each non-negative $v \in \mathscr{C} \cap \mathscr{X}$ on X and open sets V, W such that $X-\bar{G}_0 \supset V \supset (\mathrm{supp}\, v) \cap \bar{G}_1$, $X-\bar{G}_1 \supset W \supset (\mathrm{supp}\, v) \cap \bar{G}_0$ and $\bar{V} \cap \bar{W} = \emptyset$, we can find sequences $\{\phi_n\}$, $\{\phi'_n\}$ in $\mathscr{V}$ with support in V and $\{\psi_n\}$, $\{\psi'_n\}$ in $\mathscr{W}$ with support in W such that $\phi_n \uparrow v$, $\phi'_n \downarrow v$ uniformly on $\bar{G}_1$ and $\psi_n \uparrow v$, $\psi'_n \downarrow v$ uniformly on $\bar{G}_0$ as $n \to \infty$. Then we see by (4.1) that

$$\int \phi_n d\nu - \int \psi'_n d\tau \leq \langle \nabla \Phi(u), v \rangle \leq \int \phi'_n d\nu - \int \psi_n d\tau .$$

Letting $n \to \infty$ yields that

$$\langle \nabla \Phi(u), v \rangle = \int v d\nu - \int v d\tau .$$

This shows that u is a potential whose associated measure is $\nu - \tau$. Thus the theorem is proved. q.e.d.

REMARK. As was seen in the above proof of Theorem 4.1, in addition to the assumptions on Φ made in §1 we needed only the property that $\Phi(T_k u) \leq \Phi(u)$ for any $u \in \mathscr{X}$. Fowler [11: Theorem 4.1] showed the condenser principle in the case of $\Phi(u) = \|u\|^2$. Our theorem is a generalization of it.

THEOREM 4.2. (*Balayage principle*) *Given a pure potential u_μ and an open set G in X, there exists a unique pure potential $u_{\mu'}$ such that*

(1) *$u_{\mu'} = u_\mu$ on G;*

(2) *μ' is supported by $\bar{G}$;*

(3) *if u_ν is a pure potential and $u_\nu \geq u_\mu$ on G, then $u_\nu \geq u_{\mu'}$ on X.*

PROOF. Define a closed and convex set $\mathscr{K}$ in $\mathscr{X}$ by

$$\mathscr{K} = \{v \in \mathscr{X};\, v \geq u_\mu \quad on\, G\} .$$

Then it is non-empty and there exists a unique $v_0 \in \mathscr{K}$ such that $\langle \nabla \Phi(v_0), v - v_0 \rangle \geq 0$ for any $v \in \mathscr{K}$. Since $v + v_0 \in \mathscr{K}$ for any $v \in \mathscr{X}$ such that $v \geq 0$, we have $\langle \nabla \Phi(v_0), v \rangle \geq 0$ for such v. Hence v_0 is a pure potential by Theorem 3.1. Let u_ν be any pure potential such that $u_\nu \geq u_\mu$ on G. Then $v_0 \wedge u_\nu \in \mathscr{K}$ and $\langle \nabla \Phi(v_0 \wedge u_\nu), v_0 - v_0 \wedge u_\nu \rangle = \langle \nabla \Phi(u_\nu - (u_\nu - v_0)^+), (u_\nu - v_0)^- \rangle \geq \langle \nabla \Phi(u_\nu), (u_\nu - v_0)^- \rangle \geq 0$ because of $(\Phi C_\infty)'$. So we have

$$\langle \nabla \Phi(v_0) - \nabla \Phi(v_0 \wedge u_\nu), v_0 - v_0 \wedge u_\nu \rangle \leq 0 .$$

Therefore we have $v_0 = v_0 \wedge u_\nu$, that is, $v_0 \leq u_\nu$ by property (b), and especially $v_0 \leq u_\mu$. It remains to show that the associated measure μ' of v_0 is supported by

$\bar{G}$. For this, take any function $\phi \in \mathscr{C} \cap \mathscr{X}$ such that $\mathrm{supp}\,\phi \subset X - \bar{G}$. Noting that $v_0 \pm \phi \in \mathscr{X}$, we see that $\int \phi d\mu' = \langle \nabla\Phi(v_0), \phi \rangle = 0$. It follows from Lemma 2.3 that $\mathrm{supp}\,\mu' \subset \bar{G}$. Thus v_0 is a function having the required properties.

Finally we note that properties (1) and (3) imply the uniqueness of such a $u_{\mu'}$. q.e.d.

§ 5. Capacity with respect to Φ

Let us assume in this section that $\mathscr{X}$ is a regular functional space in which T_∞ operates. We introduce a notion of capacity with respect to Φ and discuss the refinement of functions in $\mathscr{X}$.

Let K be any compact set in X. We define

$$\mathrm{Cap}(K) = \inf\{\Phi(\phi); \phi \in \mathscr{C} \cap \mathscr{X}, \phi \geq 1 \text{ on } K\}.$$

Note that such ϕ exists by Axiom (b) and that $0 \leq \mathrm{Cap}(K) < \infty$. Since T_∞ operates in $\mathscr{X}$, $\mathrm{Cap}(K) = \inf\{\Phi(\phi); \phi \in \mathscr{C} \cap \mathscr{X}, \phi \geq 0 \text{ on } X \text{ and } \geq 1 \text{ on } K\}$. For any open set G in X, we define

$$\mathrm{Cap}(G) = \sup\{\mathrm{Cap}(K); K \subset G, K \text{ is compact}\}.$$

LEMMA 5.1. *Let K be any compact set in X. Then*

$$\mathrm{Cap}(K) = \inf\{\mathrm{Cap}(G); K \subset G, G \textit{ is open}\}.$$

PROOF. Since $\mathrm{Cap}(K) \leq \inf\{\mathrm{Cap}(G); K \subset G, G \text{ is open}\}$ is trivial, we show the converse inequality. Given $\varepsilon > 0$, choose $\phi \in \mathscr{C} \cap \mathscr{X}$ such that $\phi \geq 1$ on K and $\Phi(\phi) \leq \mathrm{Cap}(K) + \varepsilon$. Set $G_\eta = \{x \in X; \phi(x) > \eta\}$ for any number η, $0 < \eta < 1$. Then, since G_η is open and contains K, $\inf\{\mathrm{Cap}(G); K \subset G, G \text{ is open}\} \leq \mathrm{Cap}(G_\eta) \leq \Phi(\phi/\eta)$ for each η. By the continuity of Φ, $\Phi(\phi/\eta) \to \Phi(\phi)$ as $\eta \uparrow 1$. Hence we have

$$\inf\{\mathrm{Cap}(G); K \subset G, G \textit{ is open}\} \leq \Phi(\phi) \leq \mathrm{Cap}(K) + \varepsilon.$$

Since ε is arbitrary, the lemma is valid. q.e.d.

The above lemma allows us to give the following definition:

DEFINITION 5.1. *For any set A in X we define*

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(G) : A \subset G, G \textit{ is open}\}.$$

and call it the capacity of A (with respect to Φ).

Using condition (ΦC_∞), we can easily prove the following two lemmas.

LEMMA 5.2. *For any set A_1 and A_2 in X, we have*

$$\mathrm{Cap}(A_1 \cup A_2)+\mathrm{Cap}(A_1 \cap A_2) \leq \mathrm{Cap}(A_1)+\mathrm{Cap}(A_2).$$

LEMMA 5.3. $\mathrm{Cap}(\cdot)$ *is countably subadditive, i.e., for any countable family $\{A_n\}$ of sets in X we have*

$$\mathrm{Cap}\,(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathrm{Cap}\,(A_n)\,.$$

LEMMA 5.4. *Let A be a Borel set in X and let μ be the associated measure of a pure potential. Then*

$$\mu(A) \leq \|\nabla \Phi(u_\mu)\| \cdot \left(\frac{\mathrm{Cap}(A)}{C}\right)^{1/p}.$$

PROOF. According to the definition of capacity and Lemma 5.1 it suffices to show the case where A is a compact set K in X. Given $\varepsilon>0$, choose a function $\phi \in \mathscr{C} \cap \mathscr{X}$ so that $\phi \geq 0$ on X, $\phi \geq 1$ on K and $\Phi(\phi) \leq \mathrm{Cap}(K)+\varepsilon$. Then by the definition of u_μ and assumption (1.1),

$$\mu(K) \leq \int \phi \mathrm{d}\mu = <\nabla \Phi(u_\mu),\, \phi> \,\leq \|\nabla \Phi(u_\mu)\|\, \|\phi\|$$

$$\leq \|\nabla \Phi(u_\mu)\| \left(\frac{\Phi(\phi)}{C}\right)^{1/p} \leq \|\nabla \Phi(u_\mu)\| \left(\frac{\mathrm{Cap}(K)+\varepsilon}{C}\right)^{1/p},$$

so the lemma is obtained. q.e.d.

COROLLARY 1. *If A is a Borel set in X such that* $\mathrm{Cap}(A)=0$ *and if μ is the associated measure of a pure potential, then $\mu(A)=0$.*

COROLLARY 2. *If A is as in the above Corollary, then $\xi(A)=0$.*

PROOF. Setting $\mu = f\mathrm{d}\xi$ for any non-negative function $f \in \mathscr{M}$, we see from Theorem 1.1 that μ is an associated measure of a pure potential. Therefore, by Corollary 1, we have $\int_A f\mathrm{d}\xi = 0$. It then follows that $\xi(A)=0$. q.e.d.

THEOREM 5.1. *With each function $v \in \mathscr{X}$, it is possible to associate a function $\tilde{v}$ (refinement of v) such that*

(1) *$\tilde{v}=v$ a.e. on X;*

(2) *there exists a decreasing sequence $\{G_n\}$ of open sets in X such that* $\mathrm{Cap}(G_n) \downarrow 0$ *as $n \to \infty$ and $\tilde{v}$ is continuous as a function on $X-G_n$ for each n;*

(3) *$\tilde{v}$ is measurable with respect to the associated measure μ of any pure potential and*

$$<\nabla\Phi(u_\mu), v> \ = \int \tilde{v} d\mu .$$

PROOF. By Axiom (b) and the continuity of Φ, we can take a sequence $\{\phi_n\}$ in $\mathscr{C} \cap \mathscr{X}$ strongly convergent to $v \in \mathscr{X}$ in $\mathscr{X}$ such that

$$\|\sum_{n=1}^{\infty}\{\Phi(2^n(\phi_n-\phi_{n+1}))+\Phi(2^n(\phi_{n+1}-\phi_n))\}\| < \infty . \tag{5.1}$$

We then set $G'_j=\{x\in X; |\phi_{j+1}(x)-\phi_j(x)|>1/2^j\}$ for each j, $G_n=\cup_{j=n}^{\infty}G'_j$ for each n and $A=\cap_{n=1}^{\infty}G_n$. Now, note that $\lim_{n\to\infty}\phi_n(x)$ exists for each $x\in X-A$, and define

$$\tilde{v}(x) = \begin{cases} \lim_{n\to\infty}\phi_n(x) & \text{if } x\in X-A, \\ 0 & \text{if } x\in A. \end{cases}$$

From (5.1) and Lemma 5.3 it follows that $\mathrm{Cap}(G_n)\downarrow 0$ as $n\to\infty$, so that $\mathrm{Cap}(A)=0$. Therefore, using Corollaries 1 and 2 to Lemma 5.4, we see that $\tilde{v}$ fulfills (1) and (2) in the theorem and that $\tilde{v}$ is measurable with respect to the associated measure μ of any pure potential. Next we observe from (C_∞), (ΦC_∞) and (1.1) that

$$\begin{aligned}\int|\phi_n-\phi_m|d\mu &= <\nabla\Phi(u_\mu), |\phi_n-\phi_m|> \\ &\le \|\nabla\Phi(u_\mu)\|\,\||\phi_n-\phi_m|\| \\ &\le \|\nabla\Phi(u_\mu)\|\left(\frac{\Phi(\phi_n-\phi_m)+\Phi(\phi_m-\phi_n)}{C}\right)^{1/p}.\end{aligned}$$

Since $\{\phi_n\}$ is a Cauchy sequence in $\mathscr{X}$, it is also a Cauchy sequence in $L^1(X;\mu)$. Therefore there are a μ-measurable function v_0 and a subsequence $\{\phi_{n_j}\}$ of $\{\phi_n\}$ convergent to v_0 in $L^1(X;\mu)$ such that $\phi_{n_j}\to v_0$ μ-a.e. on X as $j\to\infty$. Clearly $v_0=\tilde{v}$ μ-a.e. on X and $<\nabla\Phi(u_\mu), v> =\lim_{j\to\infty}<\nabla\Phi(u_\mu), \phi_{n_j}> =\lim_{j\to\infty}\int\phi_{n_j}d\mu$ $=\int\tilde{v}d\mu$. Thus (3) is proved.

REMARK. Recently, Fowler [12] introduced a notion of a capacity with respect to the norm in a regular functional space in which the unit contraction T_1 operates and discussed the refinement of functions.

§6. Examples

EXAMPLE 1. (*Discrete case*) Let X be a finite set, say $\{1, 2,\ldots, N\}$, equipped with the discrete topology. In this case, $\mathscr{C}$ can be identified with R^N. We put

$$\|u\| = (\sum_{i=1}^{N} |u_i|^p)^{1/p}, \qquad u = (u_1, u_2, \dots, u_N) \in R^N$$

and $\mathscr{X} = R^N$. Clearly, $\mathscr{X}$ is a regular functional space. Let Φ be a convex and continuously differentiable function on R^N with the property (1.1). Then we observe that $\nabla\Phi$ is a mapping of R^N into R^N such that $\nabla\Phi(u) = (\Phi_1(u), \Phi_2(u), \dots, \Phi_N(u))$ for $u \in R^N$, where $\Phi_i = (\partial/\partial x_i)\Phi$, $i = 1, 2, \dots, N$. Let us denote by e_0 the element $(1, 1, \dots, 1)$ in R^N, and define $r^+ = \max(r, 0)$ for a real number r. For any $u = (u_1, u_2, \dots, u_N)$ and $v = (v_1, v_2, \dots, v_N) \in R^N$, we write $u \leq v$ when $u_i \leq v_i$ for all i, and write $u \wedge v$ for $(w_1, w_2, \dots, w_N)$ with $w_i = \min(u_i, v_i)$ for all i.

PROPOSITION 6.1. *The following statements are equivalent to each other*:

(1) *T_∞ operates in R^N.*

(2) *$\nabla\Phi(u \wedge v) \geq \nabla\Phi(u) \wedge \nabla\Phi(v)$ for any $u, v \in R^N$.*

(3) *If $u = (u_1, u_2, \dots, u_N)$ and $v = (v_1, v_2, \dots, v_N) \in R^N$ and if*

$$\sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(v))^+ (u_i - v_i)^+ = 0,$$

then $u \leq v$.

In fact, assertions (1)→(2) and (2)→(3) are already shown in Theorems 3.2 and 3.3, respectively. For a proof of the assertion (3)→(1), see Kenmochi-Mizuta [16].

PROPOSITION 6.2. *Assume that T_∞ operates in R^N, and let k be a positive number. Then the following statements are equivalent to each other*:

(a) *T_k operates in R^N.*

(b) *$\nabla\Phi(u \wedge (v + ke_0)) \geq \nabla\Phi(u) \wedge \nabla\Phi(v)$ for any $u, v \in R^N$.*

(c) *If $u = (u_1, u_2, \dots, u_N)$ and $v = (v_1, v_2, \dots, v_N) \in R^N$ and if*

$$\sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(v))^+ (u_i - v_i - k)^+ = 0,$$

then $u \leq v + ke_0$.

Assertions (a)→(b) and (b)→(c) follow from Theorems 3.4 and 3.5, respectively. For a proof of the assertion (c)→(a), see Kenmochi-Mizuta [16] in which the assertion is proved in a more general form.

EXAMPLE 2. Let X be a bounded domain Ω in the n-dimensional Euclidean space R^n and ξ be the Lebesgue measure dx. We consider Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ (=the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) with norm $\|u\|_{1,p} = \|u\|_{L^p(\Omega)}$

$+\sum_{k=1}^{n}\left\|\frac{\partial u}{\partial x_k}\right\|_{L^p(\Omega)}$. For these Banach spaces, Axiom (a) can be verified. Besides, we see that $W_0^{1,p}(\Omega)$ is regular, but $W^{1,p}(\Omega)$ is not.

Let α_0 and α_1 be bounded measurable functions such that $\alpha_0, \alpha_1 \geq c$ *a.e.* on Ω for a positive constant c. Then functions Φ_0 on $W_0^{1,p}(\Omega)$ and Φ_1 on $W^{1,p}(\Omega)$ defined by

$$\Phi_0(v) = \frac{1}{p}\int_\Omega \alpha_1 |\nabla v|^p \mathrm{dx}$$

and

$$\Phi_1(v) = \Phi_0(v) + \frac{1}{p}\int_\Omega \alpha_0 |v|^p \mathrm{dx}$$

satisfy all of the assumptions for $\Phi=\Phi_0$ and $\Phi=\Phi_1$, respectively, where ∇v is the gradient of v. It is also easy to see that all T_k, $0\leq k\leq\infty$, operate in each space with respect to the corresponding function.

By definition, a function u in $W_0^{1,p}(\Omega)$ is a potential (with respect to Φ_0) if and only if for some measure μ on Ω, it satisfies the following differential equation in the distribution sense on Ω:

$$-\mathrm{div}(\alpha_1 |\nabla v|^{p-2}\nabla u) = \mu .$$

In this case, μ is the associated measure of u.

EXAMPLE 3. Let Ω be a bounded domain in R^n with smooth boundary Γ. We consider the trace space $W^{1/p',p}(\Gamma)$ $(1/p+1/p'=1)$ with norm

$$\|\hat{v}\|_{1/p',p} = \|\hat{v}\|_{L^p(\Gamma)} + \left(\int_\Gamma\int_\Gamma \frac{|v(x')-v(y')|^p}{|x'-y'|^{n+p-2}} \mathrm{d}\Gamma_{x'}\mathrm{d}\Gamma_{y'}\right)^{1/p},$$

where $\mathrm{d}\Gamma$ means the surface measure on Γ, and we denote by γ the trace operator which is a linear continuous operator from $W^{1,p}(\Omega)$ onto $W^{1/p',p}(\Gamma)$ (cf. Lions-Magenes [18]). Clearly, $W^{1/p',p}(\Gamma)$ is a regular functional space; in this case, we take Γ and $\mathrm{d}\Gamma$ as X and ξ, respectively.

Now, let Φ_1 be as in Example 2. Then we see that

$$<\nabla\Phi_1(v),w> = \sum_{k=1}^{n}\int_\Omega \alpha_1 |\nabla v|^{p-2}\frac{\partial v}{\partial x_k}\frac{\partial w}{\partial x_k}\mathrm{dx} + \int_\Omega \alpha_0 |v|^{p-2} vw\mathrm{dx},$$

and

$$\Phi_1(v) = \frac{1}{p} <\nabla\Phi_1(v), v> .$$

Consider the boundary value problem for given $\hat{u}\in W^{1/p',p}(\Gamma)$: Find $u\in$

$W^{1,p}(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(\alpha_1 |\nabla u|^{p-2}\nabla u)+\alpha_0 |u|^{p-2}u = 0 \qquad \text{on } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(in the distribution sense)}, \\ \gamma u = \hat{u}\,. \end{cases}$$

This problem has a unique solution u for any $\hat{u} \in W^{1/p',p}(\Gamma)$; in fact, u is the function at which $\inf\{\Phi_1(v); v \in W^{1,p}(\Omega), \gamma v = \hat{u}\}$ is attained (cf. [15; § 3]). Hence we can define an operator S: $W^{1/p',p}(\Gamma) \to W^{1,p}(\Omega)$ by setting $S\hat{u} = u$. Notice that for each $v \in W^{1,p}(\Omega)$ the value $<\nabla\Phi_1(S\hat{u}), v>$ depends only on γv, that is, $<\nabla\Phi_1(S\hat{u}), \phi> = 0$ if $\phi \in W_0^{1,p}(\Omega)$. This allows us to define an operator $\mathscr{A}$: $W^{1/p',p}(\Gamma) \to W^{-1/p',p'}(\Gamma)$ (=the dual space of $W^{1/p',p}(\Gamma)$) by

$$<\mathscr{A}\hat{u}, \gamma v>_\Gamma = <\nabla\Phi_1(S\hat{u}), v>\,, \qquad v \in W^{1,p}(\Omega)\,,$$

where $<\cdot,\cdot>_\Gamma$ denotes the duality pairing between $W^{-1/p',p'}(\Gamma)$ and $W^{1/p',p}(\Gamma)$. In view of a result in [15; § 6] the operator S is bounded and continuous, so that $\mathscr{A}$ is also bounded and continuous. Define the function $\hat{\Phi}$ on $W^{1/p',p}(\Gamma)$ by $\hat{\Phi}(u) = \Phi_1(S\hat{u})$. Then we have

PROPOSITION 6.3. (i) *$\hat{\Phi}$ is strictly convex and bounded on bounded sets in $W^{1/p',p}(\Gamma)$.*

(ii) *$\hat{\Phi}$ is everywhere differentiable in the sense of Gateaux and $\nabla\hat{\Phi}(\hat{u}) = \mathscr{A}\hat{u}$ for every $\hat{u} \in W^{1/p',p}(\Gamma)$.*

(iii) *For some positive constant C',*

$$\hat{\Phi}(\hat{u}) \geq C'\|\hat{u}\|_{1/p',p} \qquad \textit{for every } \hat{u} \in W^{1/p',p}(\Gamma),$$

and $\hat{\Phi}(0) = 0$.

PROOF. The fact that $\hat{\Phi}$ is bounded on bounded sets in $W^{1/p',p}(\Gamma)$ follows from the definition of $\hat{\Phi}$ and the boundedness of S. Let $\hat{u}$ and $\hat{v}$ be any functions in $W^{1/p',p}(\Gamma)$ and t be a number such that $0<t<1$. Then we see that if $\hat{u} \neq \hat{v}$, then

$$\begin{aligned} \hat{\Phi}(t\hat{u}+(1-t)\hat{v}) &= \Phi_1(S(t\hat{u}+(1-t)\hat{v})) \\ &= \inf\{\Phi_1(v); v \in W^{1,p}(\Omega), \gamma v = t\hat{u}+(1-t)\hat{v}\} \\ &\leq \Phi_1(tS\hat{u}+(1-t)S\hat{v}) < t\Phi_1(S\hat{u})+(1-t)\Phi_1(S\hat{v}) \\ &= t\hat{\Phi}(\hat{u})+(1-t)\hat{\Phi}(\hat{v}). \end{aligned}$$

Thus (i) is proved.

Next, to prove (ii) we observe that for any $\hat{u}, \hat{v} \in W^{1/p',p}(\Gamma)$

$$(6.1) \quad \lim_{t \downarrow 0} \frac{1}{t} \{\hat{\Phi}(\hat{u}+t\hat{v}) - \hat{\Phi}(\hat{u})\} \leq \lim_{t \downarrow 0} \frac{1}{t} \{\Phi_1(S\hat{u}+tS\hat{v}) - \Phi_1(S\hat{u})\}$$

$$= \langle \nabla \hat{\Phi}_1(S\hat{u}), S\hat{v} \rangle = \langle \mathscr{A}\hat{u}, \hat{v} \rangle_\Gamma .$$

On the other hand, by virtue of a result in [19], there is $\hat{u}^* \in W^{-1/p',p'}(\Gamma)$ for each $\hat{u} \in W^{1/p',p}(\Gamma)$ such that

$$(6.2) \qquad \langle \hat{u}^*, \hat{v} - \hat{u} \rangle_\Gamma \leq \hat{\Phi}(\hat{v}) - \hat{\Phi}(\hat{u}) \qquad \textit{for all } \hat{v} \in W^{1/p',p}(\Gamma) .$$

Therefore, by (6.1) and (6.2) we have

$$\langle \hat{u}^*, \hat{v} \rangle_\Gamma \leq \lim_{t \downarrow 0} \frac{1}{t} \{\hat{\Phi}(\hat{u}+t\hat{v}) - \hat{\Phi}(\hat{u})\} \leq \langle \mathscr{A}\hat{u}, \hat{v} \rangle_\Gamma$$

for any $\hat{v} \in W^{1/p',p}(\Gamma)$, so that $\hat{u}^* = \mathscr{A}\hat{u}$ holds. This shows (ii). (iii) is clear.
q.e.d.

PROPOSITION 6.4. *For every* $k \in [0, \infty]$, T_k *operates in* $W^{1/p',p}(\Gamma)$ *with respect to* $\hat{\Phi}$.

PROOF. Since it is clear that condition (C_k) is satisfied for every $k \in [0, \infty]$, we shall show (ΦC_k) with Φ replaced by $\hat{\Phi}$.

Notice that for any $k \in [0, \infty]$ and any $u, v \in W^{1,p}(\Omega)$,

$$\Phi_1(u) + \Phi_1(v) \geq \Phi_1(u + T_k(v-u)) + \Phi_1(v - T_k(v-u)) .$$

From this inequality and the definition of $\hat{\Phi}$ we see that condition (ΦC_k) with $\Phi = \hat{\Phi}$ is satisfied for every $k \in [0, \infty]$.

REMARK. Such an operator $\mathscr{A}$ was also treated by Lions [17; Chapter 2] so as to formulate initial value problems on Γ.

References

[1] A. Beurling and J. Deny, Espaces de Dirichlet, I. Le cas élémentaire, Acta Math. **99** (1958), 203–224.

[2] A. Beurling and J. Deny, Dirichlet spaces, Proc. Nat. Acad. Sci. U. S. A. **45** (1959), 208–215.

[3] J. Bliedtner, Functional spaces and their exceptional sets, Lecture Notes in Math. **226**, Springer-Verlag, Berlin, 1971.

[4] J. Bliedtner, Dirichlet forms on regular functional spaces, Lecture Notes in Math. **226**, Springer-Verlag, Berlin, 1971.

[5] F. E. Browder, On a theorem of Beurling and Livingston, Canad. J. Math. **17** (1965), 367–372.

[6] F. E. Browder, Nonlinear variational inequalities and maximal monotone mappings in Banach spaces, Math. Ann. **183** (1969), 213–231.

[7] B. Calvert, Potential theoretic properties for nonlinear monotone operators, Boll. Un. Ital. **5** (1972), 473–489.

[8] B. Calvert, On T-accretive operators, Ann. Mat. Pura Appl. **94** (1972), 473–489.

[9] J. Deny, Sur les espaces de Dirichlet, Séminaire de théorie du potentiel, dirigé par M. Brelot et G. Choquet; **1**re année: 1957. Secrétariat mathématique, **11**rue Pierre Curie, Paris, 1958.

[10] J. Deny, Principe complet du maximum et contractions, Ann. Inst. Fourier **15** (1965), 258–272.

[11] P. Fowler, Potential theory in Banach spaces of functions, a condenser theorem, J. Math. Anal. Appl. **33** (1971), 310–327.

[12] P. Fowler, Capacity theory in Banach spaces, Pacific J. Math. **48** (1973), 365–385.

[13] M. Itô, A note on extended regular functional spaces, Proc. Japan Acad. **43** (1967), 435–440.

[14] M. Itô, Topics on the Dirichlet space theory (in Japanese), Sûgaku **19** (1967/68), 65–76.

[15] N. Kenmochi, Pseudo-monotone operators and nonlinear elliptic boundary value problems, (submitted).

[16] N. Kenmochi and Y. Mizuta, Potential theoretic properties of the gradient of a convex function on a functional space, (submitted).

[17] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod Gauthier-Villars, Paris, 1969.

[18] J. L. Lions and E. Magenes, Problemi ai limiti non omogenei (III), Ann. Scuola Norm. Sup. Pisa **15** (1961), 41–103.

[19] J. J. Moreau, Sém. Équations aux Dérivées Partielles, 1966/67, Fasc. 2: Fonctionnelles convexes, Collège de France, Paris, 1966/67.

[20] I. Namioka, Partially ordered linear topological spaces, Mem. Amer. Math. Soc. **24** (1957).

Department of Mathematics,
Faculty of Science,
Hiroshima University