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On Claw-decomposition of Complete Graphs and Complete Bigraphs

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Introduction

It has been found in our course of study on the design of balanced file organization schemes of order two [2] that, in the terminology of graphs, the decomposition of a complete graph into a union of line disjoint claws or stars provides us an optimal file organization scheme in the sense such that it has the least redundancy among the schemes of order two for every probability distribution of records having property of invariance in the permutation of attributes. As far as the present authors know, no information about such a claw-decomposition problem of complete graphs has yet been obtained. In this paper, a complete answer to the problem which may be called the claw-decomposition theorem of complete graphs will be given. A similar theorem of complete bigraphs will also be given.

In processing those decomposition theorems, it has been found useful to provide an existence theorem and a construction algorithm of bigraphs having preassigned degrees of points. The existence of bigraphs having preassigned degrees is equivalent to that of 0–1 matrices having preassigned row and column sum vectors. In the terminology of the latter, a necessary and sufficient condition for the existence of a 0–1 matrix has been given by Ryser [1]. In this paper, a straightforward construction algorithm to decide the existence of a 0–1 matrix will be given. An alternative proof of Ryser's theorem will also be given.

§1. Existence of bigraphs

A bigraph (or bipartite graph) $G_{m,n}$ is an (unordered) graph whose point set can be partitioned into two subsets V_1 and V_2 with m and n points each, such that every line of $G_{m,n}$ joins V_1 with V_2. If it contains every line joining V_1 and V_2, then it is called a complete bigraph and denoted by $K_{m,n}$. Specifically, a complete bigraph $K_{1,c}$ is called a *claw* or a *star* with c lines.

Upon labelling those points in V_1 and V_2 of $G_{m,n}$ by $u_1, u_2,\cdots, u_m$ and $v_1, v_2,\cdots, v_n$, the adjacency of points in $G_{m,n}$ can be represented uniquely by an $m\times n$ 0–1 matrix $A=\|a_{ij}\|$ in which $a_{ij}=1$ if u_i is adjacent with v_j and $a_{ij}=0$ otherwise.

The set of row and column sum vectors

$$\{(d_1, d_2, \cdots, d_m), (e_1, e_2, \cdots, e_n)\} \tag{1.1}$$

of A corresponds to an arrangement of the degrees of points of the corresponding bigraph $G_{m,n}$, where $d_i = \sum_{j=1}^{n} a_{ij}$ and $e_j = \sum_{i=1}^{m} a_{ij}$. We may note that permutation of the labels in V_1 and in V_2 and the interchange of V_1 and V_2 are, of course, irrelevant to $G_{m,n}$; they correspond to permutation of the rows, of the columns, and transposition of the matrix A, respectively.

Since the number of lines in $G_{m,n}$ is given by $N = \sum_{i=1}^{m} d_i = \sum_{j=1}^{n} e_j$, (1.1) can also be considered as a pair of m and n partitions of a nonnegative integer N.

A pair of m and n partitions

$$\Pi_{m,n} = \{(r_1, r_2, \cdots, r_m), (s_1, s_2, \cdots, s_n)\} \tag{1.2}$$

of a nonnegative integer N will be called *bigraphical* if there exists a bigraph $G_{m,n}$ whose arrangement of degrees is $\Pi_{m,n}$ or, equivalently, if there exists a 0–1 matrix A of size $m \times n$ whose set of row and column sum vectors is $\Pi_{m,n}$, where r_i and s_j are nonnegative integers and satisfy $N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$.

The following theorem provides us an algorithm to decide whether $\Pi_{m,n}$ is bigraphical or not.

THEOREM 1.1. (Algorithm) *A pair of m and n partitions*

$$\Pi_{m,n} = \{(r_1, r_2, \cdots, r_m), (s_1, s_2, \cdots, s_n)\} \tag{1.3}$$

of a nonnegative integer N with $s_1 \geq s_2 \geq \cdots \geq s_n$ *is bigraphical if and only if a modified pair of* $m-1$ *and n partitions*

$$\Pi_{m-1,n} = \{(r_2, r_3, \cdots, r_m), (s_1 - 1, \cdots, s_{r_1} - 1, s_{r_1+1}, \cdots, s_n)\} \tag{1.4}$$

of the nonnegative integer $N - r_1$ *exists and is bigraphical.*

PROOF. If $\Pi_{m-1,n}$ in (1.4) exists and is bigraphical, a 0–1 matrix having $\Pi_{m,n}$ in (1.3) can be obtained by adding a row which has r_1 ones followed by $n - r_1$ zeros to the 0–1 matrix having (1.4). $\Pi_{m,n}$ in (1.3) is, therefore, bigraphical.

Conversely, suppose $\Pi_{m,n}$ in (1.3) to be bigraphical. If the first row of $A = \|a_{ij}\|$ corresponding to (1.3) is a vector composed of r_1 ones followed by $n - r_1$ zeros, then $\Pi_{m-1,n}$ in (1.4) exists and a 0–1 matrix having (1.4) can be obtained from A by deleting the first row. If $a_{1j} = 0$ for some j satisfying $1 \leq j \leq r_1$, then there exists some j' satisfying $r_1 < j' \leq n$ such that $a_{1j'} = 1$. In this case there exists some i for which $a_{ij} = 1$ and $a_{ij'} = 0$, since $s_j \geq s_{j'}$. Interchanging zeros and ones with those four elements in A, a 0–1 matrix A^* with the same $\Pi_{m,n}$ is obtained, in which $a^*_{1j}=1$ and $a^*_{1j'}=0$. Repeated application of such an interchange will yield a 0–1 matrix $\tilde{A}$ with $\Pi_{m,n}$ in which the first row is composed of r_1 ones followed by $n-r_1$ zeros. Removing the first row of $\tilde{A}$, a 0–1 matrix $\tilde{A}_1$ of size $(m-1)\times n$ having $\Pi_{m-1,n}$ in (1.4) is obtained. The modified pair of partitions $\Pi_{m-1,n}$ exists and is bigraphical.

This completes the proof.

Note: The family of 0–1 matrices having $\Pi_{m,n}$, if it exists, is essentially unique in that any two are transformable each other by repeated interchange of zeros and ones.

THEOREM 1.2. (Ryser) *A pair of m and n partitions*

$$\Pi_{m,n} = \{(r_1, r_2,\cdots, r_m), (s_1, s_2,\cdots, s_n)\} \tag{1.5}$$

of a nonnegative integer N with $r_1 \geq r_2 \geq \cdots \geq r_m$ *is bigraphical if and only if the inequalities*

$$\sum_{i=1}^{k} r_i \leq \sum_{j=1}^{n} \min(k, s_j) \tag{1.6}$$

hold for all $k=1, 2,\cdots, m$.

PROOF. If $\Pi_{m,n}$ in (1.5) is bigraphical, then there exists a 0–1 matrix A of size $m\times n$ having $\Pi_{m,n}$. Consider a submatrix A_k composed of the first k rows of A for each $k=1, 2,\cdots, m$. The number of ones in the j-th column of A_k is not greater than $\min(k, s_j)$ for each $j=1, 2,\cdots, m$. The total number of ones in A_k, $\sum_{i=1}^{k} r_i$, is, therefore, not greater than $\sum_{j=1}^{n} \min(k, s_j)$. This implies that the inequalities (1.6) are necessary.

The sufficiency of (1.6) will be proved by induction on m.

For $m=1$, since $\sum_{j=1}^{n} s_j = r_1 \leq \sum_{j=1}^{n} \min(1, s_j)$ and $\sum_{j=1}^{n} s_j \geq \sum_{j=1}^{n} \min(1, s_j)$ hold by assumption, it follows that $s_j=1$ or 0 for all $j=1, 2,\cdots, n$. A 0–1 matrix $A=\|a_{ij}\|$ of size $1\times n$, in which $a_{1j}=1$ if $s_j=1$ and $a_{1j}=0$ otherwise, has the required set of row and column sum vectors. Hence, $\Pi_{1,n}$ is bigraphical.

Suppose (1.6) to be sufficient for any pair of partitions $\Pi_{m,n}$ with $m=t$, and assume that the inequalities

$$\sum_{i=1}^{k} r_i \leq \sum_{j=1}^{n} \min(k, s_j) \tag{1.7}$$

hold for all $k=1, 2,\cdots, t+1$ with respect to a pair of $t+1$ and n partitions

$$\Pi_{t+1,n} = \{(r_1, r_2,\cdots, r_{t+1}), (s_1, s_2,\cdots, s_n)\}. \tag{1.8}$$

Without loss of generality, we may rearrange s_j in a way such that they satisfy

$s_1 \geq s_2 \geq \cdots \geq s_n$. Since it follows from (1.7) that $r_1 \leq \sum_{j=1}^{n} \min(1, s_j)$, s_{r_1} must be a positive integer. Thus a modified pair of t and n partitions

$$\Pi_{t,n} = \{(r'_1, r'_2, \cdots, r'_t), (s'_1, s'_2, \cdots, s'_n)\} \tag{1.9}$$

can be constructed from (1.8), where $r'_i = r_{i+1}$ for $i=1, 2, \cdots, t$; $s'_j = s_j - 1$ for $j=1, 2, \cdots, r_1$ and $s'_j = s_j$ for $j = r_1+1, \cdots, n$.

For every k satisfying $1 \leq k < s_{r_1}$, it follows that

$$\begin{aligned} r'_1 + r'_2 + \cdots + r'_k &= r_2 + \cdots + r_{k+1} \leq k r_1 \\ &= \sum_{j=1}^{r_1} \min(k, s'_j) \leq \sum_{j=1}^{n} \min(k, s'_j) \end{aligned} \tag{1.10}$$

and for every k satisfying $s_{r_1} \leq k \leq t$, it follows that

$$\begin{aligned} r'_1 + r'_2 + \cdots + r'_k &= r_1 + r_2 + \cdots + r_{k+1} - r_1 \\ &\leq \sum_{j=1}^{n} \min(k+1, s_j) - r_1 \\ &= \sum_{j=1}^{r_1} \min(k+1, s_j) + \sum_{j=r_1+1}^{n} \min(k+1, s_j) - r_1 \\ &= \sum_{j=1}^{r_1} \min(k, s'_j) + r_1 + \sum_{j=r_1+1}^{n} \min(k, s'_j) - r_1 \\ &= \sum_{j=1}^{n} \min(k, s'_j) \end{aligned} \tag{1.11}$$

The inequalities (1.10) and (1.11) show that $\Pi_{t,n}$ in (1.9) is bigraphical by the induction hypothesis. So is $\Pi_{t+1,n}$ in (1.8).

This completes the proof.

COROLLARY 1.3. *A pair of m and n partitions*

$$\Pi_{m,n} = \{(r_1, r_2, \cdots, r_m), (s, s, \cdots, s)\} \tag{1.12}$$

of a nonnegative integer N with $r_1 \geq r_2 \geq \cdots \geq r_m$ *is bigraphical if and only if*

$$r_1 \leq n. \tag{1.13}$$

PROOF. It is sufficient to show that (1.13) is equivalent to (1.6) with $s_1 = s_2 = \cdots = s_n = s$. The latter can be reduced to

$$\sum_{i=1}^{k} r_i \leq n \min(k, s) \tag{1.14}$$

for all $k = 1, 2, \cdots, m$.

From (1.14) with $k=1$, we have $r_1 \leq n \min(1, s) \leq n$. Hence we have (1.13). Conversely, for every k satisfying $1 \leq k \leq s$, it follows that

$$n \min(k, s) = n k \geq r_1 k \geq r_1 + r_2 + \cdots + r_k,$$

and for every k satisfying $s+1 \leq k \leq m$, it follows that

$$n \min(k, s) = ns = r_1 + r_2 + \cdots + r_m \geq r_1 + r_2 + \cdots + r_k.$$

This completes the proof.

§ 2. Claw-decomposition theorems

Now we shall state the claw-decomposition theorem of complete graphs:

THEOREM 2.1. *A complete graph, K_l, with l points and $\binom{l}{2}$ lines can be decomposed into a union of line disjoint $\binom{l}{2}/c$ claws, $K_{1,c}^{(\alpha)}$, with c lines each if and only if*

(i) *$\binom{l}{2}$ is an integral multiple of c, and*

(ii) *$l \geq 2c$.*

PROOF. (Necessity) The condition (i) is obviously necessary. Suppose $l < 2c$ and assume that K_l can be decomposed into a union of line disjoint $b = \binom{l}{2}/c$ claws. Since $b < l-1$, there exists a point which cannot be the root (or a point of degree c) of any claw. Its degree must be less than $l-1$. This contradicts the fact that K_l is regular of degree $l-1$. The condition (ii) is, therefore, necessary.

(Sufficiency) The set of $\binom{l}{2}$ lines of a complete graph K_l can be identified with the triangular set

$$T = \{(i, j) | 1 \leq i < j \leq l\} \tag{2.1}$$

of $\binom{l}{2}$ lattice points (i, j). The set of c lines of a claw $K_{1,c}$ which is a subgraph of the K_l can be identified with a subset of T composed of c lattice points standing together on the same i-th row and/or i-th column. Such a subset may be called a *claw-type subset* of T. The proof of sufficiency will be completed by giving an algorithm of the decomposition of T into mutually disjoint $b = \binom{l}{2}/c$ claw-type subsets assuming that (i) and (ii) hold. This algorithm will be given by dividing it into the subsequent three cases.

Case 1. $2c \leq l < 3c$

Put $l=2c+r$ and $b=\binom{l}{2}/c=2c-1+2r+b_1$. Since $0\le r<c$ and $b_1=r(r-1)/(2c)$, b_1 is zero if $r=0$ or 1 and an integer satisfying $0<b_1<\frac{r-1}{2}$ otherwise.

The set T of $\binom{l}{2}$ lattice points will be decomposed into the following subsets A, B, C, D and E:

$$\begin{aligned} A &= \{(i,j)\,|\,1 \le i < j \le c+1\} \\ B &= \{(i,j)\,|\,c+1 \le i < j \le c+r+1\} \\ C &= \{(i,j)\,|\,c+r+1 \le i < j \le l\} \\ D &= \{(i,j)\,|\,1 \le i \le c, c+2 \le j \le c+r+1\} \\ E &= \{(i,j)\,|\,1 \le i \le c+r, c+r+2 \le j \le l\}\,. \end{aligned} \tag{2.2}$$

Among those subsets, D can be decomposed into r claw-type subsets by dividing it into r rows. In order to decompose the remaining T–D into claw-type subsets, every point in E will first be classified into those either labelled (r) or labelled (c) in a way such that the number of points labelled (r) in each column ranging from the 1st to the $(c+r)$-th column will be $c-1, c-2,\cdots, 1, 0, r-1, r-2,\cdots, 1, 0$, respectively and that the number of points labelled (r) in each row ranging from the $(c+r+2)$-nd to the l-th row will be $c-1, c-2,\cdots, b_1+1, b_1+c, b_1-1+c,\cdots, 1+c$, respectively. The remaining points will be labelled (c). This labelling will be performed by the algorithm given in Theorem 1.1. As will be seen presently, the pair of $c-1$ and $c+r$ partitions

$$\begin{aligned} &\{(c-1, c-2,\cdots, b_1+1, b_1+c,\cdots, 1+c), \\ &\quad (c-1, c-2,\cdots, 1, 0, r-1, r-2,\cdots, 1, 0)\} \end{aligned} \tag{2.3}$$

of $c(c-1)/2+r(r-1)/2$ satisfies the condition of Theorem 1.2.

Since $c+b_1>c+b_1-1>\cdots>c+1>c-1>c-2>\cdots>b_1+1$, the sum of the largest k integers, R_k, of the left hand member of (1.6) will be

$$R_k=\begin{cases} k(c+b_1)-\dfrac{k(k-1)}{2} & \text{for} \quad 1\le k\le b_1 \\ ck+b_1(k+1)-\dfrac{k(k+1)}{2} & \text{for} \quad b_1+1\le k\le c-1. \end{cases}$$

The right hand member of (1.6), S_k, will be

$$S_k=\begin{cases} k(r+c-k-1) & \text{for} \quad 1\le k\le r-1 \\ \dfrac{r(r-1)}{2}+ck-\dfrac{k(k+1)}{2} & \text{for} \quad r\le k\le c-1\,. \end{cases}$$

When $r=0$ or 1, since $b_1=0$, we have $S_k-R_k=0$ for all $1\leq k\leq c-1$. When $2\leq r<c$, since $0<b_1<\frac{r-1}{2}<r-1<c-1$, we have

$$S_k-R_k=\begin{cases}\frac{k}{2}(2r-k-2b_1-3)>\frac{k}{2}(r-1-k-1)\geq 0 & \text{for } 1\leq k\leq b_1\\ \frac{1}{2}k\{(r-1-2b_1)+(r-k)\}-b_1\geq k-b_1>0 & \text{for } b_1+1\leq k\leq r-1\\ \frac{r(r-1)}{2}-b_1(k+1)\geq\frac{r(r-1)}{2}-b_1c=0 & \text{for } r\leq k\leq c-1.\end{cases}$$

Thus we have $S_k-R_k\geq 0$ for all $k=1, 2,\cdots, c-1$, and, consequently, (2.3) is bigraphical.

After labelling the points in E, the subsets A and B are divided vertically into c subsets containing c, $c-1,\cdots$, 1 points, respectively and r subsets containing r, $r-1,\cdots$, 1 points, respectively. Combining those points labelled (c) in E which are standing on the corresponding columns to the above subsets, we have $c+r$ claw-type subsets, since there are 0, $1,\cdots, c-1$, $c-r$, $c-r+1,\cdots, c-1$ points labelled (c) in the corresponding columns of E, respectively.

The subset C is divided horizontally into $c-1$ subsets. These subsets contain $1, 2,\cdots, c-1$ points, respectively. Combining those $c-1$, $c-2,\cdots, 2$, 1 points labelled (r) in E to the corresponding subsets, we have $c-1$ claw-type subsets. The remaining $b_1\times c$ points labelled (r) can easily be divided into b_1 claw-type subsets. This completes the decomposition of T into $b=2c+2r-1+b_1$ claw-type subsets.

Case 2. $3c\leq l<4c$

Put $l=3c+r$ and $b=\binom{l}{2}/c=4c+3r-1+b_2$. Since $0\leq r<c$ and $b_2=\{c(c-1)+r(r-1)\}/(2c)$, b_2 is a positive integer satisfying $\frac{c-1}{2}\leq b_2<c-1$. In this case, T will be divided into the following subsets:

$$\begin{aligned} A_1 &= \{(i,j)\,|\,1\leq i<j\leq c+1\}\\ A_2 &= \{(i,j)\,|\,c+1\leq i<j\leq 2c+1\}\\ B &= \{(i,j)\,|\,2c+1\leq i<j\leq 2c+r+1\}\\ C &= \{(i,j)\,|\,2c+r+1\leq i<j\leq l\}\\ D_1 &= \{(i,j)\,|\,1\leq i\leq c,\, c+2\leq j\leq 2c+1\}\end{aligned} \tag{2.4}$$

$$D_2 = \{(i,j) \mid 1 \le i \le c, 2c+2 \le j \le 2c+r+1\}$$

$$D_3 = \{(i,j) \mid c+1 \le i \le 2c, 2c+2 \le j \le 2c+r+1\}$$

$$E = \{(i,j) \mid 1 \le i \le 2c+r, 2c+r+2 \le j \le l\}.$$

The subsets D_1, D_2 and D_3 can be divided horizontally into $c+2r$ claw-type subsets. As in Case 1, the labelling of points in E will be performed first by determining a 0–1 matrix of size $(c-1)\times(2c+r)$ with row totals $c-1, c-2, \cdots, b_2+1, c+b_2, c+b_2-1, \cdots, c+1$ and column totals $c-1, c-2, \cdots, 1, 0, c-1, c-2, \cdots, 1, 0, r-1, r-2, \cdots, 1, 0$, respectively. It can be shown that the labelling is possible by the similar manner shown in Case 1, since those totals satisfy the condition (1.6) of Theorem 1.2. Those subsets A_1, A_2 and B will be divided vertically into $2c+r$ subsets and combining those points labelled (c) of corresponding columns, we have $2c+r$ claw-type subsets. The subset C will be divided horizontally into $c-1$ subsets and combining those points labelled (r) of corresponding rows we have $c-1$ claw-type subsets. The remaining $b_2\times c$ points labelled (r) will easily be divided into b_2 claw-type subsets. This completes the decomposition of T into $b=4c+3r-1+b_2$ claw-type subsets.

Case 3. $l \ge 4c$

There exist positive integers n and l_0 satisfying $l=2nc+l_0$ and $2c \le l_0 < 4c$. In this case, T can be divided into $2n+1$ subsets:

$$T_0 = \{(i,j) \mid 1 \le i < j \le l_0\}$$

$$U_p = \{(i,j) \mid 1 \le i \le l_0+2(p-1)c-1, l_0+2(p-1)c+1 \le j \le l_0+2pc\} \quad (2.5)$$

$$V_p = \{(i,j) \mid l_0+2(p-1)c \le i < j \le l_0+2pc\}; \quad p = 1, 2, \cdots, n.$$

Since $2c \le l_0 < 4c$, the decomposition of T_0 will be reduced to Case 1 or Case 2 described above. The decomposition of V_p can be performed by the method described in Case 1 since it is the same with that of $l=2c+1$. The decomposition of U_p can be performed by dividing them vertically since there stand $2c$ points vertically in each of the columns.

This completes the proof of Theorem 2.1.

The claw-decomposition theorem of complete bigraphs will be given in the following:

THEOREM 2.2. *A complete bigraph, $K_{m,n}$, with m and n points and mn lines can be decomposed into union of mn/c line disjoint claws, $K_{1,c}^{(\alpha)}$, with c lines each if and only if m and n satisfy one of the following three conditions*:

(i) $n \equiv 0 \pmod{c}$ *when* $m < c$

(ii) $m \equiv 0 \pmod{c}$ *when* $n < c$

(iii) $mn \equiv 0 \pmod{c}$ *when* $m \ge c$ *and* $n \ge c$.

Before entering the proof of Theorem 2.2, we may note that the set of mn lines of a complete bigraph $K_{m,n}$ with m and n points can be identified with the rectangular set

$$R = \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \tag{2.6}$$

of mn lattice points $(i;j)$. The set of c lines of a claw $K_{1,c}$ which is a subgraph of the $K_{m,n}$ can, in this case, be identified with a subset of R composed of c lattice points standing together on the same row or the same column. Such a subset of R may be called a claw-type subset. The claw-decomposition problem of a complete bigraph $K_{m,n}$ is, therefore, equivalent to the decomposition problem of the rectangular set R of mn lattice points into the union of mn/c mutually disjoint claw-type subsets with c points each. The proof of Theorem 2.2 will, therefore, be performed by using the latter expressions.

PROOF of Theorem 2.2.

When $m<c$ and $n<c$, there is no claw-type subset in R, since the number of points on the same row or the same column is less than c. Hence the claw-decomposition of R is impossible in this case.

When $m<c$ and $n\geq c$, since the lattice points of any claw-type subset of R must be on the same row, the condition $n\equiv 0 \pmod c$ is necessary. Evidently, this is also sufficient for the claw-decomposition.

When $n<c$ and $m\geq c$, the condition $m\equiv 0 \pmod c$ is also necessary and sufficient.

When $m\geq c$ and $n\geq c$, the condition $mn\equiv 0 \pmod c$ is necessary, since the number of lattice points of R must be an integral multiple of c. The condition is also sufficient, as will be seen presently.

Let $m=m_0+pc$ and $n=n_0+qc$ where p and q are nonnegative integers and m_0 and n_0 are positive integers satisfying the inequalities $c\leq m_0<2c$ and $c\leq n_0<2c$. Then R can be decomposed into the union of three mutually disjoint subsets R_0, R_1 and R_2:

$$\begin{aligned} R_0 &= \{(i,j) \mid 1 \leq i \leq m_0, 1 \leq j \leq n_0\} \\ R_1 &= \{(i,j) \mid 1 \leq i \leq m_0, n_0+1 \leq j \leq n_0+qc\} \\ R_2 &= \{(i,j) \mid m_0+1 \leq i \leq m_0+pc, 1 \leq j \leq n_0+qc\}. \end{aligned} \tag{2.7}$$

Since the number of lattice points in each row of R_2 as well as in each column of R_1 is an integral multiple of c, both R_1 and R_2 can be decomposed into unions of mutually disjoint claw-type subsets. Thus it is sufficient to show that R_0 with $m_0\times n_0$ lattice points is claw-decomposable when $m_0n_0\equiv 0 \pmod c$.

If either m_0 or n_0 is equal to c, R_0 is clearly claw-decomposable by dividing it horizontally or vertically. If $m_0>c$ and $n_0>c$, then $t=\frac{m_0(n_0-c)}{c}$ is a positive integer. In this case R_0 will be divided into two subsets R_{01} and R_{02}:

$$(2.8)\qquad \begin{aligned} R_{01} &= \{(i,j) \mid 1 \leq i \leq m_0, 1 \leq j \leq t\} \\ R_{02} &= \{(i,j) \mid 1 \leq i \leq m_0, t+1 \leq j \leq n_0\}. \end{aligned}$$

Each of the $m_0 \times t$ points in R_{01} can be labelled (r) or (c) in a way such that the column sum vector of the number of points labelled (r) is $(n_0-c, n_0-c, \cdots, n_0-c)$ and the row sum vector of them is $(c, c, \cdots, c)$, since the set of these vectors satisfies the condition of Corollary 1.3. After labelling those points in R_{01}, R_{02} will be divided vertically into m_0 subsets with n_0-t points each. Adding $t-n_0+c$ points labelled (c) on the corresponding column of R_{01} to each of the m_0 subsets, we have m_0 claw-type subsets in R_0. The remaining points labelled (r) in R_{01} can be divided horizontally into t claw-type subsets. This completes the claw-type decomposition of R_0. The condition is, therefore, sufficient.

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