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A Note on Coreflexive Coalgebras

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Introduction

E. J. Taft [6] has introduced the concept of coreflexive coalgebras. Finite-dimensional coalgebras are coreflexive and the coalgebra of divided powers is coreflexive. The latter is a cocommutative coconnected coalgebra and its space of primitive elements is 1-dimensional. Taft has shown that if a cocommutative coconnected coalgebra is coreflexive, then the space of primitive elements is finite-dimensional. In this paper we show the converse of this result.

To this end, following D. E. Radford's idea in discussing coreflexivity in [3], we introduce a topology in the dual algebra of a coalgebra and give a necessary and sufficient condition for a coalgebra to be coreflexive.

Throughout this paper we employ the notations and terminology used in [4] and [6]. All vector spaces are over a fixed field k. For a vector space V and a subspace X of V

$$X^{\perp} = \{v^* \in V^*: \langle v^*, X\rangle = 0\}$$

and for a subspace Y of V^*

$$Y^{\perp} = \{v \in V: \langle Y, v\rangle = 0\}.$$

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1. The following lemma was indicated in [4], p. 240.

LEMMA 1. *Let $\{C_\mu, \sigma_\nu^\mu\}$ be an inductive system with a directed set M. If every C_μ has a coalgebra structure and every σ_ν^μ is a coalgebra map, then $C = \varinjlim C_\mu$ has a coalgebra structure such that every canonical map σ^μ: $C_\mu \to C$ is a coalgebra map.*

Furthermore, the dual algebra C^* is isomorphic to $\varprojlim C_\mu^*$ as algebras by the canonical map.

PROOF. We denote by Δ_μ and ε_μ the coalgebra structure of C_μ. Since σ_ν^μ is a coalgebra map the maps Δ_μ induce a map Δ': $C \to \varinjlim (C_\mu \otimes C_\mu)$ such that

$$\begin{tikzcd} C \arrow[r] & \varinjlim (C_\mu \otimes C_\mu) \\ C_\mu \arrow[u, "\sigma^\mu"] \arrow[r] & C_\mu \otimes C_\mu \arrow[u, "\text{cano.}"'] \end{tikzcd}$$

is a commutative diagram. Further, we have a map θ induced by $\sigma^\mu \otimes \sigma^\mu$: $C_\mu \otimes C_\mu \to C \otimes C$ such that a diagram

$$\begin{tikzcd} \varinjlim (C_\mu \otimes C_\mu) \arrow[rr, "\theta"] & & C \otimes C \\ & C_\mu \otimes C_\mu \arrow[ul, "\text{cano.}"] \arrow[ur, "\sigma^\mu \otimes \sigma^\mu"'] & \end{tikzcd}$$

commutes. Put $\Delta = \theta \Delta'$. Then the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] & C \otimes C \\ C_\mu \arrow[u, "\sigma^\mu"] \arrow[r] & C_\mu \otimes C_\mu \arrow[u, "\sigma^\mu \otimes \sigma^\mu"'] \end{tikzcd}$$

Similarly, we have a map ε such that a diagram

$$\begin{tikzcd} C \arrow[rr, "\varepsilon"] & & k \\ & C_\mu \arrow[ul, "\sigma^\mu"] \arrow[ur, "\varepsilon_\mu"'] & \end{tikzcd}$$

commutes. It is then easily verified that (C, Δ, ε) is a coalgebra.

Let $\phi: C^* \to \varprojlim C_\mu^*$ be the canonical linear isomorphism. Then for c^*, $d^* \in C^*$, we have

$$\begin{aligned} \sigma_\mu \phi(c^* d^*) &= (\sigma^\mu)^*(c^* d^*) = (\sigma^\mu)^*(c^*)(\sigma^\mu)^*(d^*) \\ &= \sigma_\mu \phi(c^*) \sigma_\mu \phi(d^*), \\ \sigma_\mu \phi \varepsilon &= (\sigma^\mu)^* \varepsilon = \varepsilon \sigma^\mu = \varepsilon_\mu, \end{aligned}$$

where σ_μ denotes the projection of $\varprojlim C_\mu^*$ to C_μ^*. q.e.d.

Let A be an algebra and let $\mathfrak{R}$ be the Jacobson radical of A. If σ_m^n denotes the canonical map of $A/\mathfrak{R}^m$ to $A/\mathfrak{R}^n$, then $\{A/\mathfrak{R}^n, \sigma_m^n\}$ and $\{(A/\mathfrak{R}^n)^{0*}, (\sigma_m^n)^{0*}\}$ are projective systems and $\{(A/\mathfrak{R}^n)^0, (\sigma_m^n)^0\}$ is an inductive system.

LEMMA 2. *Let $\pi_n: A \to A/\mathfrak{R}^n$ be the canonical map, and let ψ be a map induced by $(\pi_n)^0: (A/\mathfrak{R}^n)^0 \to A^0$ such that a diagram*

$$\begin{tikzcd} \varinjlim (A/\mathfrak{R}^n)^0 \arrow[rr, "\psi"] & & A^0 \\ & (A/\mathfrak{R}^n)^0 \arrow[ul, "\sigma^n"] \arrow[ur, "(\pi_n)^0"'] & \end{tikzcd}$$

commutes. Then ψ is a coalgebra isomorphism.

PROOF. We denote the coalgebra structures of $\varinjlim (A/\mathfrak{R}^n)^0$, A^0 and $(A/\mathfrak{R}^n)^0$ by $\{\Delta, \varepsilon\}$, $\{\bar{\Delta}, \bar{\varepsilon}\}$ and $\{\Delta_n, \varepsilon_n\}$ respectively. First we show that ψ is a coalgebra map. If $a^0 \in \varinjlim (A/\mathfrak{R}^n)^0$, then $\sigma^n(a_n^0)=a^0$ for some n and $a_n^0 \in (A/\mathfrak{R}^n)^0$. We have

$$\begin{aligned}
\bar{\Delta}\psi(a^0) &= \bar{\Delta}\psi\sigma^n(a_n^0) = \bar{\Delta}(\pi_n)^0(a_n^0) \\
&= ((\pi_n)^0 \otimes (\pi_n)^0)\Delta_n(a_n^0) \\
&= (\psi\otimes\psi)(\sigma^n\otimes\sigma^n)\Delta_n(a_n^0) \\
&= (\psi\otimes\psi)\Delta\sigma^n(a_n^0) \\
&= (\psi\otimes\psi)\Delta(a^0), \\
\bar{\varepsilon}\psi(a^0) &= \bar{\varepsilon}\psi\sigma^n(a_n^0) = \bar{\varepsilon}(\pi_n)^0(a_n^0) \\
&= \varepsilon_n(a_n^0) = \varepsilon\sigma^n(a_n^0) \\
&= \varepsilon(a^0).
\end{aligned}$$

Since $(\pi_n)^0$ is injective, ψ is injective. Finally, we show that ψ is surjective. Let $a^0 \in A^0$ and let $\mathfrak{a}$ be a cofinite ideal contained in $\operatorname{Ker} a^0$. Then for some $n>0$ $\mathfrak{R}^n \subseteq \mathfrak{a}$ and so $\mathfrak{a}^\perp \subseteq (\mathfrak{R}^n)^\perp$. We take an $a_n^0 \in (A/\mathfrak{R}^n)^0$ such that $(\pi_n)^0(a_n^0)=a^0$. Then $a^0=\psi\sigma^n(a_n^0)$. This completes the proof.

REMARK. By the last part of the above proof, it is clear that if A is a proper algebra then $\bigcap_{n\geq 0}\mathfrak{R}^n=\{0\}$. This is an extended result of Theorem 2.5 (b) in [1].

By Lemma 1, $\phi\colon (\varinjlim(A/\mathfrak{R}^n)^0)^* \to \varprojlim (A/\mathfrak{R}^n)^{0*}$ is an algebra isomorphism. Let Λ be a map induced by $\Lambda_{A/\mathfrak{R}^n}$ such that a diagram

$$\begin{array}{ccc}
\varprojlim A/\mathfrak{R}^n & \xrightarrow{\Lambda} & \varprojlim (A/\mathfrak{R}^n)^{0*} \\
{\scriptstyle \text{proj.}}\downarrow & & \downarrow{\scriptstyle \text{proj.}} \\
A/\mathfrak{R}^n & \xrightarrow[\Lambda_{A/\mathfrak{R}^n}]{} & (A/\mathfrak{R}^n)^{0*}
\end{array}$$

commutes. We denote by Φ the composite map

$$\varprojlim A/\mathfrak{R}^n \longrightarrow \varprojlim (A/\mathfrak{R}^n)^{0*} \xrightarrow{\phi^{-1}} (\varinjlim (A/\mathfrak{R}^n)^0)^* \xrightarrow{(\psi^{-1})^*} A^{0*}.$$

LEMMA 3. *The diagram*

$$\begin{array}{ccc}
A & \xrightarrow{\Lambda_A} & A^{0*} \\
{\scriptstyle \pi}\searrow & & \nearrow{\scriptstyle \Phi} \\
& \varprojlim A/\mathfrak{R}^n &
\end{array}$$

commutes, where π denotes the canonical map.

Proof. For $a \in A$ and $a^0 \in A^0$ we have

$$\begin{aligned} < \Phi\pi(a), a^0 > &= <(\psi^{-1})^* \phi^{-1} \Lambda\pi(a), a^0 > \\ &= < \phi^{-1} \Lambda\pi(a), \psi^{-1}(a^0) > . \end{aligned}$$

Since $\psi^{-1}(a^0) = \sigma^n(a_n^0)$ for some $a_n^0 \in (A/\mathfrak{N}^n)^0$,

$$\begin{aligned} < \Phi\pi(a), a^0 > &= <(\sigma^n)^* \phi^{-1} \Lambda\pi(a), a_n^0 > \\ &= <(\Lambda_{A/\mathfrak{N}^n})\pi_n(a), a_n^0 > \\ &= < a_n^0, \pi_n(a) > \; = \; <(\pi_n)^0(a_n^0), a > \\ &= < \psi\sigma^n(a_n^0), a > \; = \; < a^0, a > \\ &= < \Lambda_A(a), a^0 > . \end{aligned}$$

q.e.d.

2. Let A be an algebra. Then $\mathfrak{I}[A] = \{\mathfrak{a}\colon \mathfrak{a}$ is a cofinite ideal of $A\} \neq \emptyset$ and is a filter base, and so we can define a uniform topology on A. With this topology A is a topological algebra. If A is proper then it is a Hausdorff space. We denote the closure of a subset X of A by $\bar{X}$. We can prove the following lemma by a way similar to Lemma 2.

Lemma 4. *Let A be a proper algebra. Then $\varinjlim (A/\mathfrak{N}^n)^0$ is isomorphic to A^0 as coalgebras.*

Let C be a coalgebra and let R be its coradical. Then C^* is a proper algebra and $R^\perp$ is the Jacobson radical of C^* ([1], Theorem 2.5). We introduce into C^* the linear weak topology determined by a dual pair $< C, C^* >$. If X is a subspace of C^*, then the linear weak closure of X is $X^{\perp\perp}$ and clearly $\bar{X} \subseteq X^{\perp\perp}$. In particular, if C is a coreflexive coalgebra then $\bar{X} = X^{\perp\perp}$. With this topology C^* is again a topological algebra. We define $C_n = \wedge^{n+1} R$, $n \geq 0$, as usual.

Let $\iota_n\colon C_n \to C$ be the inclusion map. Then we have the maps Ψ and λ induced respectively by $(\iota_n)^{*0}\colon (C_n)^{*0} \to C^{*0}$ and $\Lambda_{C_n}\colon C_n \to C_n^{*0}$ such that a diagram

$$\begin{tikzcd} C \arrow[r, "\lambda"] & \varinjlim C_n^{*0} \arrow[r, "\Psi"] & C^{*0} \\ C_n \arrow[u, "\iota_n"] \arrow[r, "\Lambda_{C_n}"'] & C_n^{*0} \arrow[u, "\text{cano.}"'] \arrow[ur, "(\iota_n)^{*0}"'] & \end{tikzcd}$$

commutes. Then the following lemma is easily verified.

Lemma 5. *A diagram*

$$
\begin{tikzcd}
C \arrow[rr, "\Lambda_C"] \arrow[dr, "\lambda"'] & & C^{*0} \\
& \varinjlim C_n^{*0} \arrow[ur, "\Psi"'] &
\end{tikzcd}
$$

commutes.

THEOREM 6. *Let C be a coalgebra. Then C is coreflexive if and only if* (i) *all C_n are coreflexive and* (ii) $C_n{}^{\perp}=\overline{\mathfrak{R}^{n+1}}$, *where $\mathfrak{R}$ denotes the Jacobson radical of C^*.*

PROOF. Suppose that C is coreflexive. Then (i) is clear by Proposition 6.4 of [6]. We show (ii) by induction. If $n=1$, then by Proposition 9.0.0b) of [4]

$$C_1{}^{\perp}=(R\wedge R)^{\perp}=(R^{\perp}R^{\perp})^{\perp\perp}=(\mathfrak{R}^2)^{\perp\perp}=\overline{\mathfrak{R}^2}.$$

Assuming (ii) for $n-1$ we get

$$
\begin{aligned}
C_n{}^{\perp}&=(R\wedge C_{n-1})^{\perp}=(R^{\perp}(C_{n-1})^{\perp})^{\perp\perp}=(\mathfrak{R}\overline{\mathfrak{R}^n})^{\perp\perp}\\
&=\overline{\mathfrak{R}\overline{\mathfrak{R}^n}}=\overline{\mathfrak{R}^{n+1}}.
\end{aligned}
$$

Conversely, we assume the conditions (i) and (ii). We prove the coreflexivity of C. First we show that Ψ is an isomorphism. The composite

$$C^*\xrightarrow{\text{cano.}}C^*/\overline{\mathfrak{R}^{n+1}}=C^*/C_n{}^{\perp}\xrightarrow{\text{cano.}}C_n^*$$

coincides with ι_n^*, and so by the universality of inductive systems and by a commutative diagram

$$
\begin{tikzcd}
C_n^{*0} \arrow[r, "\text{iso.}"] \arrow[d] & (C^*/\overline{\mathfrak{R}^{n+1}})^0 \arrow[r] \arrow[d] & C^{*0} \\
\varinjlim C_n^{*0} \arrow[r, "\text{iso.}"'] & \varinjlim (C^*/\overline{\mathfrak{R}^{n+1}})^0 \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

the composite $\varinjlim C_n^{*0}\to\varinjlim(C^*/\overline{\mathfrak{R}^{n+1}})^0\to C^{*0}$ and Ψ coincide with each other. By assumption Λ_{C_n} is an isomorphism, and so λ is also an isomorphism. Thus we see that Λ_C is an isomorphism, i.e., C is coreflexive. q.e.d.

LEMMA 7. *Let C be a cocommutative coalgebra satisfying the minimum condition on subcoalgebras. Then every ideal of C^* is linearly closed and C^* is a Noetherian algebra.*

PROOF. For $c^*\in C^*$ we denote by c_R^* the right translation by c^*. Then $c_R^*\colon C^*\to C^*$ is a continuous linear map with respect to the linear topology. Since C^* is linearly compact ([2], § 10, 10), this implies that the ideal generated by c^* is linearly closed. Hence it is sufficient for us to show the last part.

Let $\mathfrak{a}$ be an ideal of C^*. Then $\{\mathfrak{b}\colon \mathfrak{b}$ is a finitely generated ideal contained in $\mathfrak{a}\}$ is a non-empty family consisting of linearly closed ideals, and so it has a maximal element $\mathfrak{a}' \subseteq \mathfrak{a}$ by assumption. Then $\mathfrak{a}' = \mathfrak{a}$. In fact, if $\mathfrak{a}' \neq \mathfrak{a}$, then for any $c^* \in \mathfrak{a} - \mathfrak{a}'$, $\mathfrak{a}' + (c^*)$ is a finitely generated ideal contained in $\mathfrak{a}$ and contains $\mathfrak{a}'$ properly. Thus we see that C^* is Noetherian. This completes the proof.

THEOREM 8. *Let C be a cocommutative coalgebra satisfying the minimum condition on subcoalgebras. Then C is coreflexive and C^* is $\mathfrak{R}$-adically complete, where $\mathfrak{R}$ is the Jacobson radical of C^*.*

PROOF. By Lemma 7, $(\mathfrak{R}^{n+1})^{\perp\perp} = \mathfrak{R}^{n+1}$, and so $C_n^{\perp} = (\mathfrak{R}^{n+1})^{\perp\perp} = \mathfrak{R}^{n+1} = \overline{\mathfrak{R}^{n+1}}$. By assumption R is finite-dimensional, and so $\mathfrak{R}$ is cofinite and all $\mathfrak{R}^n$ are cofinite ideals. This implies that each C_n is finite-dimensional and therefore it is coreflexive. By Theorem 6, this implies that C is coreflexive.

Further since all $\mathfrak{R}^n$ are cofinite, $C^*/\mathfrak{R}^n$ are finite-dimensional and so they are reflexive algebras. Hence Φ in Lemma 3 is an isomorphism. Therefore π: $C^* \to \varprojlim C^*/\mathfrak{R}^n$ is also an isomorphism since so is Λ_{C^*} ([6], Proposition 6.1). This completes the proof.

COROLLARY 9. (i) *Let C be a cocommutative coconnected coalgebra. Then C is coreflexive if and only if the space of primitive elements $P(C)$ of C is finite-dimensional.*

(ii) *Let C_i, $i = 1, 2, \dots, n$, be cocommutative coconnected coalgebras. Then $C_1 \otimes C_2 \otimes \cdots \otimes C_n$ is coreflexive if and only if every C_i is coreflexive.*

PROOF. (i) By Heyneman's Theorem [5], if $P(C)$ is finite-dimensional C satisfies the minimum condition on subcoalgebras. Hence C is coreflexive. The converse has been shown by Taft ([6], p. 1127).

(ii) By Corollary 11.0.7 of [4], $P(C_1 \otimes \cdots \otimes C_n) = P(C_1) \oplus \cdots \oplus P(C_n)$, and so this follows from (i). q.e.d.

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