HIROSHIMA MATH. J.
5 (1975), 23–31

Nonoscillation Criteria for Fourth Order Elliptic Equations

Norio YOSHIDA

(Received July 29, 1974)

The problem of oscillation and nonoscillation of solutions of elliptic partial differential equations has been the subject of numerous investigations. For nonoscillation results we refer to Headley [2], Headley and Swanson [3], Kreith [4], Kuks [5], Piepenbrink [6], Skorobogat'ko [7], Swanson [8] and Yoshida [9]. All of these papers deal with second order elliptic equations or systems, and the author knows of no nonoscillation criteria which are applicable to equations of higher order.

Our purpose here is to develop nonoscillation criteria for the fourth order elliptic equation with real coefficients

$$Lu = \sum_{i,j,k,l=1}^{n} D_{ij}(\alpha_{ij}(x)\alpha_{kl}(x)D_{kl}u) + 2\beta(x)\sum_{k,l=1}^{n}\alpha_{kl}(x)D_{kl}u + \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) + 2\sum_{i=1}^{n} b_i(x)D_i u + c(x)u = 0 \tag{1}$$

defined in an unbounded domain R of Euclidean n-space E^n. As usual, points in E^n will be denoted by $x=(x_1,\dots, x_n)$, differentiation with respect to x_i by D_i, $i=1,\dots, n$, and successive differentiation with respect to x_i and x_j by D_{ij}, $i, j= 1,\dots, n$. The following assumptions will be made throughout:

(a) The coefficients $\alpha_{ij}\in C^2(R)$, $\beta\in C(R)$, $a_{ij}\in C^1(R)$, $b_i\in C^1(R)$ and $c\in C(R)$.

(b) The matrix (α_{ij}) is symmetric and positive definite in R.

(c) The matrix (a_{ij}) is symmetric and negative semidefinite in R.

These assumptions will be placed without further mention on the coefficients of elliptic operators of the same form as L which will be considered in the sequel.

The domain $\mathfrak{D}(L; G)$ of L relative to any subdomain G of R is defined as the set $C^4(G)\cap C^2(\bar{G})$. The notation

$$R_r = R \cap \{x\in E^n: |x| > r\}, \qquad 0 < r < \infty,$$

will be used throughout.

DEFINITION 1. A bounded subdomain G of R is called a *nodal* domain of (1) if there exists a nontrivial solution $u\in\mathfrak{D}(L; G)$ of (1) such that $u=D_i u=0$

on ∂G, $i=1,\dots, n$. Equation (1) is said to be *nonoscillatory* in R if there is a number $r>0$ such that there are no nodal domains of (1) in R_r.

Associated with the operator L in (1) is the following elliptic operator

$$(2) \qquad L_0u = \sum_{i,j,k,l=1}^{n} D_{ij}(\alpha_{ij}(x)\alpha_{kl}(x)D_{kl}u)+2\beta(x)\sum_{k,l=1}^{n}\alpha_{kl}(x)D_{kl}u + (c(x)-\nabla\cdot b(x))u.$$

DEFINITION 2. The elliptic operator L_1 defined by

$$(3) \qquad L_1u = \sum_{i,j,k,l=1}^{n} D_{ij}(A_{ij}(x)A_{kl}(x)D_{kl}u)+2B(x)\sum_{k,l=1}^{n}A_{kl}(x)D_{kl}u+C(x)u$$

is said to belong to $\mathfrak{M}[L_0; R_r]$ for some $r>0$, if for every bounded subdomain G of R_r the functional

$$V[u; G] = \int_G [(\sum_{k,l=1}^{n}\alpha_{kl}D_{kl}u)^2 -(\sum_{k,l=1}^{n}A_{kl}D_{kl}u)^2 + 2u\sum_{k,l=1}^{n}(\beta\alpha_{kl}-BA_{kl})D_{kl}u+(c-\nabla\cdot b-C)u^2]dx$$

is nonnegative for all real-valued $u\in C^2(\bar{G})$ such that $u=D_iu=0$ on ∂G, $i=1,\dots, n$.

Our first result is the following

THEOREM 1. *Equation* (1) *is nonoscillatory in* R *if for some* $r>0$ *there exist an elliptic operator* $L_1\in\mathfrak{M}[L_0; R_r]$ *and a function* $\phi\in C^2(R_r)$ *such that*

(i) $\phi>0$ [*resp.* $\geqq 0$] *in* R_r;

(ii) $\sum_{i,j=1}^{n} D_{ij}(A_{ij}\phi)+\phi^2+2B\phi+B^2-C\leqq 0$ [*resp.* <0] *in* R_r.

PROOF. Suppose to the contrary that equation (1) is not nonoscillatory in R. Then, there are a bounded domain $G\subset R_r$ and a nontrivial function $u\in\mathfrak{D}(L; G)$ such that $Lu=0$ in G and $u=D_iu=0$ on ∂G, $i=1,\dots, n$. Applying Green's formula, we obtain

$$(4) \qquad 0 = \int_G \sum_{i,j=1}^{n}[D_j(2uD_iuA_{ij}\phi)-D_i(u^2D_j(A_{ij}\phi))]dx$$
$$= \int_G \sum_{i,j=1}^{n}[2\phi uA_{ij}D_{ij}u-u^2D_{ij}(A_{ij}\phi)+2\phi A_{ij}D_iuD_ju]dx.$$

By the hypothesis $L_1\in\mathfrak{M}[L_0; R_r]$ and Green's formula we have

$$(5) \qquad 0 = \int_G uLu\,dx$$
$$= \int_G [(\sum_{i,j=1}^{n}\alpha_{ij}D_{ij}u)^2+2\beta u\sum_{i,j=1}^{n}\alpha_{ij}D_{ij}u - \sum_{i,j=1}^{n}a_{ij}D_iuD_ju$$

$$+ (c-\nabla\cdot b)u^2]dx$$

$$\geqq \int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2 + 2Bu \sum_{i,j=1}^n A_{ij}D_{ij}u + Cu^2]dx.$$

In view of (4) it holds that

$$(6) \quad \int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2 + 2Bu \sum_{i,j=1}^n A_{ij}D_{ij}u + Cu^2]dx$$

$$= \int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2 + 2(B+\phi)u \sum_{i,j=1}^n A_{ij}D_{ij}u + (C - \sum_{i,j=1}^n D_{ij}(A_{ij}\phi))u^2]dx$$

$$+ 2\int_G \phi \sum_{i,j=1}^n A_{ij}D_iuD_ju\, dx \equiv I_1 + I_2.$$

Since $u=0$ on ∂G and u is nontrivial, we see that $\nabla u \not\equiv 0$ in G. Hence, if $\phi > 0$ in R_r, then the integral I_2 is positive. From the hypothesis (ii) the integral I_1 is nonnegative. This contradicts the inequality (5). If $\phi \geqq 0$ in R_r, then I_2 is nonnegative. The hypothesis (ii) implies that I_1 is positive. This again contradicts (5). Thus the proof is complete.

Before stating the second result we prove a lemma regarding the positivity of the quadratic form

$$(7) \quad Q[\xi] = \sum_{i,j=1}^n A_{ij}(x)\xi_i(x)\xi_j(x) + 2\xi_{n+1}(x)\sum_{i=1}^n \phi_i(x)\xi_i(x) + \psi(x)\xi_{n+1}^2(x),$$

where the functions $A_{ij}(x)$, $\phi_i(x)$, $\psi(x)$, $\xi_i(x)$ are all continuous in a subdomain G of R and the matrix $(A_{ij}(x))$ is positive definite in G.

LEMMA. *If we have*

$$\psi - \phi A^{-1}\phi^* \geqq 0 \quad \text{in} \quad G,$$

then $Q[\xi]$ is nonnegative in G for each ξ and positive at some point of G for each ξ such that

$$(8) \qquad \xi' + \xi_{n+1}\phi A^{-1} \not\equiv 0 \quad \text{in} \quad G,$$

where $\phi = (\phi_1, \dots, \phi_n)$, $A = (A_{ij})$, $\xi = (\xi', \xi_{n+1})$, $\xi' = (\xi_1, \dots, \xi_n)$ and ϕ^* denotes the transpose of ϕ.

PROOF. The matrix Q associated with $Q[\xi]$ has the block form

$$Q = \begin{bmatrix} A & \phi^* \\ \phi & \psi \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ \phi A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \psi - \phi A^{-1}\phi^* \end{bmatrix} \begin{bmatrix} I_n & A^{-1}\phi^* \\ 0 & 1 \end{bmatrix},$$

where I_n is the $n\times n$ identity matrix. Letting $P=\begin{bmatrix} I_n & 0 \\ \phi A^{-1} & 1\end{bmatrix}$ and $\eta=\xi'+\xi_{n+1}\phi A^{-1}$, we have $\xi P=(\eta, \xi_{n+1})$ and

$$(9) \qquad Q[\xi] = \xi P\begin{bmatrix} A & 0 \\ 0 & \psi-\phi A^{-1}\phi^* \end{bmatrix} P^*\xi^* = \eta A\eta^* + (\psi-\phi A^{-1}\phi^*)\xi_{n+1}^2 .$$

From the hypothesis (8) we see that $\eta\not\equiv 0$ in G, hence $Q[\xi]$ is positive at some point of G. This completes the proof.

The following theorem is the main result of this paper.

THEOREM 2. *Equation* (1) *is nonoscillatory in* R *if for some* $r>0$ *there exist an elliptic operator* $L_1\in\mathfrak{M}[L_0; R_r]$ *and functions* ϕ, $\psi\in C^2(R_r)$ *such that*

(i) $\phi>0$ [*resp.* $\geqq 0$] *in* R_r;

(ii) $\sum_{i,j=1}^n D_{ij}(A_{ij}\phi)+2\sum_{i,j=1}^n D_i(A_{ij}\phi)D_j\psi-\phi^2+2B\phi-C+B^2\leqq 0$ [*resp.* <0] *in* R_r;

(iii) $\sum_{i,j=1}^n A_{ij}D_{ij}\psi+\sum_{i,j=1}^n A_{ij}D_i\psi D_j\psi+\phi\leqq 0$ *in* R_r;

(iv) *For any bounded subdomain* G *of* R_r *the condition*

$$\nabla u-u\nabla\psi\not\equiv 0 \quad in \quad G$$

holds for all nontrivial $u\in\mathfrak{D}(L; G)$ *such that* $u=D_iu=0$ *on* ∂G, $i=1,\dots, n$.

PROOF. Suppose to the contrary that equation (1) is not nonoscillatory in R. Then, there are a bounded domain $G\subset R_r$ and a nontrivial function $u\in\mathfrak{D}(L; G)$ such that $Lu=0$ in G and $u=D_iu=0$ on ∂G, $i=1,\dots, n$. Applying Green's formula, we have

$$(10) \qquad 0 = 2\int_G\sum_{i,j=1}^n D_i(\phi u^2A_{ij}D_j\psi)\,dx = \int_G\sum_{i,j=1}^n [2u^2D_i(A_{ij}\phi)D_j\psi+2\phi u^2A_{ij}D_{ij}\psi+4\phi uA_{ij}D_iuD_j\psi]dx .$$

By the hypothesis $L_1\in\mathfrak{M}[L_0; R_r]$ and Green's formula we obtain

$$(11) \qquad 0 = \int_G uLu\,dx \geqq \int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2+2Bu\sum_{i,j=1}^n A_{ij}D_{ij}u+Cu^2]dx .$$

In view of (4) and (10) it holds that

$$\int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2 + 2Bu \sum_{i,j=1}^n A_{ij}D_{ij}u + Cu^2]dx$$

$$= \int_G [(\sum_{i,j=1}^n A_{ij}D_{ij}u)^2 + 2(B+\phi)u \sum_{i,j=1}^n A_{ij}D_{ij}u$$

$$- (\sum_{i,j=1}^n D_{ij}(A_{ij}\phi) + 2\sum_{i,j=1}^n D_i(A_{ij}\phi)D_j\psi - 2\phi^2 - C)u^2]dx$$

$$+ \int_G 2\phi[\sum_{i,j=1}^n A_{ij}D_iuD_ju - 2u\sum_{i,j=1}^n A_{ij}D_i\psi D_ju - (\sum_{i,j=1}^n A_{ij}D_{ij}\psi + \phi)u^2]dx$$

$$\equiv I_1 + I_2 .$$

If $\phi > 0$ in R_r, then the hypotheses (iii), (iv) and Lemma imply that the integral I_2 is positive. From the hypothesis (ii) the integral I_1 is nonnegative. If $\phi \geqq 0$ in R_r, then I_2 is nonnegative. Since u is nontrivial, the hypothesis (ii) implies that I_1 is positive. This contradicts (11) and completes the proof.

COROLLARY 1. *Equation* (1) *is nonoscillatory in* R *if for some* $r>0$ *there exist an elliptic operator* $L_1 \in \mathfrak{M}[L_0; R_r]$ *and a function* $w \in \mathfrak{D}(L_1; R_r)$ *such that*

(i) $w > 0$ *in* $\bar{R}_r$;

(ii) $\sum_{k,l=1}^n A_{kl}D_{kl}w < 0$ [*resp.* $\leqq 0$] *in* R_r;

(iii) $L_1 w \geqq B^2 w$ [*resp.* $> B^2 w$] *in* R_r.

PROOF. Define the functions ϕ and ψ by

$$\phi = -\frac{1}{w}\sum_{k,l=1}^n A_{kl}D_{kl}w, \qquad \psi = \log w . \tag{12}$$

It is easy to verify that

$$\sum_{i,j=1}^n D_{ij}(A_{ij}\phi) + 2\sum_{i,j=1}^n D_i(A_{ij}\phi)D_j\psi - \phi^2 + 2B\phi - C + B^2 = -\frac{1}{w}L_1 w + B^2 ; \tag{13}$$

$$\sum_{i,j=1}^n A_{ij}D_{ij}\psi + \sum_{i,j=1}^n A_{ij}D_i\psi D_j\psi + \phi = 0; \tag{14}$$

$$\nabla u - u\nabla\psi = w\nabla\left(\frac{u}{w}\right). \tag{15}$$

Hence, the conclusion follows from Theorem 2.

Let us consider the following special case of (1):

$$(16) \qquad Lu \equiv \Delta(a(x)\Delta u)+\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju)+2\sum_{i=1}^{n} b_i(x)D_iu+c(x)u = 0,$$

where $a(x)$ is a positive function of class $C^2(R)$. Define the functions α and g by

$$\alpha(r) = \min_{x\in S_r} a(x),$$

$$g(r) = \min_{x\in S_r} [c(x)-\nabla\cdot b(x)],$$

where $S_r=\bar{R}\cap\{x\in E^n: |x|=r\}$, $0<r<\infty$. In this case the elliptic operator defined by (2) is

$$L_0u \equiv \Delta(a(x)\Delta u)+(c(x)-\nabla\cdot b(x))u.$$

Corollary 2. *Let $\alpha(r)\equiv\alpha>0$ and assume that*

$$(17) \qquad \liminf_{r\to\infty} g(r) > 0 \qquad \textit{for} \quad n = 1, 2, 3, 4,$$

$$(18) \qquad \liminf_{r\to\infty} r^4g(r) > -\frac{n^2(n-4)^2}{16}\alpha \qquad \textit{for } n\geqq 5.$$

Then, equation (16) *is nonoscillatory in R.*

Proof. As an elliptic operator defined by (3) we take

$$L_1u \equiv \alpha\Delta^2u+g(|x|)u.$$

It is clear that $L_1\in\mathfrak{M}[L_0; R_r]$ for some $r>0$. The function $w=|x|^m$ satisfies

$$\Delta w = m(m+n-2)|x|^{m-2},$$

$$L_1w = [\alpha f(m)+|x|^4g(|x|)]|x|^{m-4},$$

where $f(m)=m(m-2)(m+n-2)(m+n-4)$. Observing that

$$\max\{f(m): 0\leqq m\leqq 2-n\} = 0 \qquad \text{for} \quad n = 1, 2,$$

$$\max\{f(m): 2-n\leqq m\leqq 0\} = 0 \qquad \text{for} \quad n = 3, 4,$$

$$\max\{f(m): 2-n\leqq m\leqq 0\} = f\left(\frac{4-n}{2}\right)=\frac{n^2(n-4)^2}{16} \qquad \textit{for} \quad n\geqq 5,$$

and using (17) and (18), we see that there is a number $r>0$ such that $\Delta w<0$ [resp. $\leqq 0$] and $L_1w\geqq 0$ [resp. >0] in R_r. Now the conclusion follows from Corollary 1.

Corollary 3. *Assume there exists a positive function $\gamma(r)\in C^2[r_0,\infty)$ such that*

$$\gamma(r)\alpha(r) \geqq 1 \qquad \textit{for} \quad r \geqq r_0,$$

$$\gamma(r)+g(r) \geqq 0 \qquad \textit{for} \quad r \geqq r_0.$$

If the ordinary differential equation

$$\frac{d}{dr}\left(r^{n-1}\frac{dw}{dr}\right)+r^{n-1}\gamma(r)w=0 \tag{19}$$

is nonoscillatory at $r=+\infty$, *then equation* (16) *is nonoscillatory in* R.

PROOF. Let

$$L_1 u \equiv \Delta(\gamma^{-1}(|x|)\Delta u)+g(|x|)u.$$

Clearly, $L_1 \in \mathfrak{M}[L_0; R_r]$ for any $r>r_0$. The nonoscillation of (19) ensures the existence of a positive function $w=w(|x|)$ such that for some $r>r_0$

$$\Delta w+\gamma(|x|)w = 0 \qquad \text{in} \quad R_r,$$

which implies that $\Delta w<0$ in R_r. Moreover, we find

$$L_1 w = (\gamma(|x|)+g(|x|))w \geqq 0 \qquad \text{in} \quad R_r.$$

Therefore, the conclusion follows from Corollary 1.

APPENDIX. One of the main tools in the study of comparison and oscillation theory of elliptic partial differential equations is the identity of Picone type. A Picone identity for fourth order elliptic operators was obtained by Dunninger [1]. Here, we present an extension of the identity due to Dunninger to a class of formally non-self-adjoint fourth order elliptic operators. Our derivation is based on some of the relations that were used above to produce nonoscillation criteria.

Consider the following two elliptic operators:

$$L_1 u = \sum_{i,j,k,l=1}^{n} D_{ij}(A_{ij}(x)A_{kl}(x)D_{kl}u)+2B(x)\sum_{k,l=1}^{n} A_{kl}(x)D_{kl}u+C(x)u,$$

$$L_2 v = \sum_{i,j,k,l=1}^{n} D_{ij}(\alpha_{ij}(x)\alpha_{kl}(x)D_{kl}v)+2\beta(x)\sum_{k,l=1}^{n} \alpha_{kl}(x)D_{kl}v+c(x)v.$$

From (4) and (10) we have the following:

$$\int_G [(\sum_{i,j=1}^{n} A_{ij}D_{ij}u)^2+2Bu\sum_{i,j=1}^{n} A_{ij}D_{ij}u+(C+B^2)u^2]dx \tag{20}$$

$$=\int_G [(\sum_{i,j=1}^{n} A_{ij}D_{ij}u)^2+2(\phi+B)u\sum_{i,j=1}^{n} A_{ij}D_{ij}u$$

$$-\Big(\sum_{i,j=1}^{n} D_{ij}(A_{ij}\phi)+2\sum_{i,j=1}^{n} D_i(A_{ij}\phi)D_j\psi-2\phi^2-C-B^2\Big)u^2\Big]dx$$

$$+\int_G 2\phi\Big[\sum_{i,j=1}^{n} A_{ij}D_iuD_ju-2u\sum_{i,j=1}^{n} A_{ij}D_i\psi D_ju-\Big(\sum_{i,j=1}^{n} A_{ij}D_{ij}\psi+\phi\Big)u^2\Big]dx$$

$$-\int_G \sum_{i,j=1}^{n}[D_j(D_i(u^2)A_{ij}\phi)-D_i(u^2D_j(A_{ij}\phi))]dx$$

$$+2\int_G \sum_{i,j=1}^{n} D_i(\phi u^2A_{ij}D_j\psi)\ dx.$$

Put $\phi=-\frac{1}{w}\sum_{k,l=1}^{n} A_{kl}D_{kl}w$ and $\psi=\log w$ ($w>0$ on $\bar{G}$) in (20). In view of (9), (13), (14), (15) we obtain the integral identity:

$$(21)\quad \int_G\Big[\Big(\sum_{i,j=1}^{n} A_{ij}D_{ij}u\Big)^2+2Bu\sum_{i,j=1}^{n} A_{ij}D_{ij}u+(C+B^2)u^2\Big]dx$$

$$=\int_G\Big[\sum_{i,j=1}^{n} A_{ij}\Big(D_{ij}u-\frac{u}{w}D_{ij}w\Big)+Bu\Big]^2dx+\int_G \frac{L_1w}{w}u^2\ dx$$

$$-\int_G \frac{2}{w}\sum_{k,l=1}^{n} A_{kl}D_{kl}w\sum_{i,j=1}^{n} A_{ij}\Big(D_iu-\frac{u}{w}D_iw\Big)\Big(D_ju-\frac{u}{w}D_jw\Big)dx$$

$$-\int_{\partial G}\sum_{i,j,k,l=1}^{n} \frac{u^2}{w}D_j(A_{ij}A_{kl}D_{kl}w)n_i\ ds$$

$$+\int_{\partial G}\sum_{i,j,k,l=1}^{n} A_{kl}D_{kl}wA_{ij}D_j\Big(\frac{u^2}{w}\Big)n_i\ ds,$$

where $n=(n_i)$ is the unit exterior normal to the boundary ∂G. Applying Green's formula, we easily obtain the identity:

$$(22)\quad \int_{\partial G}\Big[u\sum_{i,j,k,l=1}^{n} D_j(\alpha_{ij}\alpha_{kl}D_{kl}u)n_i-\sum_{i,j,k,l=1}^{n}\alpha_{ij}\alpha_{kl}D_{kl}uD_iun_j\Big]ds$$

$$=\int_G\Big[-\Big(\sum_{i,j=1}^{n}\alpha_{ij}D_{ij}u\Big)^2-2\beta u\sum_{i,j=1}^{n}\alpha_{ij}D_{ij}u-cu^2\Big]dx+\int_G uL_2u\,dx.$$

From (21) and (22) we get the following Picone identity:

$$\int_{\partial G}\frac{u}{w}\Big[w\sum_{i,j,k,l=1}^{n} D_j(\alpha_{ij}\alpha_{kl}D_{kl}u)n_i-u\sum_{i,j,k,l=1}^{n} D_j(A_{ij}A_{kl}D_{kl}w)n_i\Big]ds$$

$$+\int_{\partial G}\Big[\sum_{i,j,k,l=1}^{n} A_{kl}D_{kl}wA_{ij}D_j\Big(\frac{u^2}{w}\Big)n_i\Big]ds-\int_{\partial G}\Big[\sum_{i,j,k,l=1}^{n}\alpha_{ij}\alpha_{kl}D_{kl}uD_jun_i\Big]ds$$

$$=\int_G\Big[\Big(\sum_{i,j=1}^{n} A_{ij}D_{ij}u\Big)^2-\Big(\sum_{i,j=1}^{n}\alpha_{ij}D_{ij}u\Big)^2+2u\sum_{i,j=1}^{n}(BA_{ij}-\beta\alpha_{ij})D_{ij}u$$

$$+ (C+B^2-c)u^2]dx$$

$$+\int_G \frac{2}{w} \sum_{k,l=1}^{n} A_{kl}D_{kl}w \sum_{i,j=1}^{n} A_{ij}\left(D_iu-\frac{u}{w}D_iw\right)\left(D_ju-\frac{u}{w}D_jw\right)dx$$

$$-\int_G\left[\sum_{i,j=1}^{n} A_{ij}\left(D_{ij}u-\frac{u}{w}D_{ij}w\right)+Bu\right]^2 dx+\int_G \frac{u}{w}[wL_2u-uL_1w]dx.$$

ACKNOWLEDGEMENT. The author wishes to thank Professor T. Kusano for his helpful suggestions.

References

[1] D. R. Dunninger, A Picone integral identity for a class of fourth order elliptic differential inequalities, Atti Accad. Naz. Lincei, Rend. **50** (1971), 630–641.
[2] V. B. Headley, Some oscillation properties of selfadjoint elliptic equations, Proc. Amer. Math. Soc. **25** (1970), 824–829.
[3] V. B. Headley and C. A. Swanson, Oscillation criteria for elliptic equations, Pacific J. Math. **27** (1968), 501–506.
[4] K. Kreith, A Picone identity for strongly elliptic systems, Duke Math. J. **38** (1971), 473–481.
[5] L. M. Kuks, Sturm's theorem and oscillation of solutions of strongly elliptic systems, Soviet Math. Dokl. **3** (1962), 24–27.
[6] J. Piepenbrink, Nonoscillatory elliptic equations, J. Differential Equations **15** (1974), 541–550.
[7] V. Ya. Skorobogat'ko, On domains of solvability of Dirichlet's problem for self-adjoint elliptic equations, Ukrain. Mat. Ž. **7** (1955), 91–95 (Russian).
[8] C. A. Swanson, Nonoscillation criteria for elliptic equations, Canad. Math. Bull. **12** (1969), 275–280.
[9] N. Yoshida, Nonoscillation of elliptic differential equations of second order, Hiroshima Math. J. **4** (1974), 279–284.

Department of Mathematics,
Faculty of Science,
Hiroshima University